FERMIONIC LADDERS IN MAGNETIC FIELD

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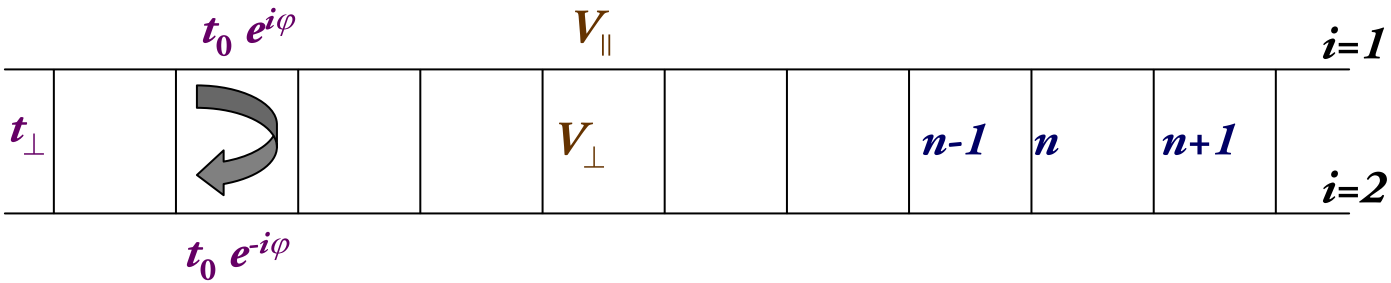
spinless fermions on a two-leg ladder



hamiltonian

$$\mathcal{H} = -\sum_n \Big[\frac{1}{2}\sum_{i=1,2}\left(t_{||}(y_i)c_i^\dagger(x_n)c_i(x_{n+1}) + h.c.\right)$$

$$+t_\perp c_1^\dagger(x_n)c_2(x_n) + h.c.\Big] + \mathcal{H}_{int},$$

$$\mathcal{H}_{int} = \sum_n \Big[V_{||}\Big(n_1(x_n)n_1(x_{n+1}) + n_2(x_n)n_2(x_{n+1})\Big)$$

$$+V_\perp n_1(x_n)n_2(x_n)\Big]$$

physical quantities

- bond current

$$j_\perp(x_n) = -it_\perp\left[c_1^\dagger(x_n)c_2(x_n) - h.c.\right]$$

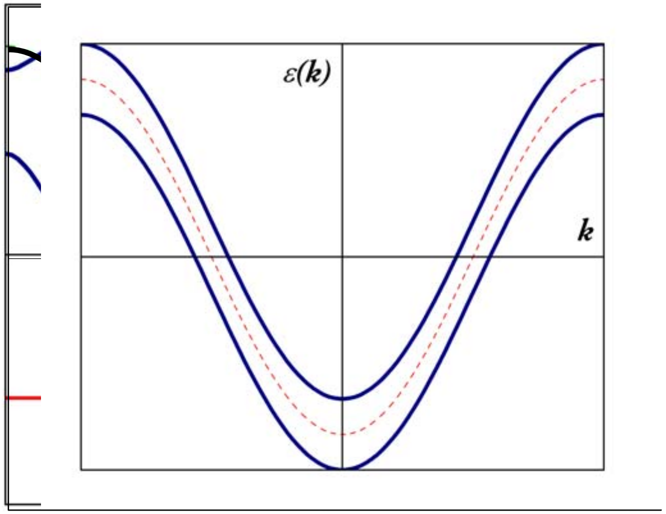
- bond density

$$\rho_\perp(x_n) = c_1^\dagger(x_n)c_2(x_n) + h.c.$$

outline

- comment on bosonization
- $\nu=1/4$: charge fractionalization
- $\nu=1/2$: field-induced phase transitions
- physics beyond bosonization –
  persistent current

bosonization in ladders



- single-chain single-particle spectrum

$$\epsilon^{(0)}(k) = -t_0 \cos k$$

- ladder spectrum

$$\epsilon^{(0)}_{\alpha(\beta)}(k) = -t_0 \left[\cos k \pm \frac{t_\perp}{t_0}\right]$$

- ladder spectrum in the presence of the magnetic field

$$\epsilon^{(0)}_{\alpha(\beta)}(k) = -t_0 \left[\cos k \cos \pi f \pm \sqrt{\sin^2 k \sin^2 \pi f + \left(\frac{t_\perp}{t_0}\right)^2}\right], \qquad t_0 \to t_0 e^{2\pi i f y/b}$$

