TIME-DEPENDENT RANDOM MATRICES: QUANTUM INTERFERENCE EFFECTS

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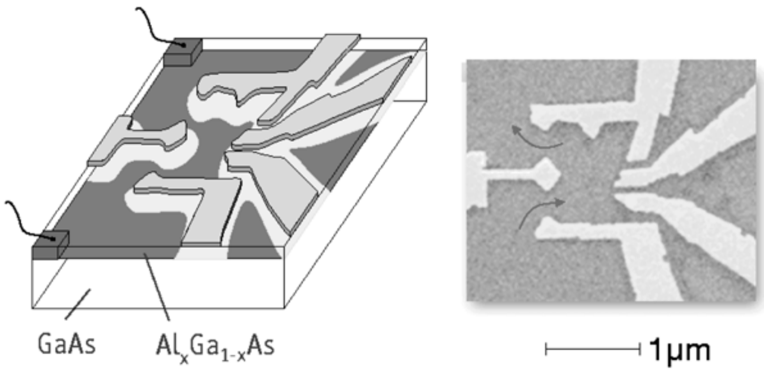
Physical motivation

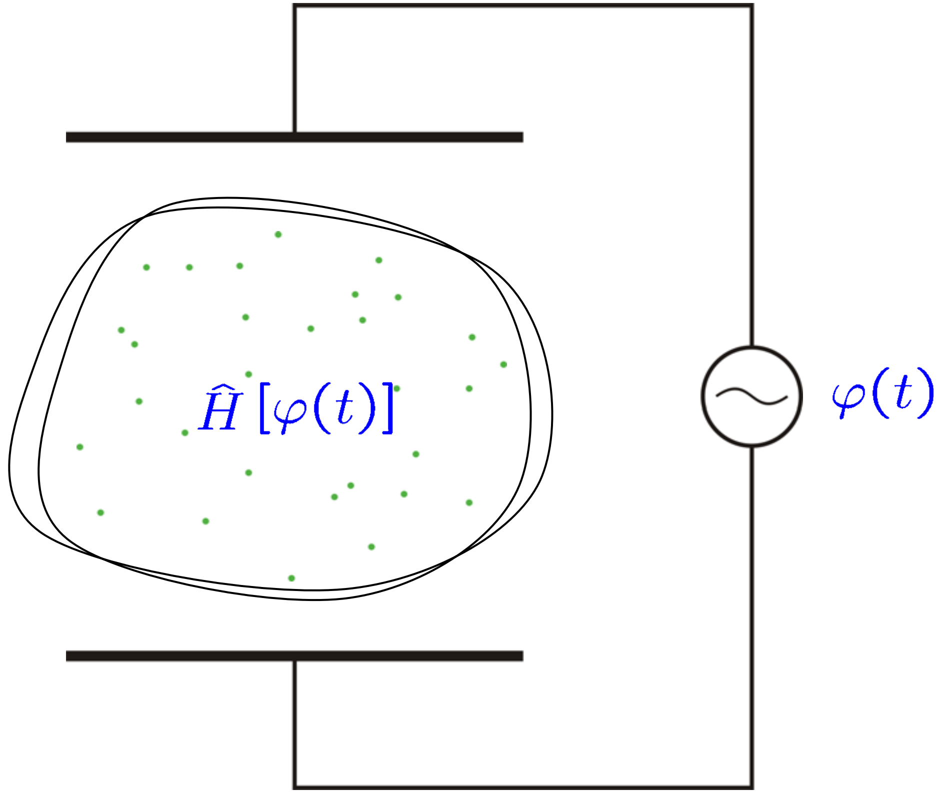
chaotic quantum dot



RMT spectral statistics

for $\omega \ll E_{\mathrm{Th}}$

$$W(t) \equiv \frac{d\langle E(t)\rangle}{dt} = ???$$



Standard approach:

Kubo formula

$$W = \frac{\pi}{\Delta^2}\left\langle \left(\frac{\partial E_i}{\partial t}\right)^2 \right\rangle = \frac{\pi}{\Delta^2}\left\langle \left(\frac{\partial E_i}{\partial \varphi}\right)^2 \right\rangle \left(\frac{d\varphi}{dt}\right)^2$$

