Many-body localization and nonlinear transport in disordered conductors

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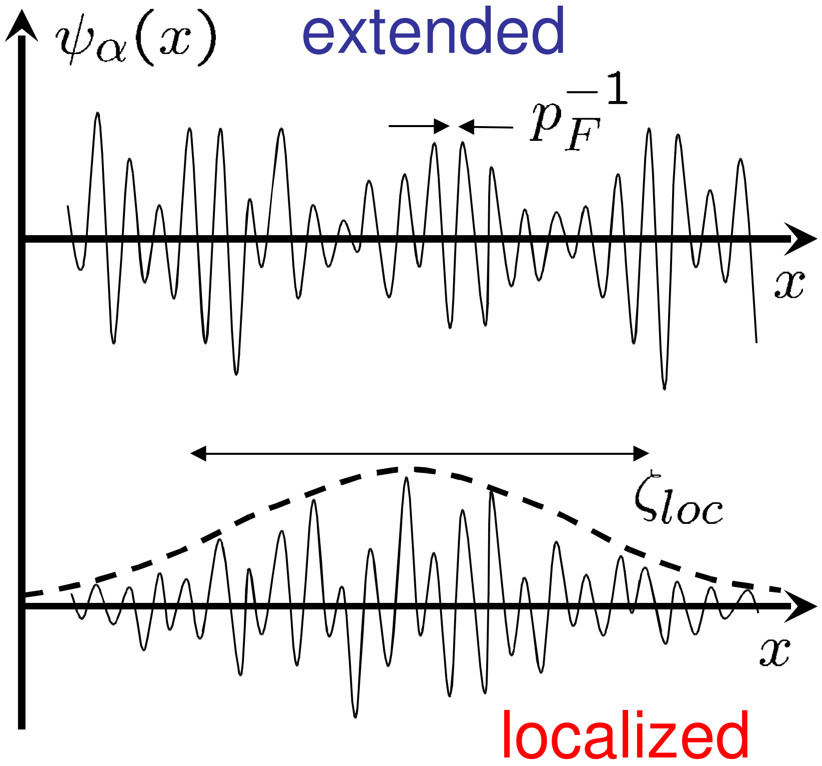
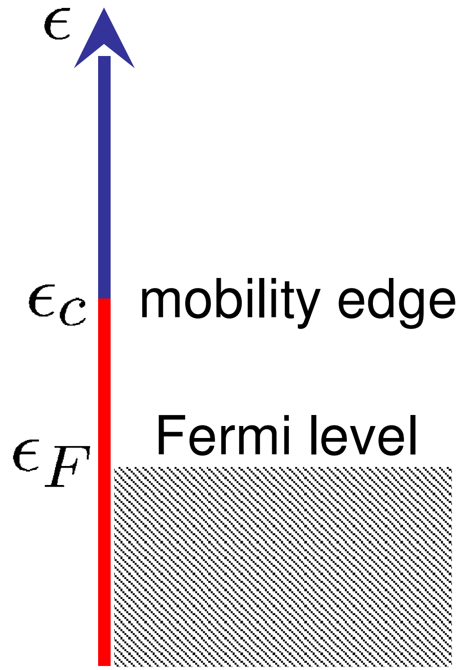
Outline

- Introduction: dc conduction in Anderson insulators at low temperatures
- Effect of electron-electron interaction: finite-*T* metal-insulator transition in the absence of electron-phonon coupling
- Effect of electron-phonon coupling: nonlinear *I*-*V* curve

Free electrons in a random potential





dc conductivity:

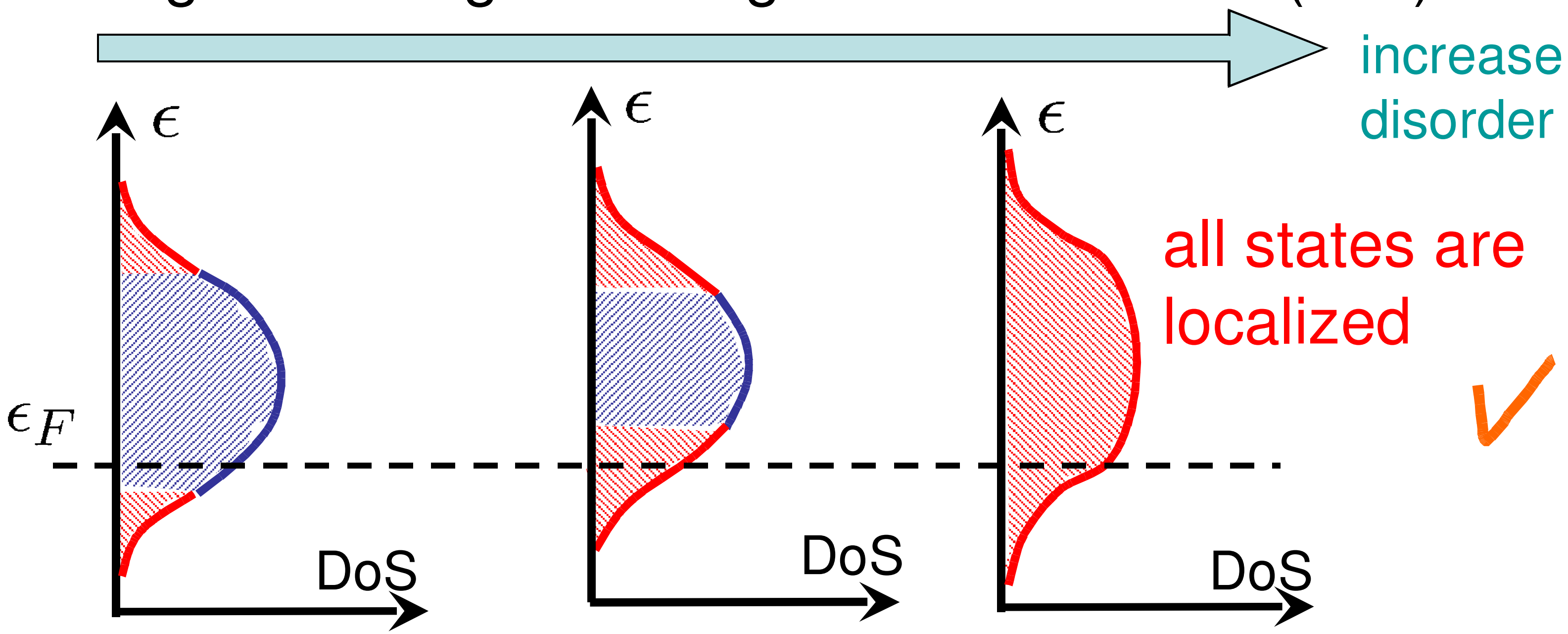
$\epsilon_c < \infty \quad \Rightarrow \quad \sigma(T) \propto \exp\left(-\frac{\epsilon_c - \epsilon_F}{T}\right)$ thermal population of extended eigenstates

$\epsilon_c = \infty \quad \Rightarrow \quad \boxed{\sigma(T) = 0}$ all eigenstates are localized

Free electrons + random potential:

d=1: all states are localized ✓
d=2: the same (?) ✓
d>2: mobility edge

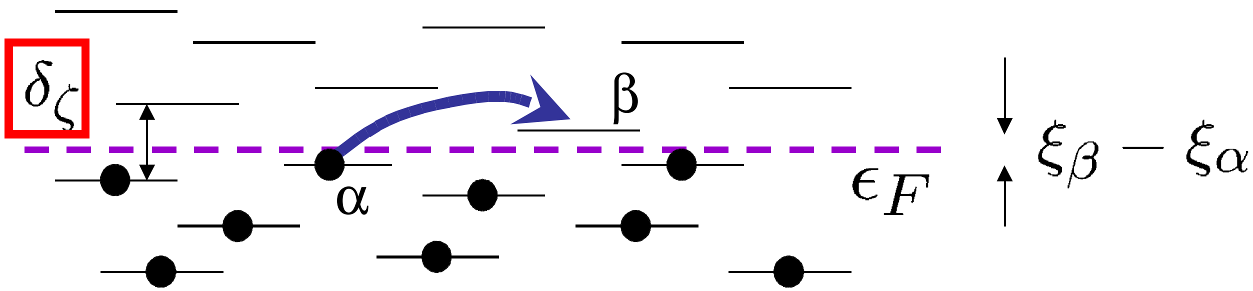
Single-band tight-binding Anderson model (*d>2*):



$\sigma(T) = const$ $\quad$ $\sigma(T) \propto e^{-\frac{\epsilon_c - \epsilon_F}{T}}$ $\quad$ $\sigma(T) = 0$

Our main assumption: all states are localized

Inelastic processes ⇒ transitions between localized states



energy mismatch

Level spacing in the localization volume: $\delta_\zeta = \dfrac{1}{\nu\, \zeta_{loc}^d}$

DoS per unit volume (ν)