- interaction terms

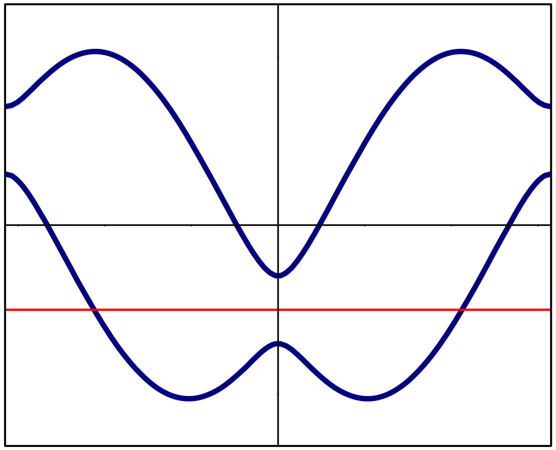
$$\mathcal{H}_{int} \to \frac{1}{N^2} \sum_{\{k_i\},j} e^{ix_j(k_1+k_3-k_2-k_4)} h(\{k_i\}) : \alpha^\dagger_{k_1} \alpha_{k_2} :: \alpha^\dagger_{k_3} \alpha_{k_4} :$$

$$\alpha_k \to \alpha_R(q)\delta_{k,k_F+q} + \alpha_L(q)\delta_{k,q-k_F}$$

$$\boxed{e^{i2x_j(k_F^\alpha - k_F^\beta)} : \alpha^\dagger_R \alpha_L :: \beta^\dagger_L \beta_R :}$$

bosonization approach to quarter-filled ladder

- single-particle spectrum



single band partially occupied

- effective low-energy hamiltonian

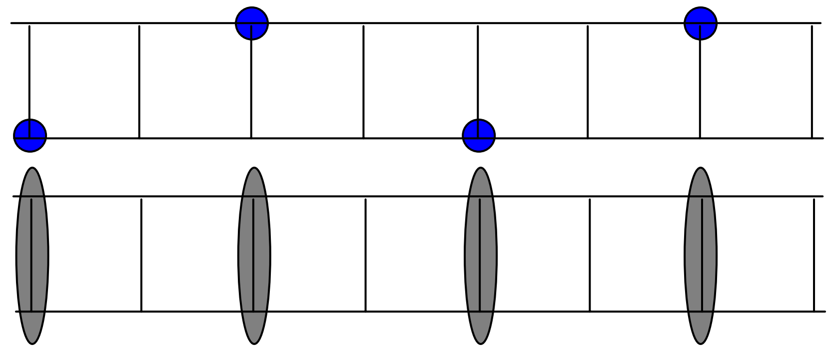
$$\mathcal{H} = \frac{v_F}{2}\left[K\Pi^2 + \frac{1}{K}(\partial_x\phi)^2\right] - \frac{g_2}{2\pi^2\alpha^2}\cos\sqrt{16\pi}\phi, \qquad K = \left[\frac{1-g_1/2\pi v_F}{1+g_1/2\pi v_F}\right]^{1/2}$$

- interaction parameters

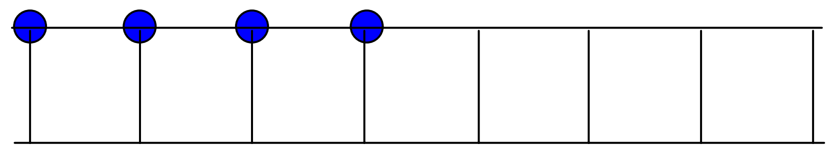
$$g_1 = 2V_{||}\frac{t_\perp^2(1-\cos 2k_F)}{t_0^2\sin^2\pi f + t_\perp^2} + V_\perp\frac{\sin^2\pi f}{\sin^2\pi f + t_\perp^2/t_0^2}, \qquad g_2 = -2V_{||}\frac{t_\perp^2\exp(i2k_F)}{t_0^2\sin^2\pi f + t_\perp^2}$$

strong-coupling cartoon

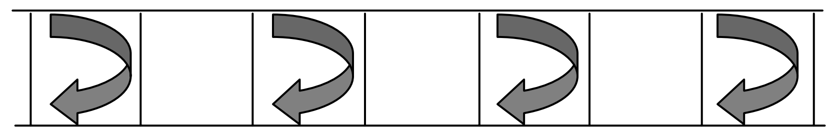
- repulsive interaction
  - strong $V_{\perp}$ - no rung doubly occupied
  - hopping delocalizes electrons on links
  - for $V_{\parallel} > 0$ – avoid neighboring sites

