Master Equation Approach to the Dynamics of Open Time-Varying Quantum Systems

M. Scala, B.D. Militello and A. Messina

MIUR and Dipartimento di Scienze Fisiche ed Astronomiche,

Università di Palermo (Italy)

Summary

- Adiabatic Approximation and Berry Phase
- Environment
- Derivation of the Master Equation
- A Simple Illustrative Application

Adiabatic Approximation

- Time-dependent hamiltonian $H(t)$

- Instantaneous eigenstates $\{ |n(t)\rangle \}$

- Evolution (non-degenerate spectrum):

$$|n(t=0)\rangle \rightarrow e^{i\phi(t)}|n(t)\rangle$$

Berry Phase

- Cyclic Evolution $H(T) = H(0)$

$$|n(0)\rangle \rightarrow e^{i\left(\phi_D(T)+\phi_G(T)\right)}|n(0)\rangle$$

- Dynamical Phase $\phi_D(T) = -\int_0^T E_n(t)\ dt$

- Berry Phase

$$\phi_G(T) = i\int_0^T \left\langle n\left(\vec{R}(t)\right)\middle|\nabla_R\ n\left(\vec{R}(t)\right)\right\rangle \cdot \frac{d\vec{R}}{dt}dt$$

Decoherence

- Fluctuations in the classical parameters

- The system is not isolated

$$H = H_S + H_{env} + H_{\text{int}}$$

System-Environment Coupling

Interaction hamiltonian

(A and B hermitian)

$$H_{int} = A \otimes B$$

Expansion of A (constant eigenvalues of Hs(t)):

$$A(\omega, t) = \sum_{n,m:\, \epsilon_m - \epsilon_n = \omega} |\epsilon_n(t)\rangle \langle \epsilon_n(t)| A |\epsilon_m(t)\rangle \langle \epsilon_m(t)|$$

Properties:

$$A(-\omega, t) = A(\omega, t)^\dagger$$

$$A = \sum_\omega A(\omega, t)$$

Interaction Picture

$$H_0 = H_S(t) + H_{env}$$

$$A^I(\omega, t) = \mathrm{e}^{-i\omega t} \sum_{n,m:\, \epsilon_m - \epsilon_n = \omega} \mathrm{e}^{i[\phi_G^m(t) - \phi_G^n(t)]} \left|\epsilon_n(0)\right\rangle \left\langle\epsilon_n(t)\right| A \left|\epsilon_m(t)\right\rangle \left\langle\epsilon_m(0)\right|$$

Evolution of the system density matrix (Born-Markov approximation):

$$\dot{\rho}(t) = -\int_0^\infty d\tau\, Tr_B \left\{ [H_I(t), [H_I(t-\tau), \rho(t) \otimes \rho_B]] \right\}$$

Master Equation (I)

Our Main Hypothesis: $A^I(\omega, t-\tau) \simeq \mathrm{e}^{-i\omega(t-\tau)}\tilde{A}(\omega, t)$

$$\tilde{A}(\omega, t) = \sum_{n,m:\, \epsilon_m - \epsilon_n = \omega} \mathrm{e}^{i[\phi_G^m(t) - \phi_G^n(t)]} \left|\epsilon_n(0)\right\rangle \left\langle\epsilon_n(t)\right| A \left|\epsilon_m(t)\right\rangle \left\langle\epsilon_m(0)\right.$$

$$\dot{\rho}(t) = \sum_{\omega,\omega'} \mathrm{e}^{i(\omega'-\omega)t}\Gamma(\omega) \left(\tilde{A}(\omega, t)\rho(t)\tilde{A}(\omega', t)^\dagger - \tilde{A}(\omega', t)^\dagger \tilde{A}(\omega, t)\rho(t)\right) + h.c$$

$$\Gamma(\omega) = \int_0^\infty d\tau\, \mathrm{e}^{i\omega\tau} \left\langle B(t)B(t-\tau)\right\rangle_B = \int_0^\infty d\tau\, \mathrm{e}^{i\omega\tau} \left\langle B(\tau)B(0)\right\rangle_B$$

Rotating Wave Approximation (RWA) →

Master Equation (II)