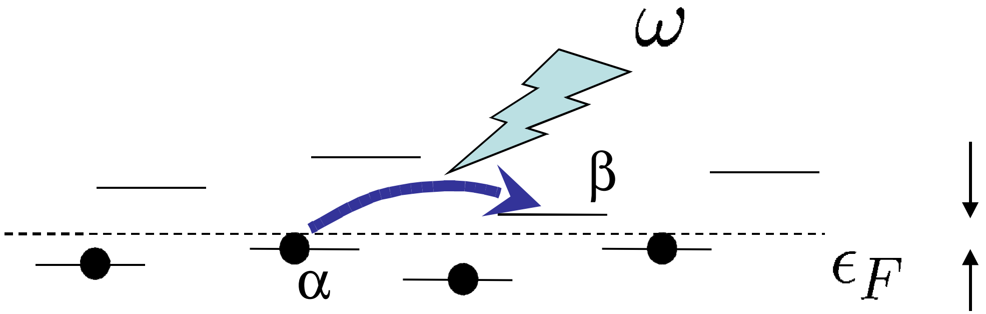
localization volume (ζ_{loc}^d)

$\sigma(T) \propto \Gamma_\alpha$ (inelastic lifetime)$^{-1}$

$T = 0 \quad \Rightarrow \quad \sigma = 0$ (any mechanism)

$T \to 0 \quad \Rightarrow \quad \sigma = ?$

Phonon-assisted hopping



ϵ_F

$$\omega = \xi_\beta - \xi_\alpha \ll \delta_\zeta$$

energy difference can always be matched by a phonon

Mott formula: $\sigma(T) \propto T^\gamma \exp\left[-\left(\frac{\delta_\zeta}{T}\right)^{1/(d+1)}\right]$

without Coulomb gap

mechanism-dependent prefactor

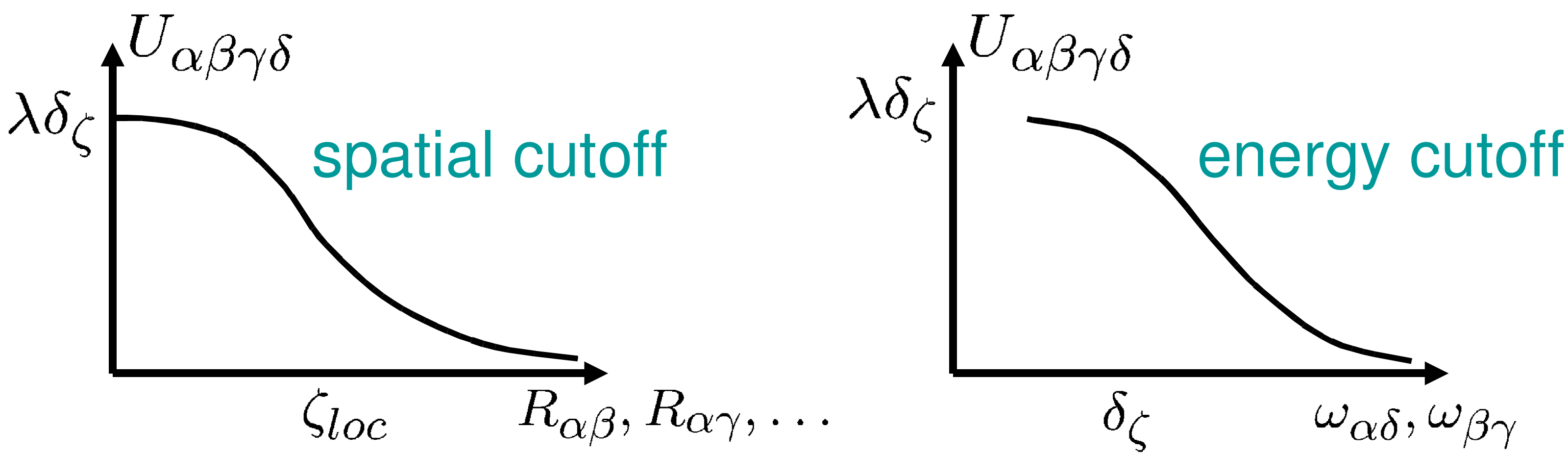
any bath with a continuous spectrum of delocalized excitations down to ***E*= *0***

No phonons, ***e-e*** interaction {weak, short-range}

$$U(\vec{r}-\vec{r}\,') = \frac{\lambda}{\nu}\,\delta(\vec{r}-\vec{r}\,')\,, \qquad \frac{1}{\nu} \equiv \delta_\zeta\,\zeta_{loc}^d$$

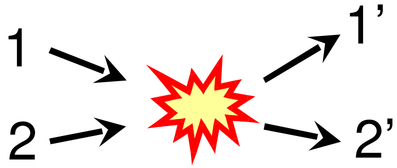
- ν: DoS per unit volume
- λ: dimensionless interaction strength (≪1)
- δ_ζ: level spacing
- ζ_{loc}^d: localization volume

Matrix elements between localized wave functions:



Energy conservation problem

emission of electron-hole pair $\leftrightarrow$ pair "collision":



T/δ_ζ – number of electrons to choose from

we need: $|\epsilon_1 + \epsilon_2 - \epsilon_{1'} - \epsilon_{2'}| \sim \lambda\delta_\zeta$

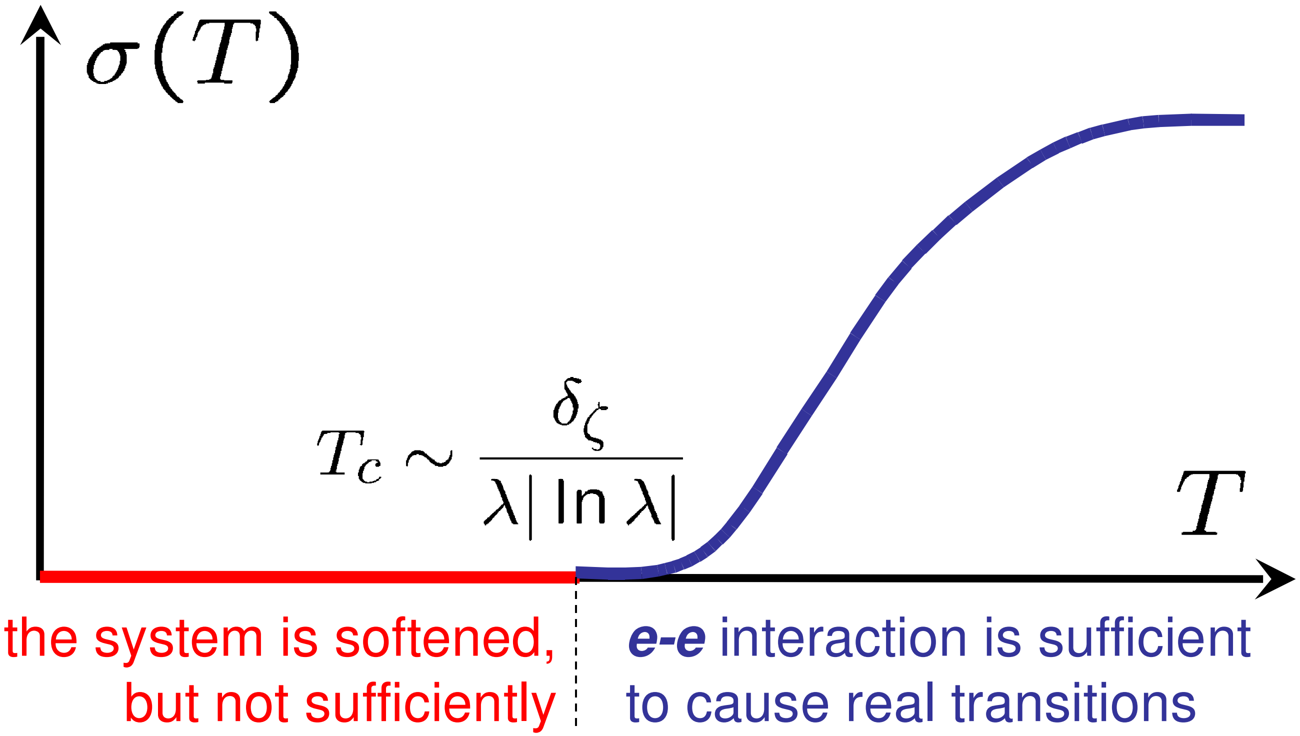
problems start at $T \ll \delta_\zeta/\lambda$ (Fleishman & Anderson, 1980)

triple "collision":

$$|\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_{1'} - \epsilon_{2'} - \epsilon_{3'}| \sim \frac{(\lambda\delta_\zeta)^2}{\delta_\zeta}$$

1. Lower temperatures $\rightarrow$ harder to conserve energy
2. Need to consider many-electron processes
   to **ALL** orders of perturbation theory

