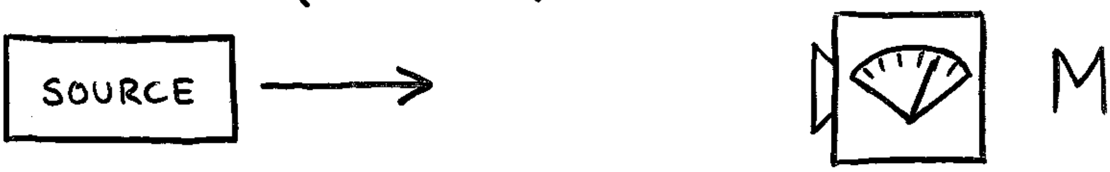
BOUNCING GEODESICS AND THE BEST MEASUREMENT

Åsa Ericsson   Stockholms Universitet

ρ_A or ρ_B ?



What measurement should we choose to determine which state — out of two given — the source produce?

I will give a new geometric characterization.

STATE + MEASUREMENT (POVM) $\longrightarrow$ PROBABILITY DISTRIBUTION

$$\rho_A + \{E_i\} \longrightarrow p_i^{(A)} = \mathrm{Tr}\, E_i \rho_A$$

$$\rho_B + \{E_i\} \longrightarrow p_i^{(B)} = \mathrm{Tr}\, E_i \rho_B$$

BHATTACHARYYA–WOOTTERS STATISTICAL DISTANCE:

$$d(p^{(A)}, p^{(B)}) = \arccos \sum_i \sqrt{p_i^{(A)}\ p_i^{(B)}}$$

For quantum states: Maximize over POVMs!

$$d(\rho_A, \rho_B) = \arccos \mathrm{Tr} \sqrt{\rho_A^{1/2}\, \rho_B\, \rho_A^{1/2}}$$

The observable corresponding to this distance:

$$M = \rho_A^{-1/2} \sqrt{\rho_A^{1/2}\, \rho_B\, \rho_A^{1/2}}\ \rho_A^{-1/2}$$

— FUCHS & CAVES (1995)

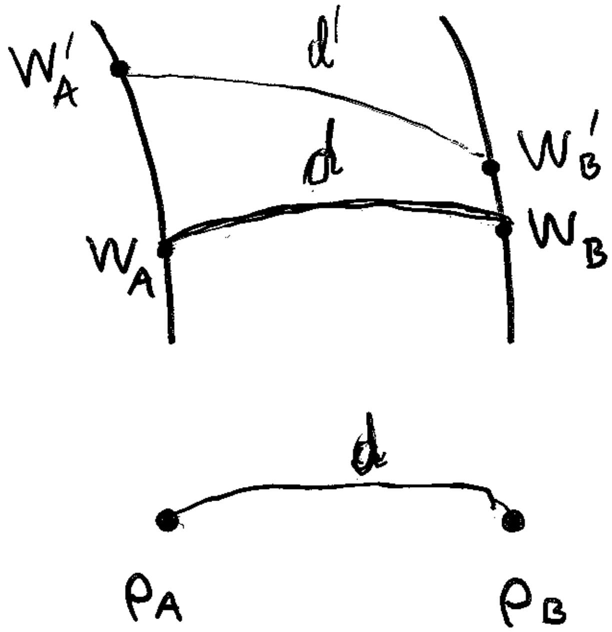
PHYSICAL INTERPRETATION:

State Purification

Mixed states are purified in $\mathcal{HS}$: W pure state

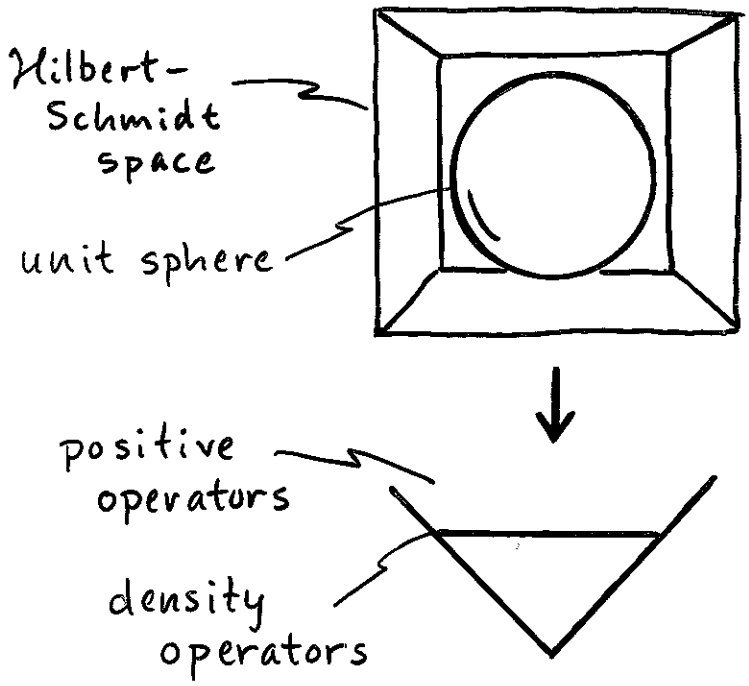
Our system is a subsystem: $\rho = WW^\dagger$ reduced state

The whole fiber WU are purifications of ρ.



The distance between ρ_A and ρ_B should never by larger than between any two purifications.