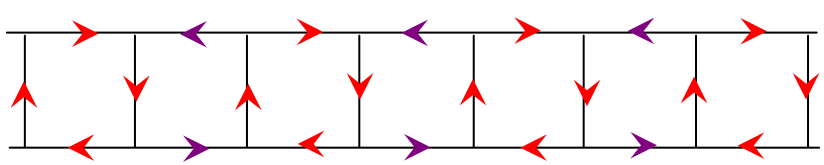


- attractive in-chain interaction
  - phase separation



- role of the magnetic field
  - delocalizes electrons around plaquettes
  - produces circulating currents





the external field is uniform!

bosonization approach to quarter-filled ladder

- possible states with long-range order (*K* < *1/2*)

  - $g_2 > 0$ - bond density wave

$$\phi_{min} = \sqrt{\pi/16} + n\sqrt{\pi/4}, \quad n = 0, \pm 1, \ldots \Rightarrow \langle \rho_{\perp}^{(s)} \rangle \sim \langle \sin\sqrt{4\pi}\phi \rangle \neq 0$$

  - $g_2 < 0$ - **staggered flux phase** (orbital anti-ferromagnet)

$$\phi_{min} = n\sqrt{\pi/4}, \quad n = 0, \pm 1, \ldots \Rightarrow \langle j_{\perp} \rangle \sim \langle \cos\sqrt{4\pi}\phi \rangle \sin \pi f \neq 0$$

- charge quantization

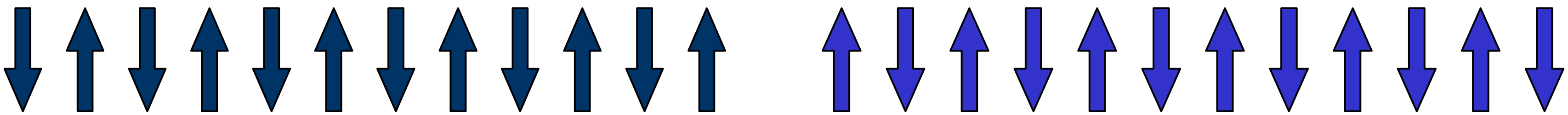
$$\rho = \frac{1}{\sqrt{\pi}}\partial_x \phi \rightarrow \boxed{Q = e\frac{\Delta\phi}{\sqrt{\pi}};} \qquad \Delta\phi = \sqrt{\pi/4} \rightarrow \boxed{Q = \pm e/2}$$

fractional quantum numbers in spin chains

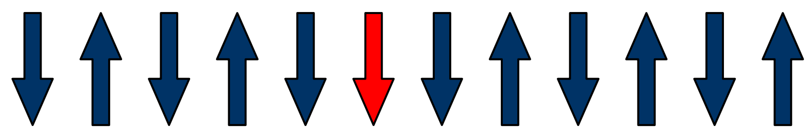
- anti-ferromagnetic Heisenberg model

$$\mathcal{H} = J \sum_i [S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z], \; \Delta > 1$$

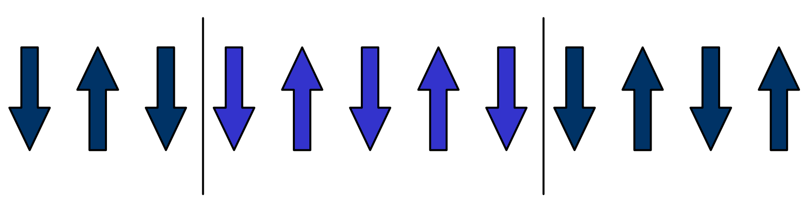
- doubly degenerate ground state



- elementary excitation – spin flip (*S=1*)