Response of $\dfrac{\partial H}{\partial t}$ to $\dfrac{\partial H}{\partial t}$

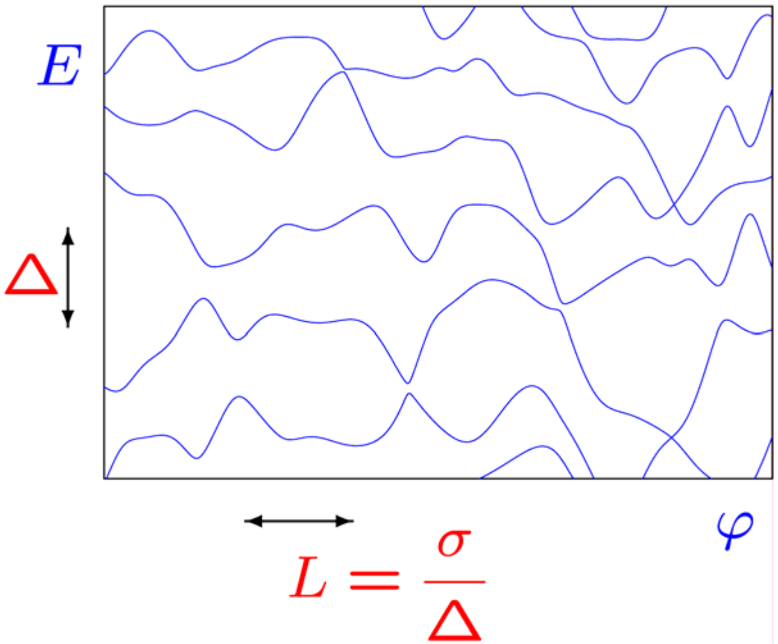
Ohmic dissipation

Outline

- Time-dependent random matrices
- Two basic phenomena in dynamics
- Keldysh σ-model for parametrically-driven systems
- First interference correction:
  - Linear perturbation
  - Dynamic localization for harmonic perturbations

Route to time-dependent random matrices

- **Eigenvalue statistics**: $H\Psi = E\Psi$ — everything is known
- **Parametric eigenvalue statistics**: $H[\varphi]\,\Psi = E[\varphi]\,\Psi$



- **Time-dependent problem**.

Let $\varphi(t)$ be a function of time:

$$i\frac{\partial \Psi(t)}{\partial t} = H[\varphi(t)]\,\Psi(t)$$

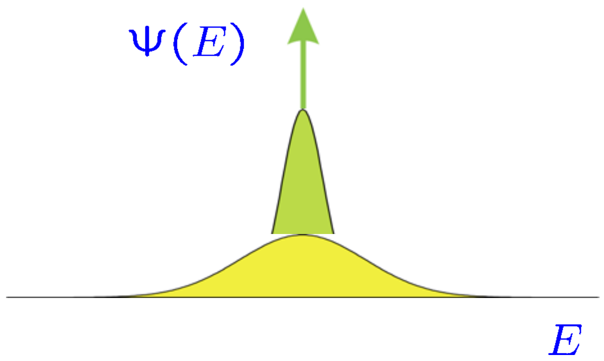
Energy is not a conserving quantity anymore

$$\langle E(t)\rangle = ???$$

What is to look for?

Spreading of the wave function due to interlevel transitions

[math]
Evolution of the initial state
$\Psi_n(0) = \delta_{n,0}$



$$\langle \psi(t) | [H(t) - E_0]^2 | \psi(t) \rangle = Dt$$

diffusion coefficient
in the energy space

[phys]
Evolution of the distribution
of noninteracting fermions

$$\langle \mathcal{E}(t) \rangle = Wt$$

energy absorption rate

$$W = \frac{D}{\Delta}$$

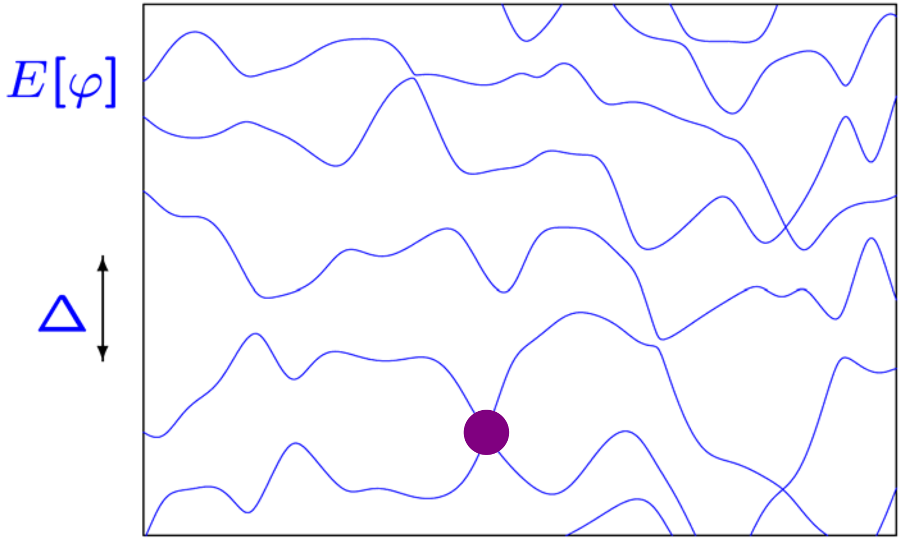
Two basic quantum phenomena

- **Adiabatic** & **Kubo** regimes of dissipation
  - ∗ distinguished by $v = d\varphi/dt$
  - ∗ local property
  - ∗ best example: $\varphi(t) = vt$

- **Dynamical localization** in the energy space
  - ∗ for re-entrant $\varphi(t)$
  - ∗ global property
  - ∗ best example: $\varphi(t) = \sin\omega t$

Slow/fast dissipation

Adiabatic spectrum



$\sigma^2 = \langle (\partial E_i / \partial \varphi)^2 \rangle$ $\qquad$ $\varphi = vt$