The answer is $\sigma(T) = 0,\ T < T_c$



finite-temperature metal-insulator transition

Anderson localization in the many-body Fock space

$$\xi_\alpha \rightarrow \xi_\gamma + \xi_\delta - \xi_\beta \rightarrow \qquad \text{(A., G., K., \& L., 1997)}$$
$$\rightarrow \xi_1 + \xi_2 + \xi_3 - \xi_4 - \xi_5 \rightarrow \ldots$$

many-body Fock states → sites with random energies

e-e interaction → coupling between sites

metal-insulator transition → Anderson transition

temperature → coordination number

Systematic treatment of many-electron transitions:

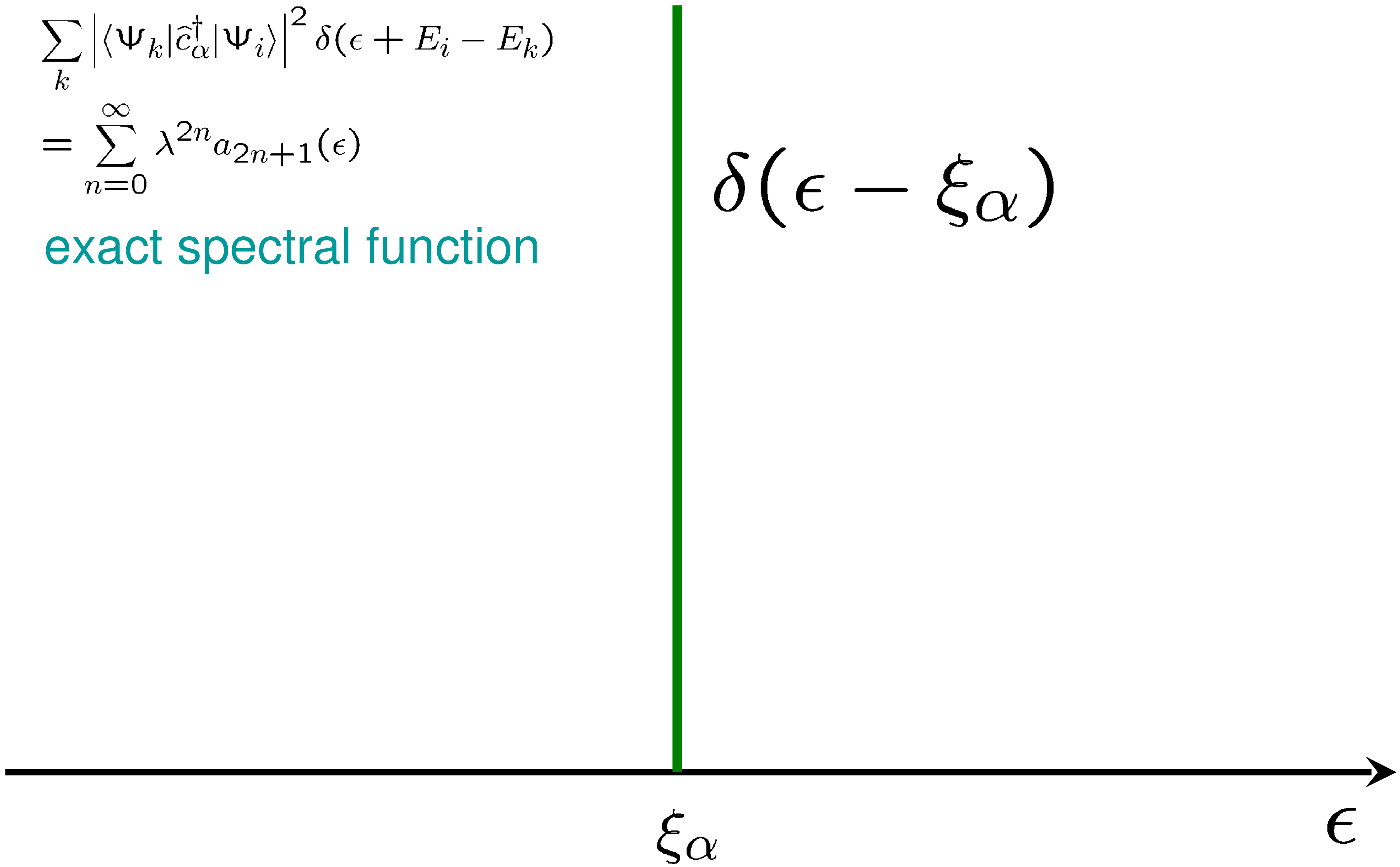
D. B., I. Aleiner, and B. Altshuler, Ann. Phys. **321**, 1126–1205 (2006)

one-particle spectral weight (λ^0)

$$\sum_k \left|\langle\Psi_k|\tilde{c}^\dagger_\alpha|\Psi_i\rangle\right|^2 \delta(\epsilon + E_i - E_k)$$

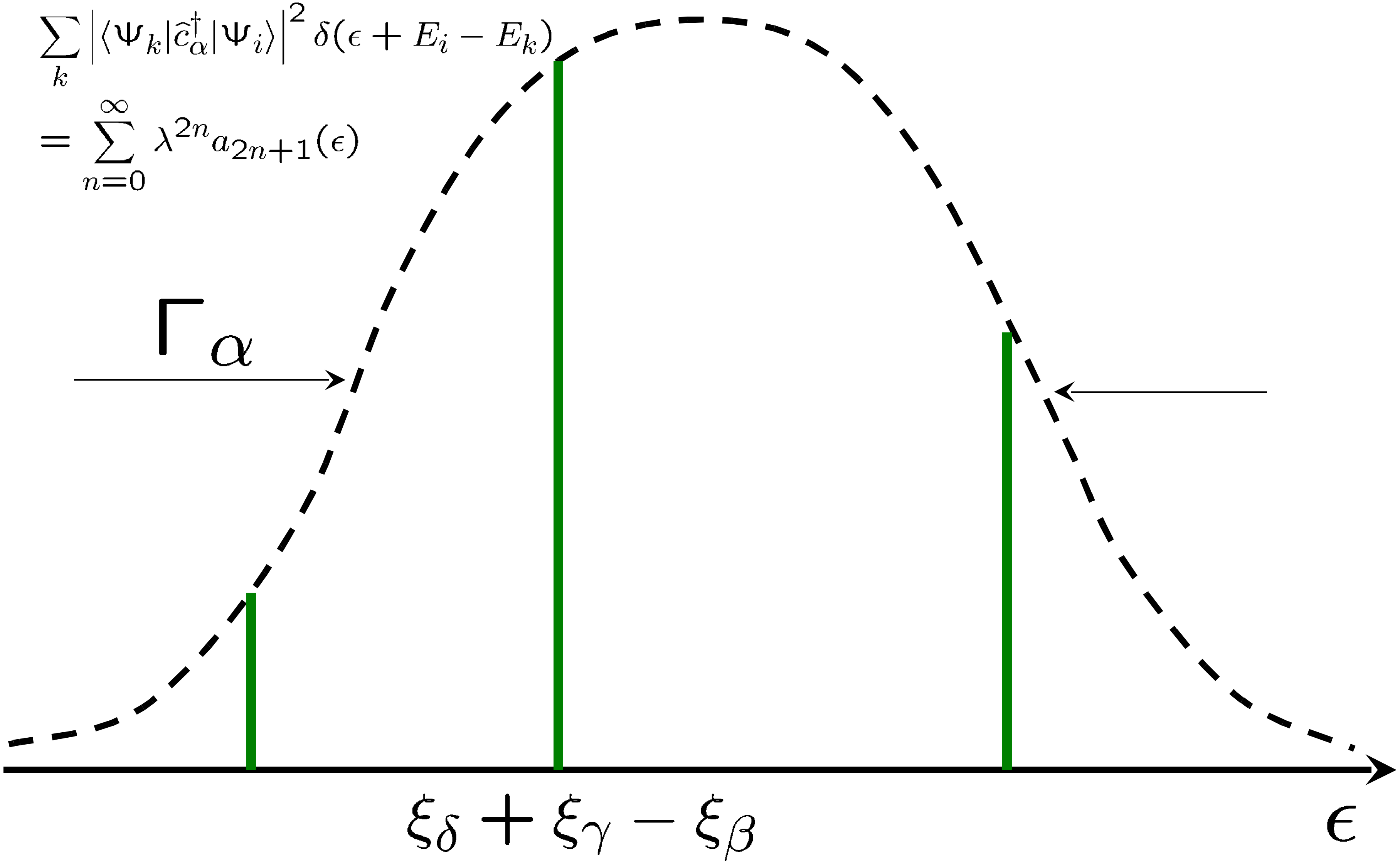
$$= \sum_{n=0}^{\infty} \lambda^{2n} a_{2n+1}(\epsilon)$$