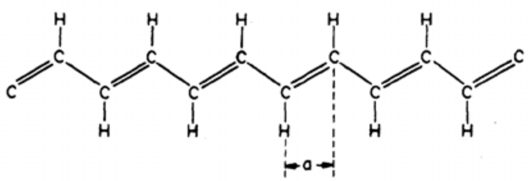


- spinons – S=1/2 excitations



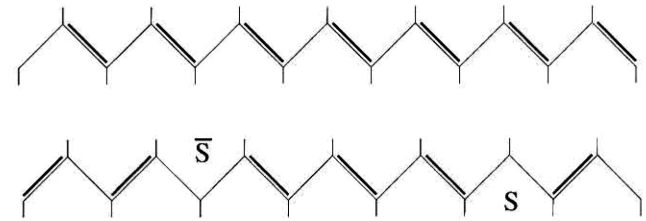
fractionalization in polyacetylene

Su, Schrieffer, Heeger *(1979)*



Trans configuration of $(CH)_x$. $a = 1.2$ Å

Brazovskii *(1978)* ; Rice *(1979)*



- hamiltonian

$$H = -\sum_{ns}(t_{n+1,n}c_{n+1,s}{}^{\dagger}c_{n,s} + \text{H.c.}) + \sum_n \tfrac{1}{2}K(u_{n+1}-u_n)^2 + \sum_n \tfrac{1}{2}M\dot{u}_n{}^2 ,$$

where to first order in the u's,

$$t_{n+1,n} = t_0 - \alpha(u_{n+1}-u_n) .$$

- electron content
  - two core (*1s*) electrons per *C*
  - two electrons in a bonding σ-orbital (*sp*^{2} hybrid) per
  - two π-electrons (out-of-plane *2p* orbital of *C*) per

- the Schrieffer counting argument
  - local neutrality : 1 σ-electron per *H*; 2 core, 3 σ, 1 π-electron per *C*
  - soliton: charge: +*e*, spin: *0* (since all electrons are paired)
  - remaining non-bonding π-orbital on central *C*: if singly occupied, the soliton is neutral with spin ½, if doubly occupied, the soliton is spinless, charge -e

conclusions for ν=*1/4*

- we have considered electrons on the two-leg ladder at arbitrary values of the external field, inter-chain hopping and interaction strength

- we have found a new ordered phase in the model – the orbital anti-ferromagnet – that exists only when the field is applied

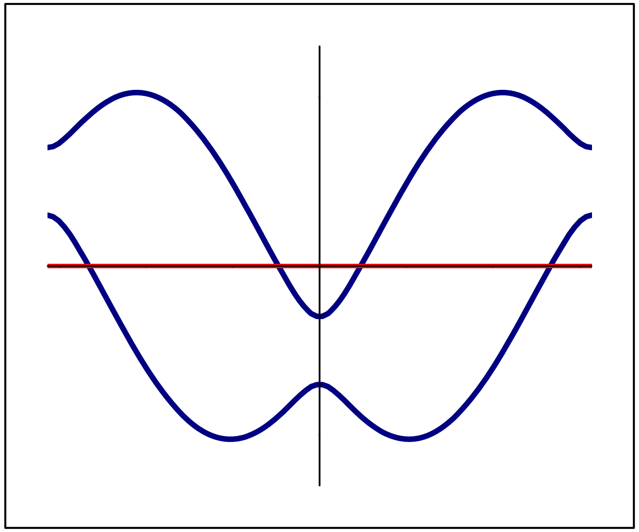
- this new ground state is doubly degenerate, so the elementary excitations carry charge ½

- we showed that fractionally charged excitations that exist in the absence of the field are stable with respect to the external magnetic field

bosonization approach to half-filled ladder

- single-particle spectrum

both bands partially occupied



- effective low-energy hamiltonian

$$\mathcal{H} = \frac{v_F^+}{2}\left[K_+\left(\partial_x\theta_+\right)^2 + \frac{1}{K_+}\left(\partial_x\phi_+\right)^2\right] - \frac{g_4}{2\pi^2\alpha_0^2}\cos\sqrt{8\pi}\phi_+$$

$$+\frac{v_F^-}{2}\left[K_-\left(\partial_x\theta_-\right)^2 + \frac{1}{K_-}\left(\partial_x\phi_-\right)^2\right] + \frac{g_5}{2\pi^2\alpha_0^2}\cos\sqrt{8\pi}\theta_- - \frac{g_6}{\pi^2\alpha_0^2}\cos\sqrt{8\pi}\phi_+\cos\sqrt{8\pi}\theta_-.$$

$$K_\pm = \left[\frac{1-(g_1\pm g_3)/4\pi v_F^\pm}{1+(g_1\pm g_3)/4\pi v_F^\pm}\right]^{1/2}$$