Critical velocity: $v_K \sim \dfrac{\Delta^2}{\sigma}$

Landau-Zener transition

- **Adiabatic regime:** $v \ll v_K$

  discrete spectrum

  LZ transitions

  depends on statistics

- **Kubo regime:** $v \gg v_K$

  continuous spectrum

  Kubo formula

  statistics independent

Linear perturbation: limiting cases

Wilkinson (1988)

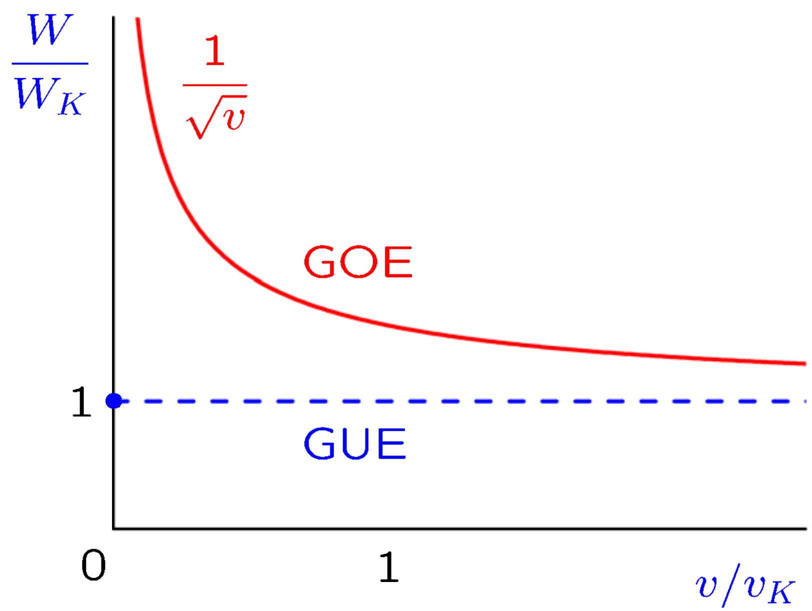
- **Kubo regime:** $W_K = \frac{\pi\sigma^2}{\Delta^2} v^2$
- **Adiabatic regime:** $W \propto v^{1+\frac{\beta}{2}}$

$$W \propto \int R_2(\varepsilon)\ \mathrm{exp}\left(-\frac{\varepsilon^2}{\sigma v}\right) d\varepsilon$$

Pair correlation function

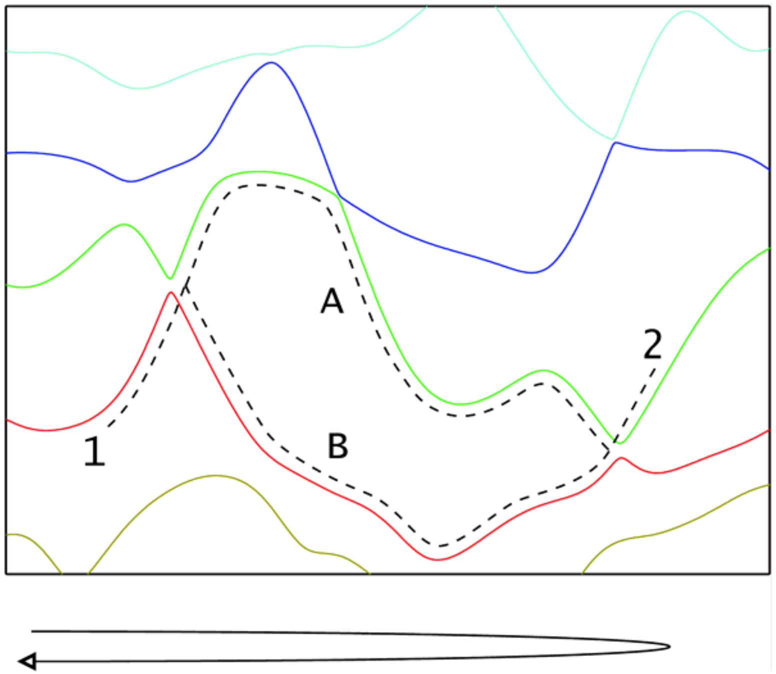
$R_2(\varepsilon) \sim \varepsilon^{\beta}$

$\varepsilon \sim \sqrt{v}$



Localization in the energy space

(Handwaving arguments in the adiabatic regime)



- **Monotonous** $\varphi(t)$
  interference is ineffective

$$|A+B|^2 = |A|^2 + |B|^2 + \cancel{2\,\mathrm{Re}(AB^*)}$$

- **Re-entrant** $\varphi(t)$
  interference *may be* important
  for well-tuned perturbations