exact spectral function



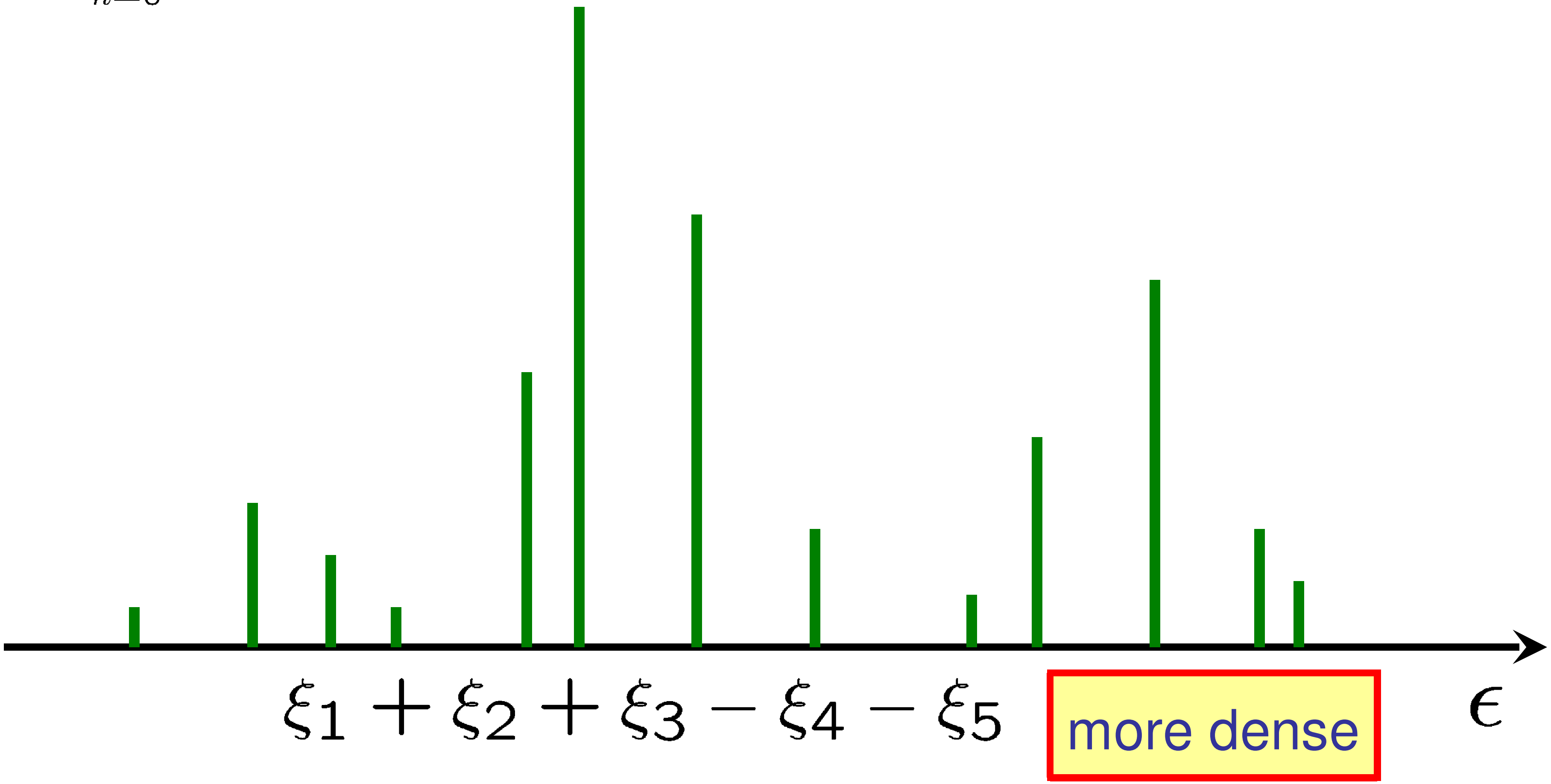
three-particle spectral weight (λ^2)

$$\sum_k \left|\langle\Psi_k|\tilde{c}^\dagger_\alpha|\Psi_i\rangle\right|^2 \delta(\epsilon + E_i - E_k)$$
$$= \sum_{n=0}^{\infty} \lambda^{2n} a_{2n+1}(\epsilon)$$



five-particle spectral weight (λ^4)

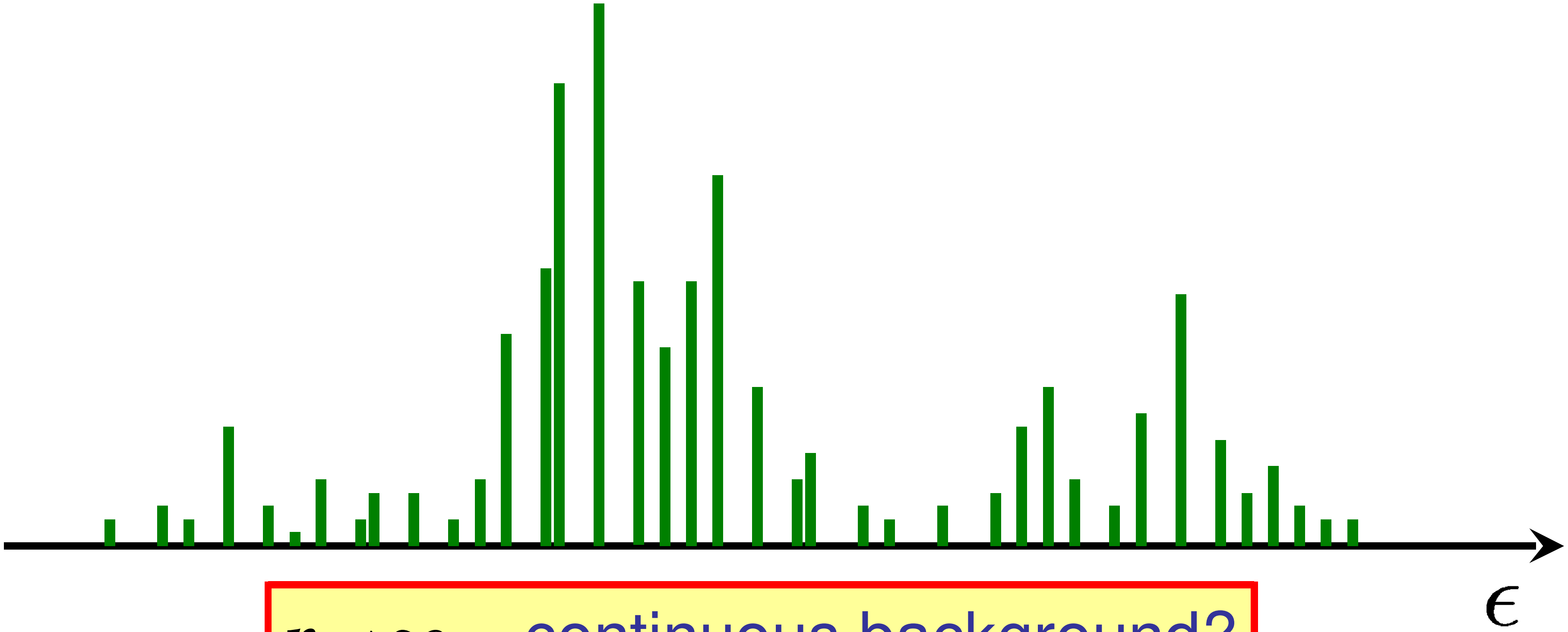
$$\sum_k \left|\langle\Psi_k|\hat{c}_\alpha^\dagger|\Psi_i\rangle\right|^2 \delta(\epsilon + E_i - E_k)$$
$$= \sum_{n=0}^{\infty} \lambda^{2n} a_{2n+1}(\epsilon)$$



seven-particle spectral weight (λ^6)

$$\sum_k \left|\langle \Psi_k | \hat{c}^\dagger_\alpha | \Psi_i \rangle\right|^2 \delta(\epsilon + E_i - E_k)$$

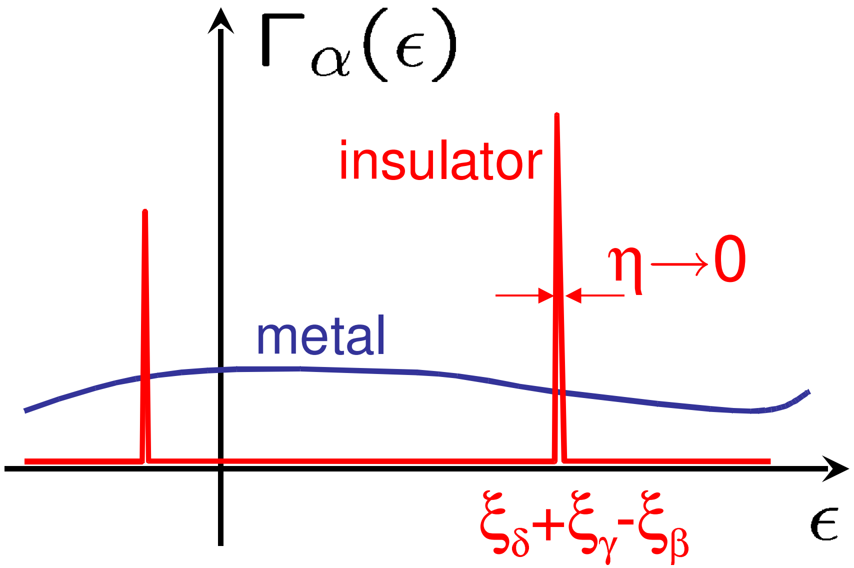
$$= \sum_{n=0}^{\infty} \lambda^{2n} a_{2n+1}(\epsilon)$$