half-filled ladder – phase diagram

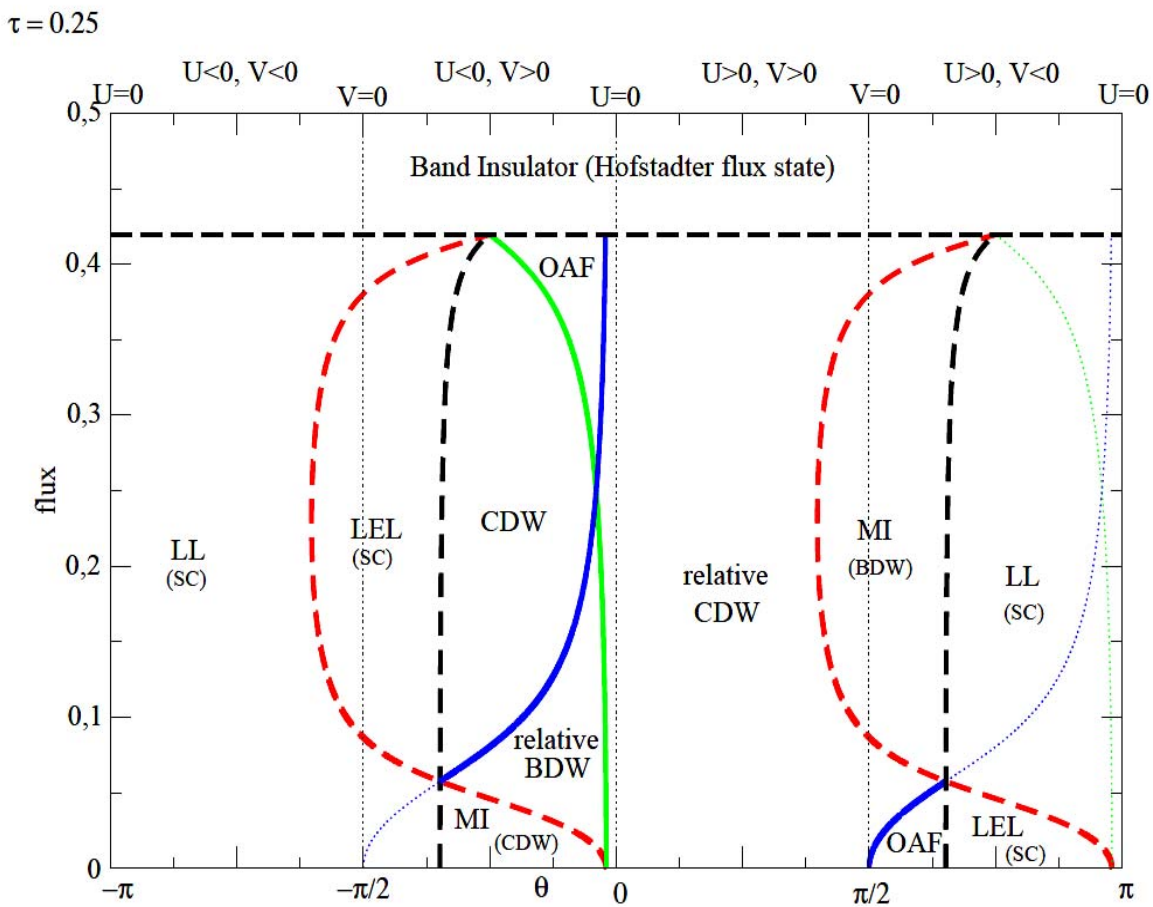
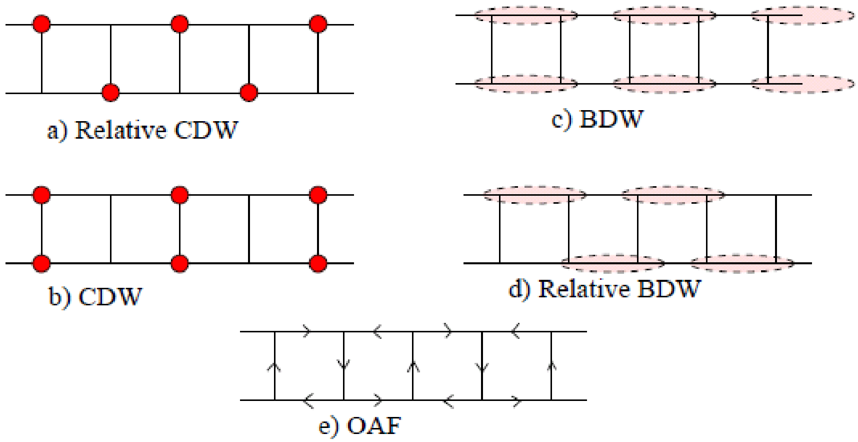