Localization in Quantum Kicked Rotor

Casati, Chirikov, Ford, Izrailev (1979)

$$\hat{H} = \frac{\hat{l}^2}{2} + K \cos\theta \sum_n \delta(t-n)$$

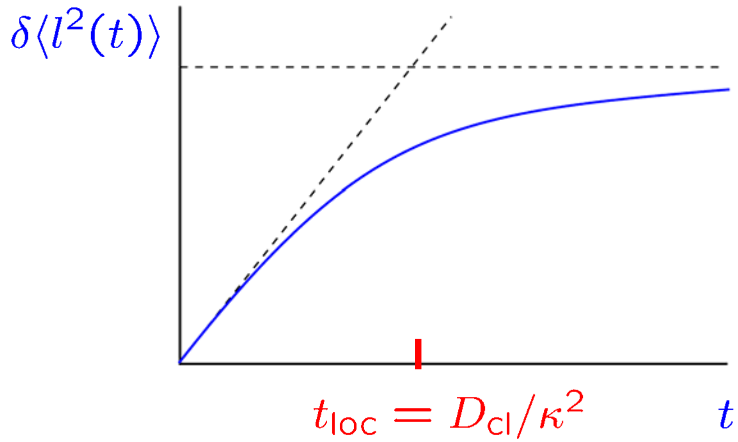
momentum operator: $\hat{l} = -i\kappa \frac{\partial}{\partial\theta}$

- **Diffusion in momentum space:**

$$\delta\langle l^2(t)\rangle \equiv \langle [l(t) - l(0)]^2\rangle = 2D_{\text{cl}}\, t$$

$$\boxed{D_{\text{cl}} \approx \frac{K^2}{4}, \qquad K \gg 1}$$

- **Dynamic localization**



The model

We consider a time-dependent matrix Hamiltonian: $H(t) = H_0 + V\varphi(t)$,

where H_0 and V are independent random $N \times N$ matrices from the GOE (GUE) chosen to give the mean level spacing Δ and the sensitivity of the spectrum to variation of φ:

$$\left\langle \left(\frac{\partial E_n}{\partial \varphi} \right)^2 \right\rangle = \frac{2\Gamma\Delta}{\pi}$$

Schrödinger equation:

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle \qquad \Longrightarrow \qquad |\psi(t)\rangle = T\exp\left(-i\int_0^t H(t')dt'\right)|0\rangle$$

Physical observable:

$$\langle\psi(t)|A(t)|\psi(t)\rangle = \langle 0|\, \underbrace{\hat{T}^{-1}\exp\left(i\int_0^t H(t')dt'\right)}_{\text{backward evolution}} A(t)\, \underbrace{\hat{T}\exp\left(-i\int_0^t H(t')dt'\right)}_{\text{forward evolution}} |0\rangle$$

DISORDER + KINETICS = ???

Keldysh σ-model

M. S. (2003)

Low-energy effective theory is formulated in terms of the matrix Q-field ($Q^2 = 1$)

$Q^{\alpha\beta}_{tt'} \in$

- **time** space: continuous index t
- 2×2 **Keldysh** space (σ_i)
- 2×2 **Particle-Hole** space (τ_i)

$$S[Q] = \frac{\pi i}{2\Delta} \mathrm{Tr}\, \tau_3 \hat{E} Q + \frac{\pi\Gamma}{8\Delta} \int dt\, dt' [\varphi(t) - \varphi(t')]^2 \,\mathrm{tr}\, Q_{tt'} Q_{t't}$$

E-term

responsible for the RMT energy **level statistics** encoded in the rich structure of $Q_{EE'}$

Altland & Kamenev (2000)

kinetic term

accounts for **interlevel transitions** of the time-dependent Hamiltonian $H[\varphi(t)]$

$$\left\langle \left(\frac{\partial E_i}{\partial \varphi} \right)^2 \right\rangle = \frac{2\Gamma\Delta}{\pi}$$

Saddle point – Kubo formula

The standard form of the Keldysh Green function ($F = 1 - 2f$):

$$Q_{tt'} = \begin{pmatrix} \delta_{tt'} & 2F_{tt'} \\ 0 & -\delta_{tt'} \end{pmatrix} \otimes \tau_3$$

The saddle point equation gives the **quantum kinetic equation**:

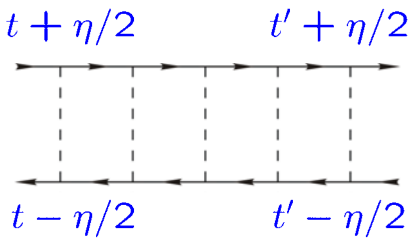
$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'}\right) f^{(0)}_{tt'} = -\Gamma(\varphi(t) - \varphi(t'))^2\, f^{(0)}_{tt'}.$$

The Wigner-transformed function $f(E,t) = \int d\tau e^{iE\tau} f_{t+\tau/2, t-\tau/2}$ after averaging over fast oscillations obeys the **diffusion equation in the energy space**:

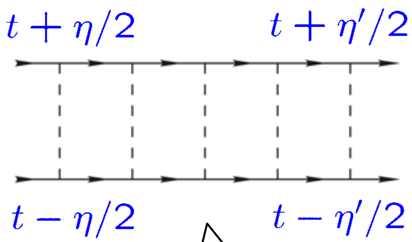
$$\frac{\partial f^{(0)}(E,t)}{\partial t} = D\frac{\partial^2 f^{(0)}(E,t)}{\partial E^2}, \qquad D = \Gamma\overline{(d\varphi/dt)^2}$$