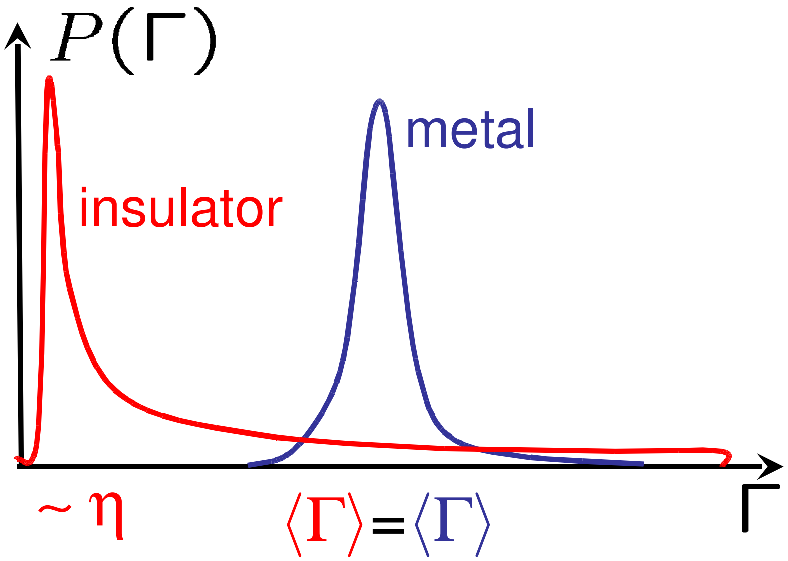
$n \to \infty$ – continuous background?

What to calculate? (Anderson, 1958)

$$\Gamma_\alpha(\epsilon) = \mathrm{Im}\,\Sigma_\alpha(\epsilon + i\eta)$$ – random quantity



behavior for a given realization

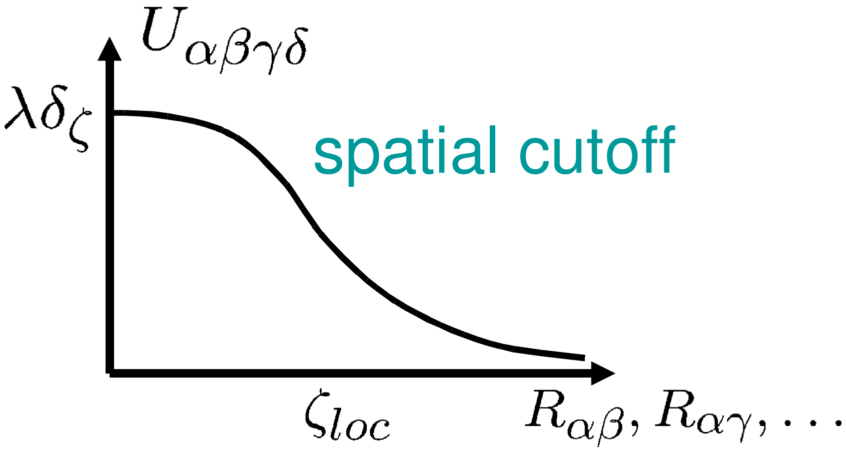


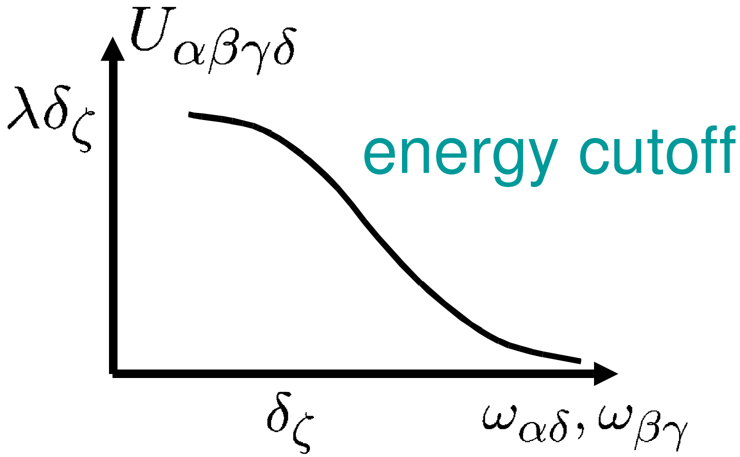
probability distribution for a fixed energy

working criterion: $$\lim_{\eta\to 0}\lim_{V\to\infty} P(\Gamma) \begin{cases} > 0 & \text{metal} \\ = 0 & \text{insulator} \end{cases}$$

$$\widehat{H} = \sum_{\alpha} \xi_{\alpha} \widehat{c}^{\dagger}_{\alpha} \widehat{c}_{\alpha} + \sum_{\alpha\beta\gamma\delta} U_{\alpha\beta\gamma\delta}\, \widehat{c}^{\dagger}_{\alpha} \widehat{c}^{\dagger}_{\beta} \widehat{c}_{\gamma} \widehat{c}_{\delta}$$

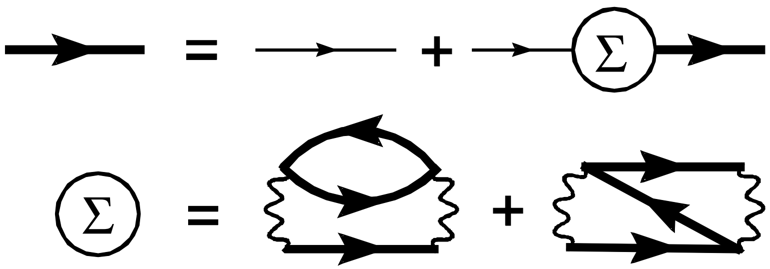
Two essential facts:





These facts can be modeled in different ways

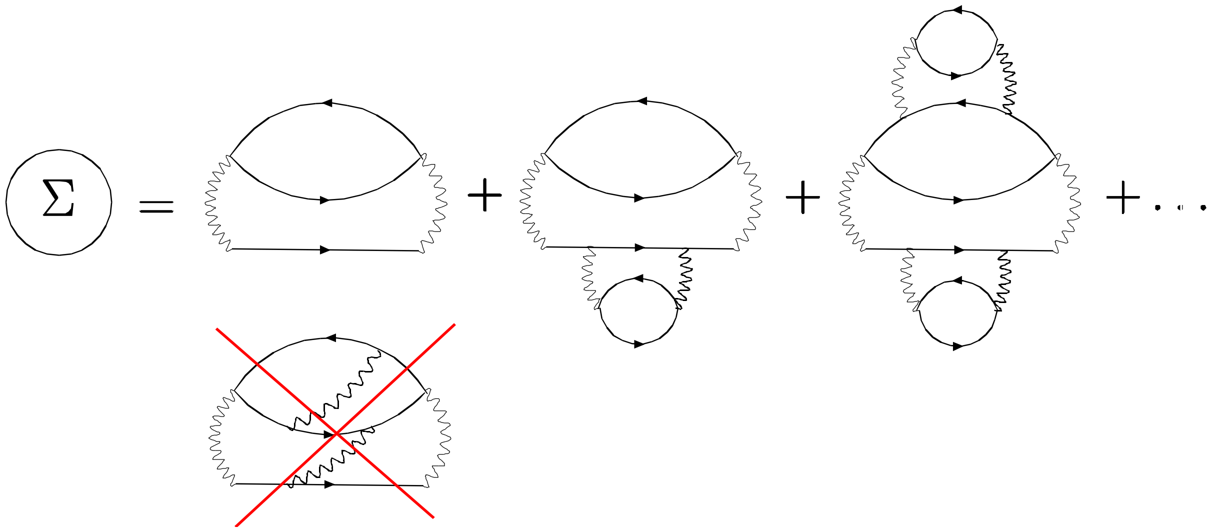
Self-consistent Born approximation



$$G_\alpha(\epsilon) = [\epsilon - \xi_\alpha - \Sigma_\alpha(\epsilon)]^{-1}$$

$$\Sigma_\alpha(\epsilon) = \ldots G_\beta G_\gamma G_\delta \ldots$$

iterations of SCBA: max number of particles in the final state



Stability of the metallic phase: finite broadening is self-consistent

- $$P(\Gamma) = \frac{1}{\sqrt{2\pi\langle\delta\Gamma^2\rangle}} \exp\left[-\frac{(\Gamma - \langle\Gamma\rangle)^2}{2\langle\delta\Gamma^2\rangle}\right]$$