FIG. 3: The weak-coupling phase diagram in the magnetic field. We plot the flux along the vertical axis and the angle θ (defined as $\theta = \tan^{-1} U/2V$) along the horizontal axis. As the phases depend on the signs of the interaction parameters, they are indicated at the top of the diagram. Ordered phases are illustrated pictorially in Fig. 1. The corresponding order parameters are listed in Table I. The disordered phases are characterized by dominant correlations (indicated in parentheses). For large values of the flux ($\sin^2 \pi f > 1 - \tau^2$), there is a band gap in the non-interacting picture. The thick solid lines (blue and green online) represent $U(1)$ Gaussian transitions between mutually dual ground states with long-range order, and the thick dotted lines (black and red online) are Berezinski-Kosterlitz-Thouless phase transitions corresponding to opening of a gap in one of the sectors.

half-filled ladder – ordered phases



- bond current (OAF)

$$j_\perp = -it_\perp \left[c_1^\dagger c_2 - h.c.\right]$$

- density (CDW)

$$\rho_+ = c_1^\dagger c_1 + c_2^\dagger c_2$$

- relative density (Rel. CDW)

$$\rho_- = c_1^\dagger c_1 - c_2^\dagger c_2$$

- bond density (BDW)

$$\rho_{||,+} = e^{i\pi f} c_1^\dagger(x_n) c_1(x_{n+1}) + e^{-i\pi f} c_2^\dagger(x_n) c_2(x_{n+1}) + h.c.$$

- bond density (Rel. BDW)

$$\rho_{||,-} = e^{i\pi f} c_1^\dagger(x_n) c_1(x_{n+1}) - e^{-i\pi f} c_2^\dagger(x_n) c_2(x_{n+1}) + h.c.$$

half-filled ladder – phase boundaries

$$
\begin{aligned}
g_1 + g_3 = 0 \ &: \ \frac{U}{2V} = -\frac{2\cos^2 \pi f - \tau^2}{\cos^2 \pi f} \\
g_1 - g_3 = 0 \ &: \ \frac{U}{2V} = \frac{\sin^2 \pi f (2\cos^2 \pi f - \tau^2) + \tau^4}{\sin^2 \pi f (\cos^2 \pi f - \tau^2) - \tau^2} \\
g_4 = 0 \ &: \ \frac{U}{2V} = -\frac{\tau^2}{\cos^2 \pi f} \\
g_5 = 0 \ &: \ \frac{U}{2V} = -\frac{\tau^2}{\sin^2 \pi f}
\end{aligned} \tag{A3}
$$

bosonization approach to half-filled ladder

- example 1: states with long-range order ($K_+ < 1, K_- > 1$)
  - $g_4 < 0$ and $g_5 < 0$ - charge density wave (CDW)

$$\phi_+ = \varphi_\pi = \sqrt{\pi/8} + n\sqrt{\pi/2}, \quad n = 0, \pm 1, \dots$$

$$\theta_- = \varphi_0 = n\sqrt{\pi/2} \Rightarrow \langle \rho_+^{(s)} \rangle \sim \tan \pi f \langle \sin \sqrt{2\pi}\phi_+ \cos \sqrt{2\pi}\theta_- \rangle \neq 0$$

  - $g_4 > 0$ and $g_5 < 0$ - staggered flux phase (OAF)

$$\phi_+ = \theta_- = \varphi_0 \Rightarrow \langle j_\perp \rangle \sim \langle \cos \sqrt{2\pi}\phi_+ \cos \sqrt{2\pi}\theta_- \rangle \neq 0$$

- example 2: states without long-range order
  - $K_+ < 1$ and $K_- < 1$ - Mott insulator (only charge sector is gapped)

$$\rho_\perp = c_1^\dagger c_2 + c_2^\dagger c_1 \sim \cos \sqrt{2\pi}\phi_+ \cos \sqrt{2\pi}\phi_- \quad (2k_F \neq \pi)$$

$$g_4 > 0 \Rightarrow \langle \cos \sqrt{2\pi}\phi_+ \rangle \neq 0 \qquad \langle \rho_\perp(\tau, x)\rho_\perp(0) \rangle \sim \frac{\cos(2k_F x/a)}{|v_-\tau - ix|^{K_-}}.$$

conclusions for $\nu = 1/2$

- we have considered electrons on the two-leg ladder at arbitrary values of the external field, inter-chain hopping and interaction strength
- at half filling the model exhibits several ordered phases as well as phases without long-range order
- we have found field-induced (sometimes re-entrant) quantum phase transitions between phases with different types of long-range order and between ordered and gapless phases
- gapless phases are characterized by the algebraic decay of dominant correlations

persistent current

- current operator

$$j_{n,\sigma} = -it_0 \left(e^{i\pi f \sigma} c^\dagger_{n,\sigma} c_{n+1,\sigma} - h.c. \right).$$