In Schroedinger Picture:

$$\dot{\rho}_S(t) = -i\left[H_S(t) + H_{LS}^S, \rho_S(t)\right] + \mathcal{D}_S\left(\rho_S(t)\right)$$

where

$$\begin{cases} H_{LS}^S = \sum_\omega S(\omega)\tilde{A}_S(\omega,t)^\dagger \tilde{A}_S(\omega,t) \\ \mathcal{D}_S\left(\rho\right) = \sum_\omega \gamma(\omega)\left(\tilde{A}_S(\omega,t)\rho\tilde{A}_S(\omega,t)^\dagger - \frac{1}{2}\left[\tilde{A}_S(\omega,t)^\dagger\tilde{A}_S(\omega,t),\rho\right]_+\right) \\ \tilde{A}_S(\omega,t) = \sum_{n,m:\,\epsilon_m-\epsilon_n=\omega} \left|\epsilon_n(t)\right\rangle\left\langle\epsilon_n(t)\right| A \left|\epsilon_m(t)\right\rangle\left\langle\epsilon_m(t)\right| \end{cases}$$

and

$$\begin{cases} \gamma(\omega) = \Gamma(\omega) + \Gamma(\omega)^* \\ S(\omega) = \frac{1}{2i}\left(\Gamma(\omega) - \Gamma(\omega)^*\right) \end{cases}$$

Prescriptions

1. Write down the Master Equation in the case of H_S independent of time

2. Insert the time dependence of the parameters directly into the stationary ME

Corrections to the Master Equation

$$\dot{\rho}(t) = \sum_\omega \Gamma(\omega) \left(\tilde{A}(\omega,t)\rho(t)\tilde{A}(\omega,t)^\dagger - \tilde{A}(\omega,t)^\dagger \tilde{A}(\omega,t)\rho(t) \right) + h.c. +$$

$$+ \sum_\omega \Gamma'(\omega) \left(\tilde{\tilde{A}}(\omega,t)\rho(t)\tilde{A}(\omega,t)^\dagger - \tilde{A}(\omega,t)^\dagger \tilde{\tilde{A}}(\omega,t)\rho(t) \right) + h.c$$

where

$$\tilde{\tilde{A}}(\omega,t) = \sum_{n,m:\, \epsilon_m - \epsilon_n = \omega} \mathrm{e}^{i[\phi_G^m(t) - \phi_G^n(t)]} \left|\epsilon_n(0)\right\rangle \left\langle\epsilon_n(t)\right| \left[\frac{\partial H(t)}{\partial t}, A \right] \left|\epsilon_m(t)\right\rangle \left\langle\epsilon_m(0)\right|$$

$$\Gamma'(\omega) = \int_0^\infty d\tau\, \tau^2\, \mathrm{e}^{i\omega\tau} \left\langle B(\tau)B(0)\right\rangle_B$$

Adiabatic Evolution

$$\dot{\rho}(t) = \mathcal{L}(t)\rho(t)$$

Lindblad-type Master Equation

- Recent criterion: (Sarandy and Lidar, PRA **71**, 012331 (2005))
- Evolution of the Eigenoperators (1-dimensional Jordan blocks):

$$\rho_i(0) \longrightarrow \mathrm{e}^{\gamma_i(t)}\rho_i(t)$$

Phase factors

$$e^{i\gamma_i(t)} = \exp\left\{\int_0^t \left[\lambda_i(t') - f_i\left(\dot{\rho}_i(t')\right)\right] dt'\right\}$$

where

$$\mathcal{L}(t)\rho_i(t) = \lambda_i(t)\rho_i(t) \qquad , \qquad f_i(t)\mathcal{L}(t) = \lambda_i(t) f_i(t)$$

and

$$f_j\left(\rho_i\right) = \left(\tilde{\rho}_j, \rho_i\right) = Tr\left\{\tilde{\rho}_j \rho_i\right\} = \delta_{ij}$$

Example: spin ½

$$H = H_S + H_{env} + H_{int}$$

$$\begin{cases} H_S(t) = \frac{1}{2}\vec{B}(t)\cdot\vec{\sigma} \\ H_{env} = \sum_k \omega_k a_k^\dagger a_k \\ H_{int} = \sigma_x \otimes \sum_k \left(g_k\, a_k + g_k^*\, a_k^\dagger\right) \end{cases}$$