  $\sqrt{\langle\delta\Gamma^2\rangle} \ll \langle\Gamma\rangle$ as long as $T \gg \frac{\delta_\zeta}{\lambda}$

- $\langle\Gamma\rangle \ll \delta_\zeta$ (levels well resolved)

- quantum kinetic equation for transitions between localized states

  $\sigma(T) \propto \lambda^2 T^\alpha$ (model-dependent)

Stability of the insulating phase: NO spontaneous generation of broadening

- $\Gamma_\alpha(\epsilon) \equiv 0$ is always a solution
- $\epsilon \longrightarrow \epsilon + i\eta$ – ***linear*** stability analysis:

$$\frac{\Gamma}{(\epsilon-\xi_\alpha)^2+\Gamma^2} \to \pi\delta(\epsilon-\xi_\alpha) + \frac{\Gamma}{(\epsilon-\xi_\alpha)^2}$$

- after n iterations of SCBA equations:

$$P_n(\Gamma) \propto \frac{\eta}{\Gamma^{3/2}}\left(\text{const}\cdot\frac{\lambda T}{\delta_\zeta}\ln\frac{1}{\lambda}\right)^n$$

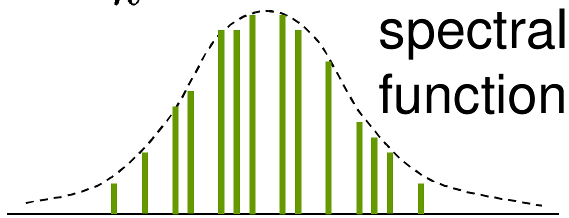
first $n \to \infty$, then $\eta \to 0$

Many-body (de)localization

eigenstates $\begin{cases} \text{one-body } |\alpha\rangle & \text{– localized in space} \\ \text{many-body } |\Psi_k\rangle & \text{– whole volume: } E_k \propto V \end{cases}$

Electron-hole excitation: $\hat{a}^\dagger_\alpha \hat{a}_\beta |\Psi_k\rangle = \sum_{k'} C^{\alpha\beta}_{kk'} |\Psi_{k'}\rangle$

IPR: $\lim_{V\to\infty} \sum_{k'} \left| C^{\alpha\beta}_{kk'} \right|^4 \begin{cases} = 0 & \text{extended} \\ > 0 & \text{localized} \end{cases}$



Developed metallic phase: $|C_{kk'}|^2 \propto \delta(E_{k'} - E_k - \omega_{\alpha\beta})$

microcanonical $\Leftrightarrow$ canonical $\Rightarrow$ temperature

Insulating phase: total energy $\boldsymbol{E_k}$

Many-body mobility edge

T

Large $E_k \Rightarrow$ high T: extended states

interaction $\longrightarrow$ dephasing $\longrightarrow$ cutoff of WL (good metal)

δ_ζ/λ^2

Fermi Golden Rule $\longrightarrow$ hopping (bad metal)

δ_ζ/λ

T_c

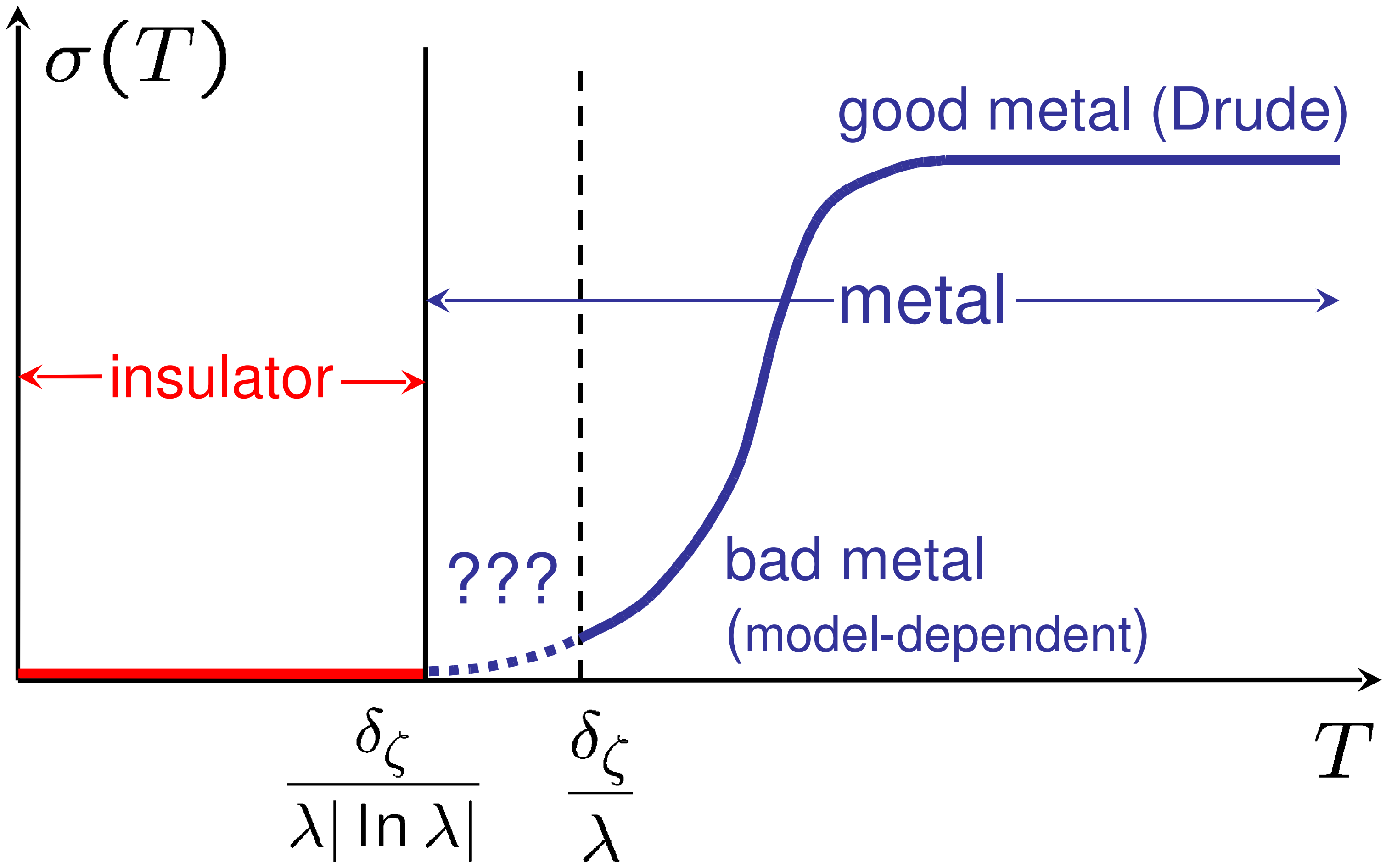
transition $\longrightarrow$ mobility edge

No activation!

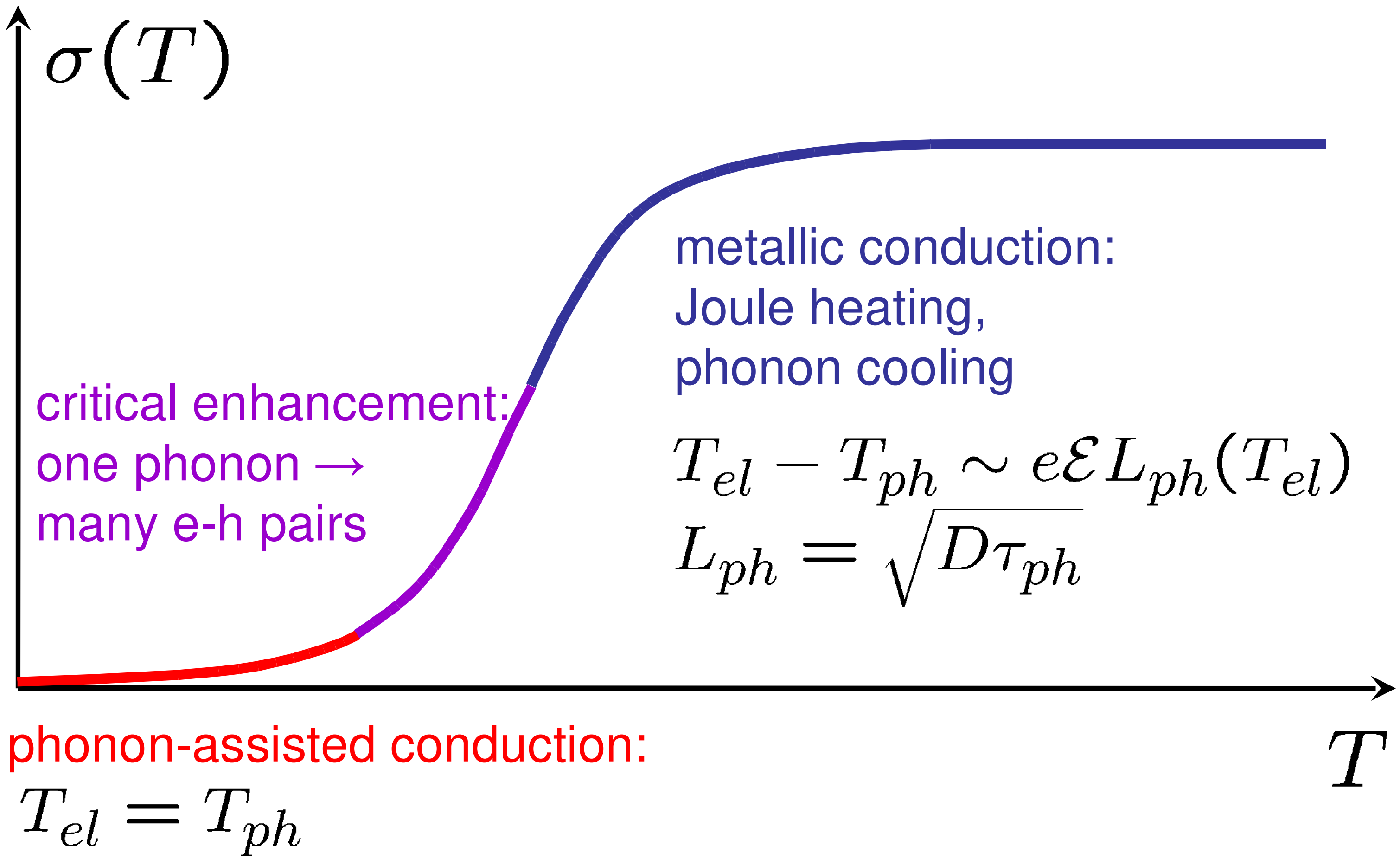
$E_c \sim \nu T_c^2 \times$ **(volume)**