Energy absorption rate: $W_K = \dfrac{\Gamma}{\Delta}\overline{\left(\dfrac{d\varphi}{dt}\right)^2}$ — Kubo formula

Soft modes: diffusons and cooperons



$$D_\eta(t,t') = \theta(t-t')\ \exp\left\{-\Gamma \int_{t'}^{t} [\varphi(\tau+\eta/2)-\varphi(\tau-\eta/2)]^2\, d\tau\right\}$$



$$C_t(\eta,\eta') = \theta(\eta-\eta')\ \exp\left\{-\frac{\Gamma}{2} \int_{\eta'}^{\eta} [\varphi(t+\tau/2)-\varphi(t-\tau/2)]^2\, d\tau\right\}$$

Each impurity line carries a nonzero frequency $\longrightarrow$ both diffusons and cooperons decay with time.

Dephasing

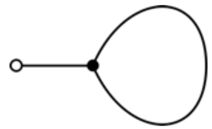
by the time-dependent perturbation

Vavilov, Aleiner (1999)

Yudson, Kanzieper, Kravtsov (2001)

One-loop quantum correction (GOE)

Fluctuations induce corrections to the distribution function $F_{tt'}$:

$\delta F =$ 

One-loop interference correction to the Kubo absorption rate W_K:

$$\delta W(t) = \frac{\Gamma}{\pi}\int_0^\infty \partial_t\varphi(t)\,\partial_t\varphi(t-\xi)\,C_{t-\xi/2}(\xi,-\xi)\,d\xi$$

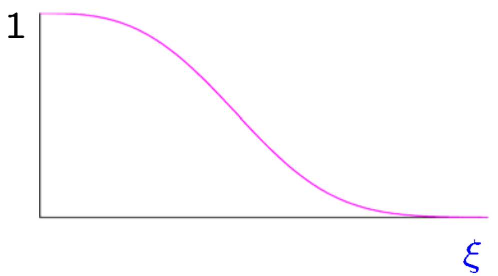
Holds for arbitrary $\varphi(t)$ and contains everything

Result for the linear perturbation

Dynamic cooperon for $\varphi = vt$:

$$C_{t-\xi/2}(\xi, -\xi) = \exp\left\{-\frac{\Omega^3}{3}\xi^3\right\}$$

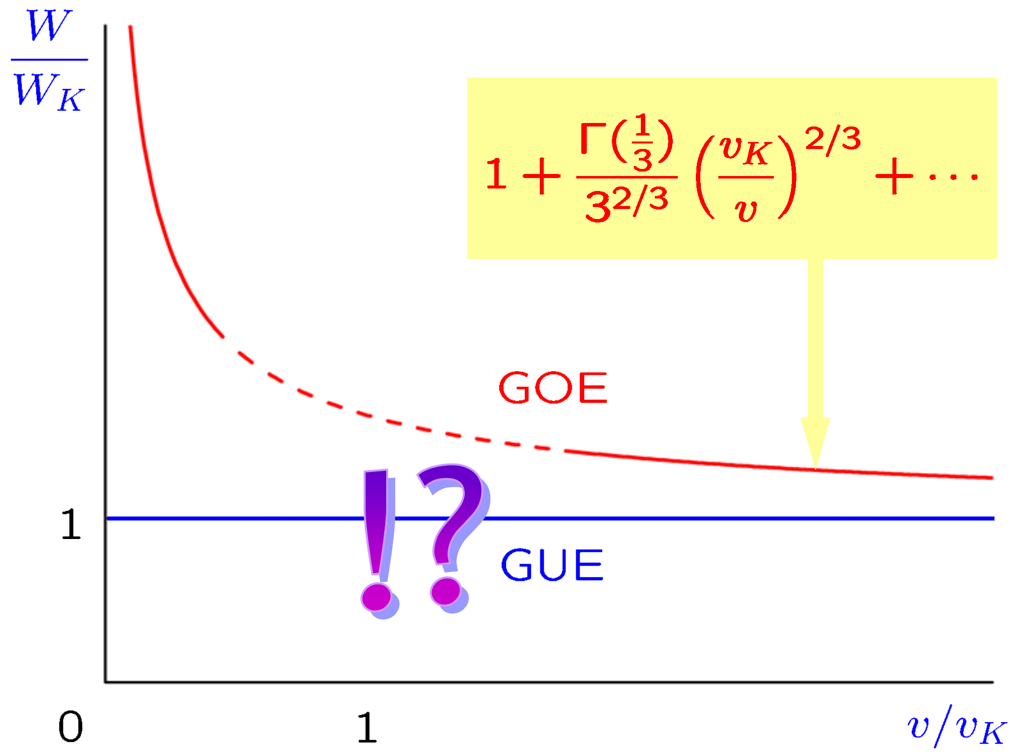
decays at the time scale Ω^{-1}, where $\Omega^3 = \Gamma v^2$.