- ground state value – relative current

$$\langle j_{rel} \rangle = -2t_0 \sin \pi f \int \frac{dk}{2\pi} \Bigg\{ \cos k \left[n_\alpha(k) + n_\beta(k) \right] - \frac{\sin^2 k \cos \pi f \left[n_\alpha(k) - n_\beta(k) \right]}{\sqrt{\sin^2 k \sin^2 \pi f + \tau^2}} \Bigg\},$$

- small flux, small inter-chain tunneling

$$\langle j_{rel} \rangle = \frac{v_F}{3a} f \tau^2 \left[1 + O(f^2, \tau^2) \right].$$

persistent current

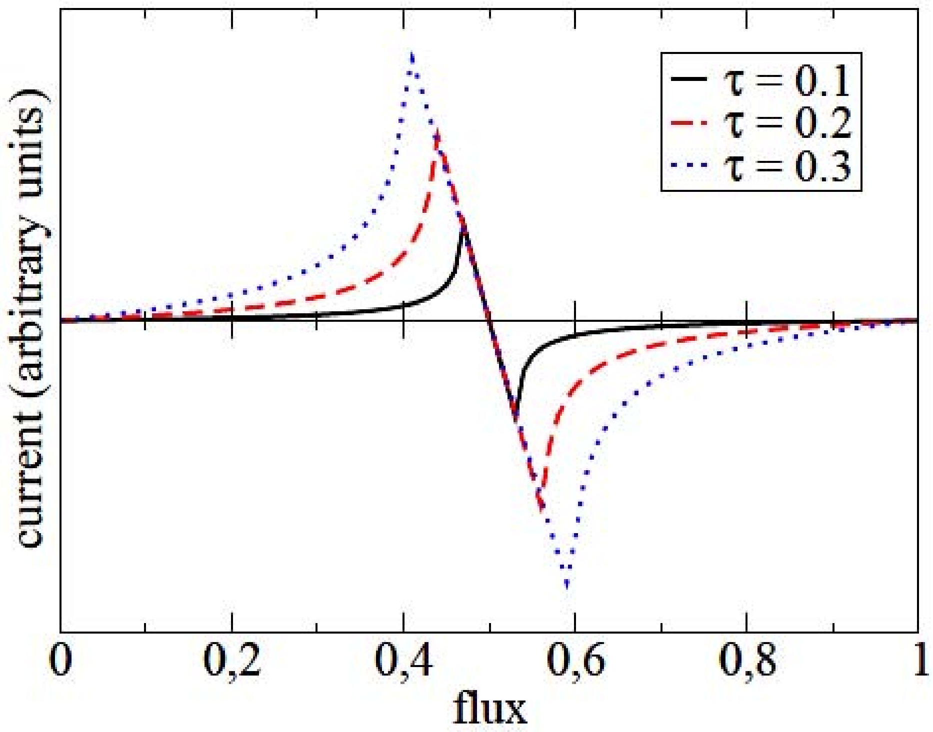


FIG. 5: Diamagnetic current as a function of flux in the absence of interaction. Only one period in f is shown. The cusps correspond to the band gap opening.

conclusions for persistent current

- persistent current is an example of a non-universal quantity contributed to by *all electrons* – not only those in the vicinity of the Fermi points
- not an infra-red quantity – non zero even in the insulating phase.
- can not be addressed in terms of any Lorentz-invariant effective low-energy field theory
- gapless phases are characterized by the algebraic decay of dominant correlations

SUMMARY

- fermionic ladders exhibit interesting physics
  - charge fractionalization a quarter-filling
  - field-induced quantum phase transitions at half-filling
- there exist physical quantities that cannot be described by means of low energy effective theory
  - such as persistent current
- possible generalizations: multiple-leg ladders, spinful fermions, ...