$$E, E_c \propto V \quad \Rightarrow \quad e^{\frac{E(T)-E_c}{T}} \underset{V\to\infty}{\longrightarrow} 0$$

Summary (no phonons)

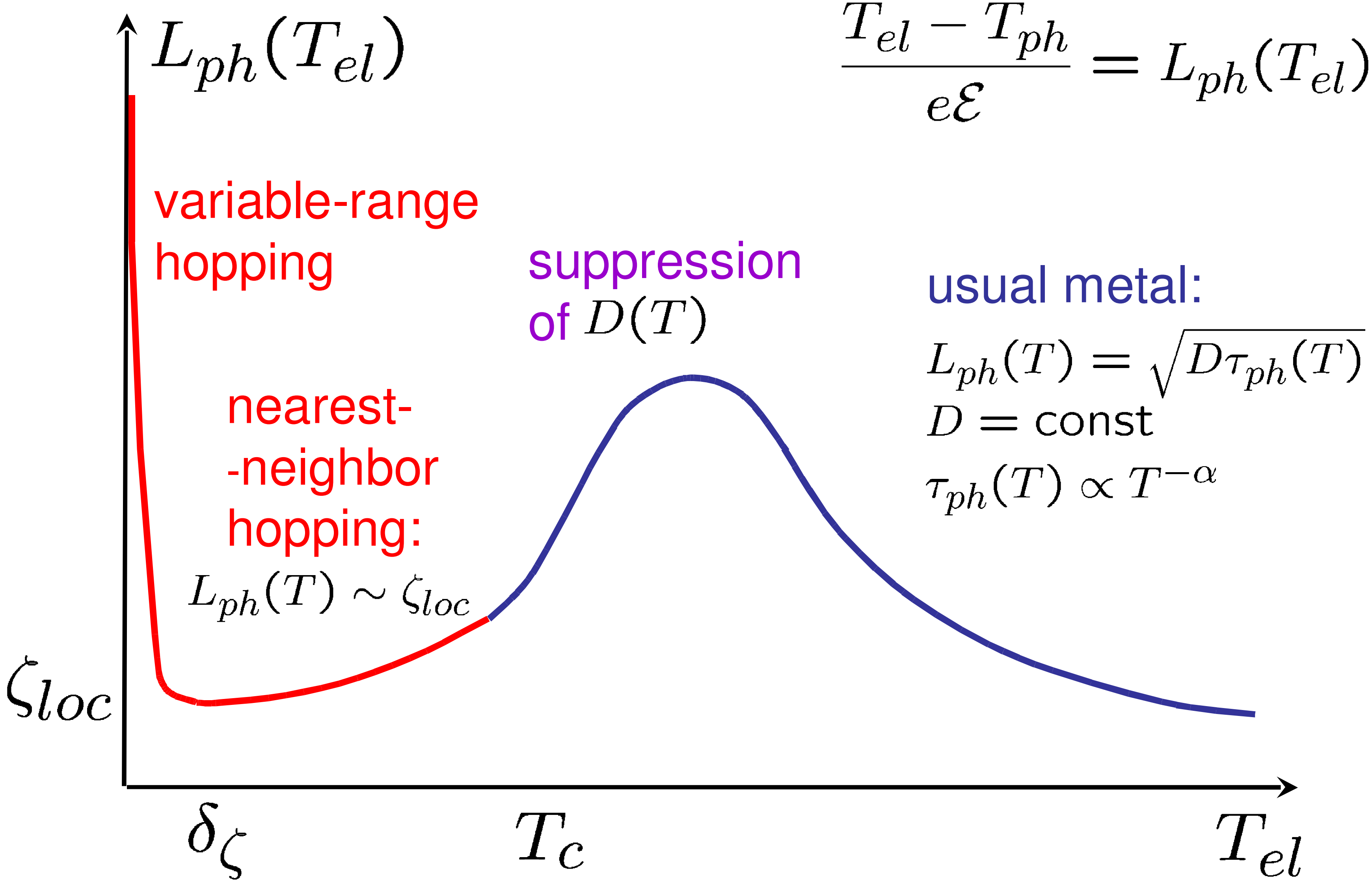


Phonons included



Electronic temperature from thermal balance:

$$\frac{T_{el} - T_{ph}}{e\mathcal{E}} = L_{ph}(T_{el})$$



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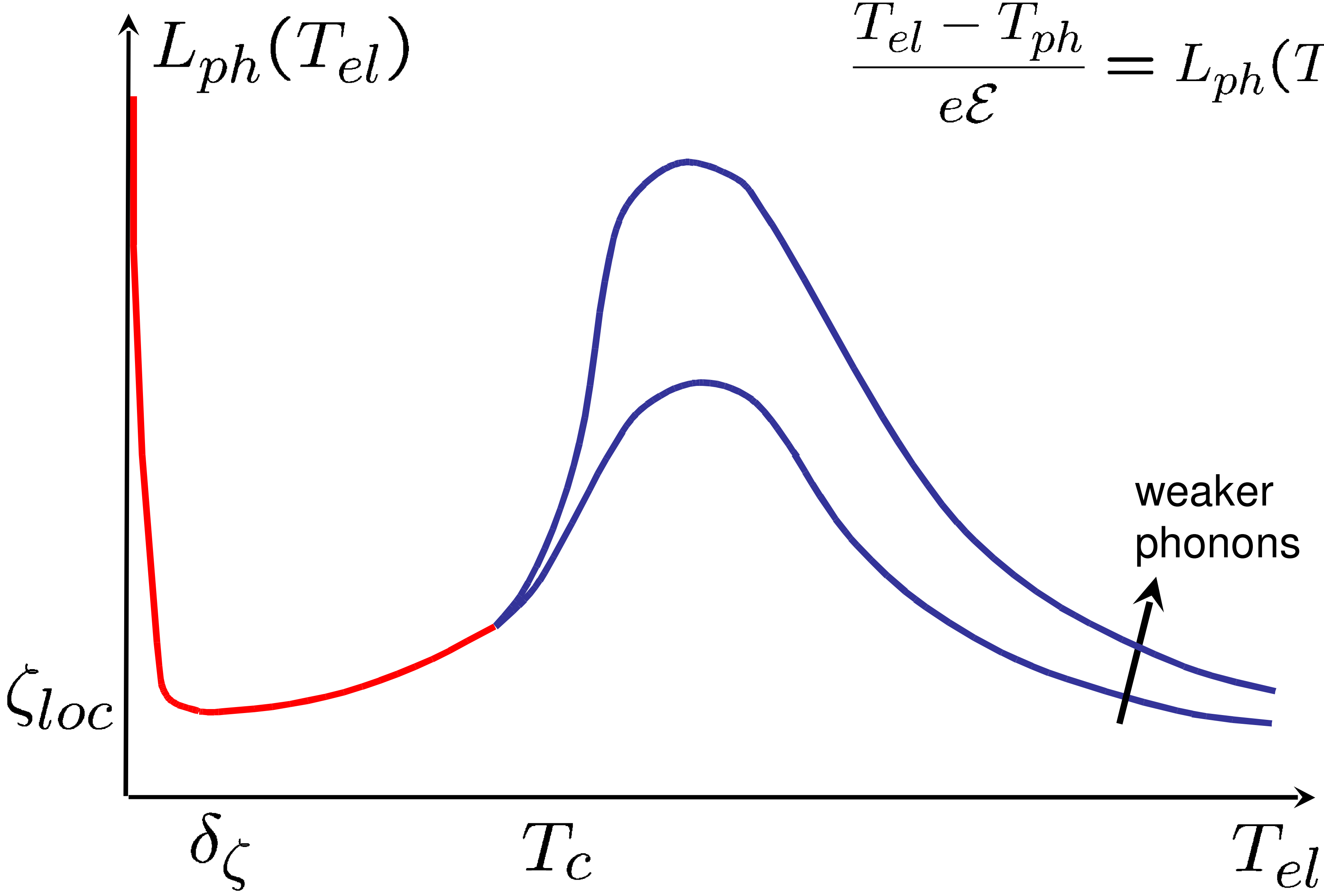