Loop expansion parameter:

$$\frac{\Delta}{\Omega} = \pi \left(\frac{v_K}{v}\right)^{2/3}$$

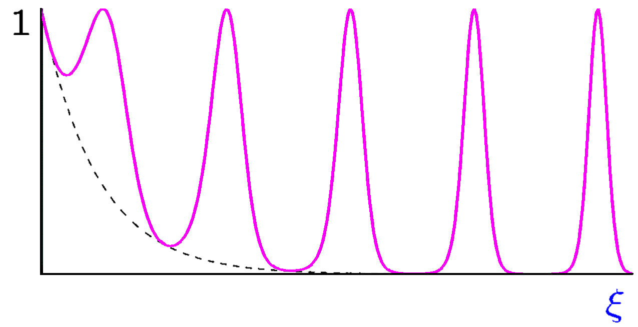
Dissipation rate vs. velocity



Result for a periodic perturbation (GOE)

For a periodic perturbation $\varphi(t) = \theta(t) \sin \omega t$
at $t, \xi \gg 1/\omega$:

$$C_{t-\xi/2}(\xi, -\xi) \approx \exp\left\{-2\Gamma \xi \cos^2[\omega(t - \xi/2)]\right\}$$



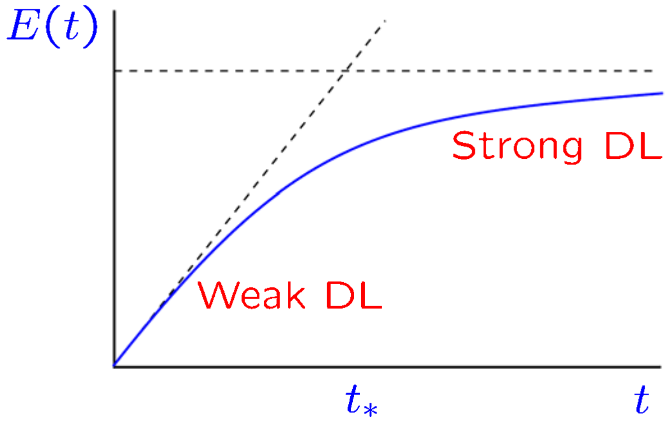
exponentially decays with ξ

no-dephasing points
Cooperon is equal to **1**
at $\xi_k = 2t - \frac{2\pi}{\omega}(k + 1/2)$

Integrating near ξ_k and summing over ξ_k we get

$$\frac{W(t)}{W_K} = 1 - \sqrt{\frac{t}{t_*}}, \qquad t_* = \frac{\pi^3 \Gamma}{2\Delta^2}$$

Weak dynamic localization (GOE)



- General *periodic* perturbation with TRS, $\varphi(-t) = \varphi(t)$

$$\frac{\delta W(t)}{W_K} = -\sqrt{\frac{t}{t_*}}$$

- General *periodic* perturbation without TRS, $\varphi(-t) \neq \varphi(t)$

$$\frac{\delta W(t)}{W_K} = -\frac{t}{t_*} \qquad \text{(2 loops)}$$

- *Aperiodic* perturbation with d incommensurate frequencies

$$d = 2: \qquad \frac{\delta W(t)}{W_K} = -\frac{\Delta}{2\pi^2\Gamma} \ln \Gamma t$$

$$d > 2: \qquad \frac{\delta W(t)}{W_K} \propto -t^{1-d/2} \longrightarrow \text{const}$$

$\Longleftrightarrow$ Weak Anderson localization in d dimensions

Analogously to the DL in the KQR with d incommensurate periods
Casati, Guarneri, Shepelyansky (1989)

GUE,
linear perturbation,
4-loops

D. Ivanov & M. S. (2005)

Field theory for GUE: summary

- Dimensional action:

$$S[Q] = -\frac{\pi}{2}\,\mathrm{tr}\int dt\,(\partial_1 - \partial_2)Q_{tt} + \frac{\pi\alpha}{4}\,\mathrm{tr}\int\!\!\int dt\,dt'\,(t-t')^2\,Q_{tt'}Q_{t't}$$

$$\boxed{\alpha = \frac{v^2}{v_K^2}}$$

- Full diffuson: $\langle Q^{K}_{t+\eta/2,t-\eta/2}\,Q^{\bar{K}}_{t'-\eta'/2,t'+\eta'/2}\rangle = \frac{2}{\pi}\delta(\eta-\eta')\,\mathcal{D}_\eta(t-t')$

- Energy diffusion coefficient: $D(\alpha) = -\frac{1}{2}\frac{\partial}{\partial t}\frac{\partial^2}{\partial\eta^2}\Big|_{\eta=0}\mathcal{D}_\eta(t)$

- Parametrization: $Q = \sigma_3\,[1 + W + W^2/2 + \ldots],\qquad W_{tt'} = \begin{pmatrix} 0 & b_{tt'} \\ -b^\dagger_{tt'} & 0 \end{pmatrix}$

- Bare diffuson: $\mathcal{D}^{(0)}_\eta(t) = \theta(t)\exp(-\alpha\eta^2 t) \;\longrightarrow\; D^{(0)} = \alpha$