BURES-UHLMANN GEOMETRY



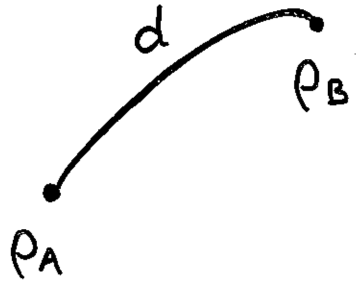
$$\begin{array}{c} W \\ \Big\downarrow \scriptstyle\text{fiber } WU \\ WW^\dagger \end{array}$$

Euclidean metric in $\mathcal{HS}$

$\longrightarrow$ Bures-Uhlmann metric in $\{\rho\}$

THE QUANTUM STATISTICAL DISTANCE!

$\rightsquigarrow$ GEODESICS:



$$\rho(t) = X(t)\,\rho_A\,X(t)$$

$$X(t) = \mathbb{1}\cos t + (M - \mathbb{1}\cos d)\frac{\sin t}{\sin d}$$

The geodesics "bounces" at the boundary of the set of quantum states at N points, where $N = \dim \mathcal{H}$.

~ When $\det \rho(t) = 0$. ~

— UHLMANN (1993)

Each boundary state singles out a pure state: the orthogonal state.

The geodesics boundary states singles out the eigenvectors of M:

$$\langle m_i | \rho(t_i) | m_i \rangle = 0$$

$$\Downarrow$$

The Bures-Uhlmann geodesic through ρ_A and ρ_B determines the optimal — in a specific sense — measurement for distinguishing the two states:

Whenever the geodesic hits the boundary, then the orthogonal state is one of the basis states of the measurement.

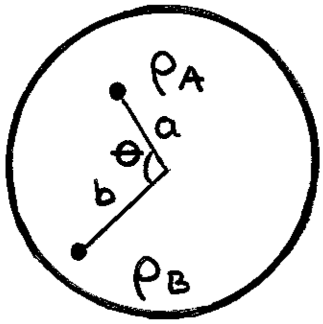
Some Refereces:

- W.K. Wootters, Phys. Rev. D 23 (1981) 357.
- C.A. Fuchs & C.M. Caves, Open Sys. & Info. Dyn. 3 (1995) 345.
- A. Uhlmann, Rep. Math. Phys. 9 (1976) 273.
  " " 24 (1986) 229.
  " " 33 (1993) 253.
- M. Hübner, Phys. Lett. A 163 (1992) 239.
- Hans-Jürgen Sommers, private commun.

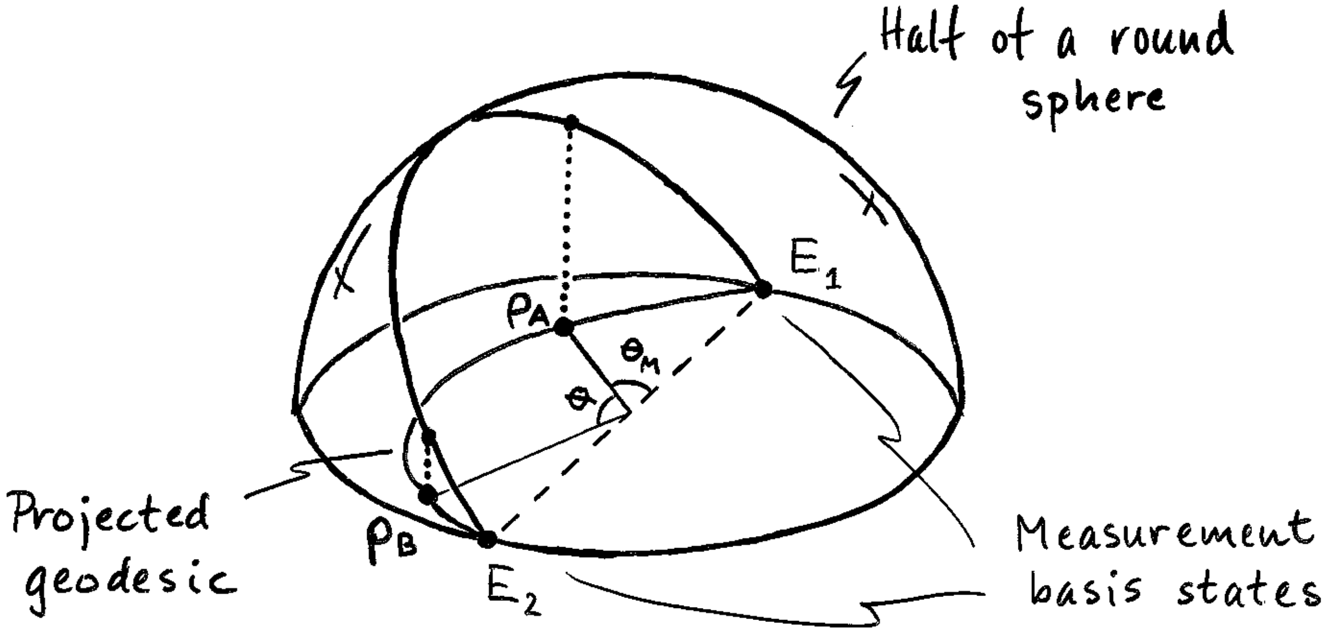
See also: Åsa Ericsson
quant-ph/0508133

And finally I would like to acknowledge Ingemar Bengtsson for helpful discussions and guidance.

Spin $\frac{1}{2}$



DISK IN THE BLOCH BALL



$$\cos d = \frac{1}{\sqrt{2}}\sqrt{1 + ab\cos\theta + \sqrt{1-a^2}\sqrt{1-b^2}}$$

$$\tan\theta_M = \frac{\sin\theta}{\frac{a}{b}\sqrt{\frac{1-b^2}{1-a^2}} - \cos\theta}$$