Master Equation for the spin

Time-dependent basis $\{|+_B(t)\rangle, |-_B(t)\rangle\}$

$$\begin{pmatrix} \dot\rho_{++} \\ \dot\rho_{--} \\ \dot\rho_{+-} \\ \dot\rho_{-+} \end{pmatrix} = \begin{pmatrix} -\tilde\gamma(\omega_0) & \tilde\gamma(-\omega_0) & 0 & 0 \\ \tilde\gamma(\omega_0) & -\tilde\gamma(-\omega_0) & 0 & 0 \\ 0 & 0 & -i\omega_0 - \left(\frac{\tilde\gamma(\omega_0)}{2} + \frac{\tilde\gamma(-\omega_0)}{2} + 2\tilde\gamma(0)\right) & 0 \\ 0 & 0 & 0 & i\omega_0 - \left(\frac{\tilde\gamma(\omega_0)}{2} + \frac{\tilde\gamma(-\omega_0)}{2} + 2\tilde\gamma(0)\right) \end{pmatrix} \begin{pmatrix} \rho_{++} \\ \rho_{--} \\ \rho_{+-} \\ \rho_{-+} \end{pmatrix}$$

where

$$\begin{cases} \tilde\gamma(\omega_0) = \gamma(\omega_0) \cdot \left(\cos^2\theta(t)\cos^2\varphi(t) + \sin^2\varphi(t)\right) \\ \tilde\gamma(-\omega_0) = \gamma(-\omega_0) \cdot \left(\cos^2\theta(t)\cos^2\varphi(t) + \sin^2\varphi(t)\right) \\ \tilde\gamma(0) = \gamma(0) \cdot \left(4\cos^2\frac{\theta(t)}{2}\sin^2\frac{\theta(t)}{2}\cos^2\varphi(t)\right) \end{cases}$$

Results (coherences) I

Eigenvalue

$$\lambda_+ = -i\omega_0 - \left(\frac{\tilde{\gamma}(\omega_0)}{2} + \frac{\tilde{\gamma}(-\omega_0)}{2} + 2\tilde{\gamma}(0) \right)$$

Right Eigenoperator

$$\rho_+ = |+_B(t)\rangle \langle -_B(t)|$$

Left Eigenoperator

$$\tilde{\rho}_+ = |-_B(t)\rangle \langle +_B(t)|$$

Adiabatic Evolution

$$|+_B(0)\rangle \langle -_B(0)| \quad \rightarrow \quad \mathrm{e}^{i[\gamma_{+\,D}(t)+\gamma_{+\,G}(t)]} |+_B(t)\rangle \langle -_B(t)|$$

Results (coherences) II

$$\theta(t) = \theta_0 \quad , \quad \varphi(t) = \varphi(0) + \frac{2\pi}{T} t \quad , \quad \left|\vec{B}(t)\right| = B_0$$

Dynamical Phase

$$\gamma_{+D}(t) = -i\omega_0 t - \frac{1}{2}\left\{\frac{\gamma(\omega_0)+\gamma(-\omega_0)}{2}\left(1+\cos^2\theta_0\right) + 2\gamma(0)\sin^2\theta_0\right\} t +$$
$$+\frac{1}{4}\left\{\frac{\gamma(\omega_0)+\gamma(-\omega_0)}{2} - 2\gamma(0)\right\}\sin^2\theta_0\left[\sin 2\varphi(t) - \sin 2\varphi(0)\right]$$

Geometric Phase

$$\gamma_{+G}(t) = -\int_0^t \left\{\langle +_B(t') | \dot{+}_B(t')\rangle - \langle -_B(t') | \dot{-}_B(t')\rangle\right\} dt' =$$
$$= \left[\phi_G^+(t) - \phi_G^-(t)\right]$$

Comparison with previous results

- Sarandy and Lidar, quant-ph/0507012.

  Similar results: no corrections to Berry phase.

- Whitney et al., PRL **94**, 070407 (2005).

  Master Eq. in a rotating frame.

  Different results: corrections to Berry phase.