Perturbation theory

- **Rational parametrization**: $Q = \sigma_3 \dfrac{1 + W/2}{1 - W/2}$, $\qquad W_{tt'} = \begin{pmatrix} 0 & b_{tt'} \\ -\bar{b}_{tt'} & 0 \end{pmatrix}$

$$S[b,\bar{b}] = \frac{\pi}{2} \int\!\!\int dt_1\, dt_2\, \bar{b}_{12} \Big[(\partial_1 + \partial_2) + \alpha (t_1 - t_2)^2 \Big] b_{21}$$
$$+ \{ \text{higher vertices with } \partial_t \} + \{ \text{higher vertices with } \alpha (t_i - t_j)^2 \}$$

- Bare diffuson: $\mathcal{D}^{(0)}_\eta(t) = \theta(t) \exp(-\alpha \eta^2 t)$

- Diffuson equation of motion: $\partial_t \mathcal{D}^{(0)}_\eta(t) = \delta(t) - \alpha \eta^2 \mathcal{D}^{(0)}_\eta(t)$

ALL vertices of the order higher than FOUR cancel in ALL orders of the perturbation theory

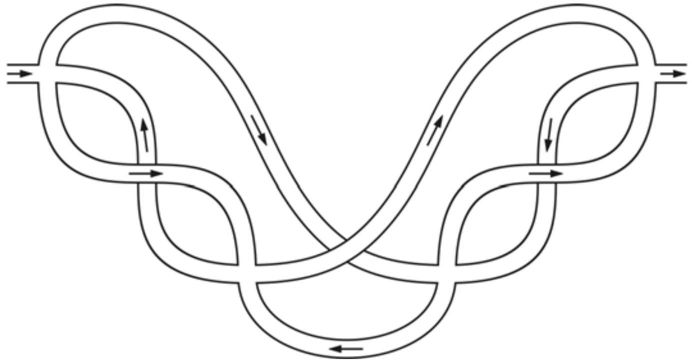
Effective matrix ϕ^4 theory

Initial Keldysh σ-model is perturbatively equivalent to the unconstrainted matrix φ^4 theory:

$$S_{\text{eff}}[b,\bar{b}] = \frac{\pi}{2}\iint dt_1\, dt_2\, \bar{b}_{12}\Big[(\partial_1 + \partial_2) + \alpha(t_1 - t_2)^2\Big] b_{21} - \frac{\pi\alpha}{8}\int dt_1\, dt_2\, dt_3\, dt_4\, (t_1 - t_2)(t_3 - t_4)\, b_{12}\bar{b}_{23}b_{34}\bar{b}_{41}$$

Four loops

- *Just 20* 4-loop diagrams, e.g.



$$\int_0^\infty \cdots \int_0^\infty dt_1 \ldots dt_7 \, P_6(t_i) \, e^{-S_3(t_i)}$$

Result for the energy diffusion coefficient:

exact zero

$(7 \pm 7) \times 10^{-5}$

$$D(\alpha) = \alpha \left(1 + \frac{d_2}{\pi^2 \alpha^{2/3}} + \frac{d_4}{\pi^4 \alpha^{4/3}} + \ldots \right)$$

Dyson-Maleev transformation

$$Q = \begin{pmatrix} 1 - bb^\dagger/2 & b - bb^\dagger b/4 \\ b^\dagger & -1 + b^\dagger b/2 \end{pmatrix}$$

violates hermiticity

Bilinear in Q action

$$S[Q] = \frac{\pi i}{\Delta} \mathrm{Tr}\, \hat{E} Q - \frac{\pi \Gamma}{4\Delta} \mathrm{Tr}[\varphi, Q]^2$$

becomes **quartic** in $b, b^\dagger$

Works for ANY unitary sigma-model

Dyson-Maleev transformation for quantum spin operators

$$\hat{S}^+ = (2S - \hat{a}^\dagger \hat{a})\hat{a}$$
$$\hat{S}^- = \hat{a}^\dagger$$
$$\hat{S}^z = S - \hat{a}^\dagger \hat{a}$$

Conclusion

- Regular perturbative expansion near the Kubo regime
- Kubo/Adiabatic crossover
- Dynamic localization
- Dyson-Maleev parametrization for σ-models

Open problems

- Mapping of periodically driven RMT onto quasi-1D AL
- Full dependence of $W(v)$ for linearly driven RMT

M. Skvortsov, PRB **68**, 041306(R) (2003)
D. Basko, M. Skvortsov, V. Kravtsov, PRL **90**, 096801 (2003)
M. Skvortsov, D. Basko, V. Kravtsov, JETP Lett. **80**, 54 (2004)
D. Ivanov and M. Skvortsov, Nucl. Phys. B **737**, 304 (2006)

The end

Derivation of the Keldysh σ-model

- Partition function via the functional integral over Grassmannian fields $\Psi(t)$:

$$Z = \int D\Psi D\Psi^* \exp\left\{ i \int_{\rightleftarrows} dt\, \Psi^\dagger(t) \left[i\tau_3 \frac{\partial}{\partial t} - H(t) \right] \Psi(t) \right\}$$

- Averaging over H_0 and V generates the *nonlocal in time* quartic term

$$\int\int dt\, dt' \left\{ \Psi_i^\dagger(t) \Psi_j(t) \right\} \left\{ \Psi_j^\dagger(t') \Psi_i(t') \right\}$$

- Decoupling by the Hubbard-Stratonovich matrix field $Q_{tt'}$
- Evaluation of the resulting Gaussian integral over $\Psi_i(t)$:

$$S[Q] = -\frac{N}{2} \operatorname{Tr} \ln \left[\frac{\pi}{N\Delta} \delta_{tt'} \frac{\partial}{\partial t'} + \gamma(t,t') Q_{tt'} \right] + \frac{N}{4} \int dt\, dt'\, \gamma(t,t') \operatorname{tr} Q_{tt'} Q_{t't}$$

$$\gamma(t,t') = 1 - \frac{\pi \Gamma}{N\Delta} [\varphi(t) - \varphi(t')]^2$$

- Expansion of the action $S[Q]$ over $1/N$

Elimination of the distribution function

In the original formulation, F enters the definition of the Q-manifold:

$$Q = U_F^{-1}\tilde{Q}U_F, \qquad U_F = \begin{pmatrix} 1 & F_0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{Q} = U^{-1}\sigma_3\tau_3 U$$

Evolution of the distribution can be read off from the Keldysh block of $\langle Q \rangle$:

$$F_{tt'} = \frac{1}{2}\langle Q^K_{tt'} \rangle, \qquad W(t) = -\frac{\pi i}{\Delta}\frac{\partial}{\partial t}\frac{\partial}{\partial \eta}\bigg|_{\eta=0} F_{t+\eta/2,t-\eta/2}.$$

$$\boxed{Q = \begin{pmatrix} R & K \\ \bar{K} & A \end{pmatrix}}$$

For a noninteracting problem one can work in terms of the $\tilde{Q}$-matrix

D. Ivanov & M. S. (2006)

- Full diffuson: $\langle \tilde{Q}^K_{t+\eta/2,t-\eta/2}\, \tilde{Q}^{\bar{K}}_{t'-\eta'/2,t'+\eta'/2} \rangle = \frac{2\Delta}{\pi}\delta(\eta-\eta')\,\mathcal{D}_\eta(t,t')$

- Energy absorption rate:

$$W(t) = -\frac{1}{2\Delta}\frac{\partial}{\partial t}\frac{\partial^2}{\partial \eta^2}\bigg|_{\eta=0} \mathcal{D}_\eta(t,t_0)$$

switch-on time
of the perturbation

Structure of the Q-manifold

The matrix $\tilde{Q} = U^{-1}\sigma_3\tau_3 U$ obeys: 1) $\tilde{Q}^\dagger = \tilde{Q}$, 2) $\tilde{Q}^T = \sigma_1\tau_2\tilde{Q}\tau_2\sigma_1$.

Parametrization of the manifold:

$$\tilde{Q} = \sigma_3\tau_3 f(W) = \sigma_3\tau_3[1 + W + W^2/2 + \ldots], \qquad f(W)f(-W) = 1$$

$$W = \left(\begin{array}{cc|cc} 0 & a & b & 0 \\ -a^\dagger & 0 & 0 & -b^T \\ \hline -\,b^\dagger & 0 & 0 & a^T \\ 0 & b^* & -a^* & 0 \end{array} \right)_K$$

$\langle bb^\dagger \rangle =$ Diffuson
$\langle aa^\dagger \rangle =$ Cooperon

parametrizations

rational	$f(W) = \dfrac{1 + W/2}{1 - W/2}$	$J = 1$
sqrt	$f(W) = \sqrt{1 + W^2} + W$	cross diagram technique
exp	$f(W) = e^W$	$\rightarrow$ global parametrization

Dynamic vs. Anderson localization

- Weak DL in periodically driven RMT ($\sqrt{t}$) = Weak DL in QKR
- Quantum kicked rotor can be mapped onto the 1D σ-model [Altland, Zirnbauer (1993)]
- Periodically driven RMT $\neq$ QKR!

  e.g., for a δ-kicked RMT localization time $t_* \to \infty$

Q: Is DL in periodically driven RMT analogous to the quasi-1D AL?

Dynamic vs. Anderson localization

If DL for the periodically driven RMT is equivalent to quasi-1D AL then [Altland (1993), Tian, Kamenev, Larkin (2004)]

$$\frac{W(t)}{W_0} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{-i\omega + 0} \frac{D(\omega)}{D_0}$$

Weak AL

$$\frac{\delta D(\omega)}{D_0} = \begin{cases} -\dfrac{1}{\sqrt{-i\omega t_{\text{loc}}}}, & \text{GOE}, \\ -\dfrac{1}{-6i\omega t_{\text{loc}}}, & \text{GUE} \end{cases}$$

Weak DL

$$\frac{\delta W(t)}{W_0} = -\begin{cases} \sqrt{\dfrac{t}{t_*}}, & \text{GOE}, \\ \dfrac{\pi t}{24 t_*}, & \text{GUE} \end{cases}$$

We believe, YES