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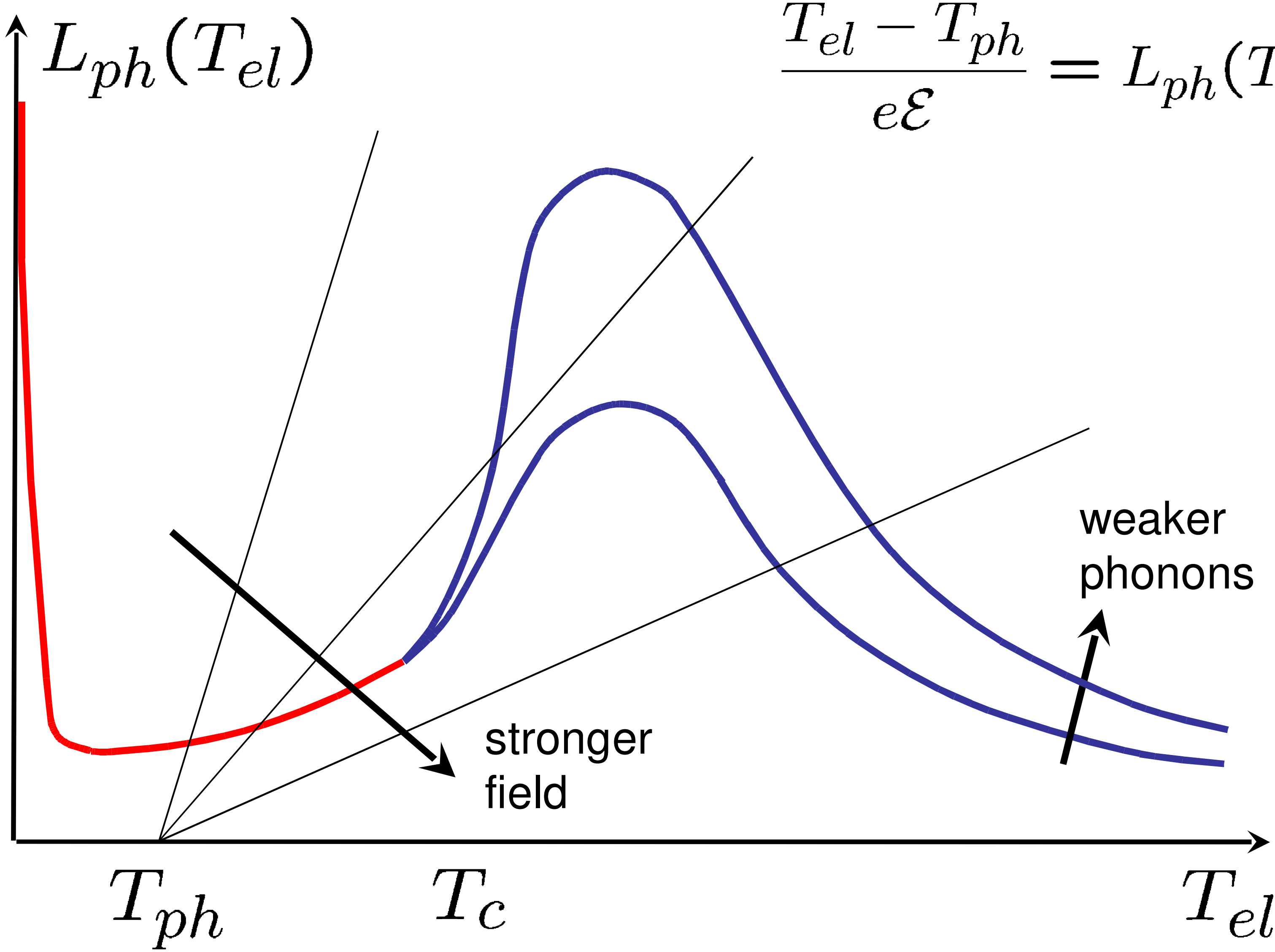

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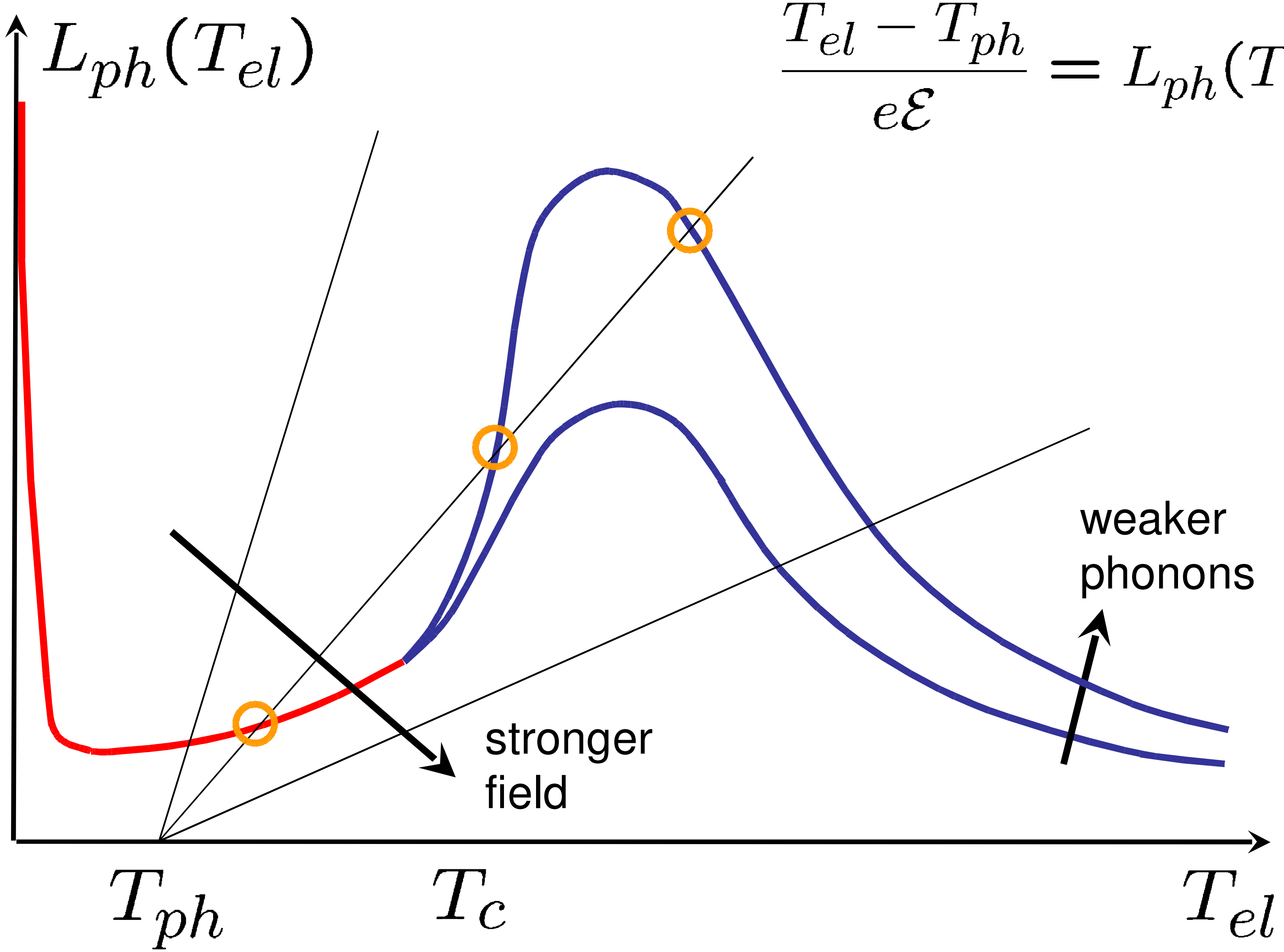
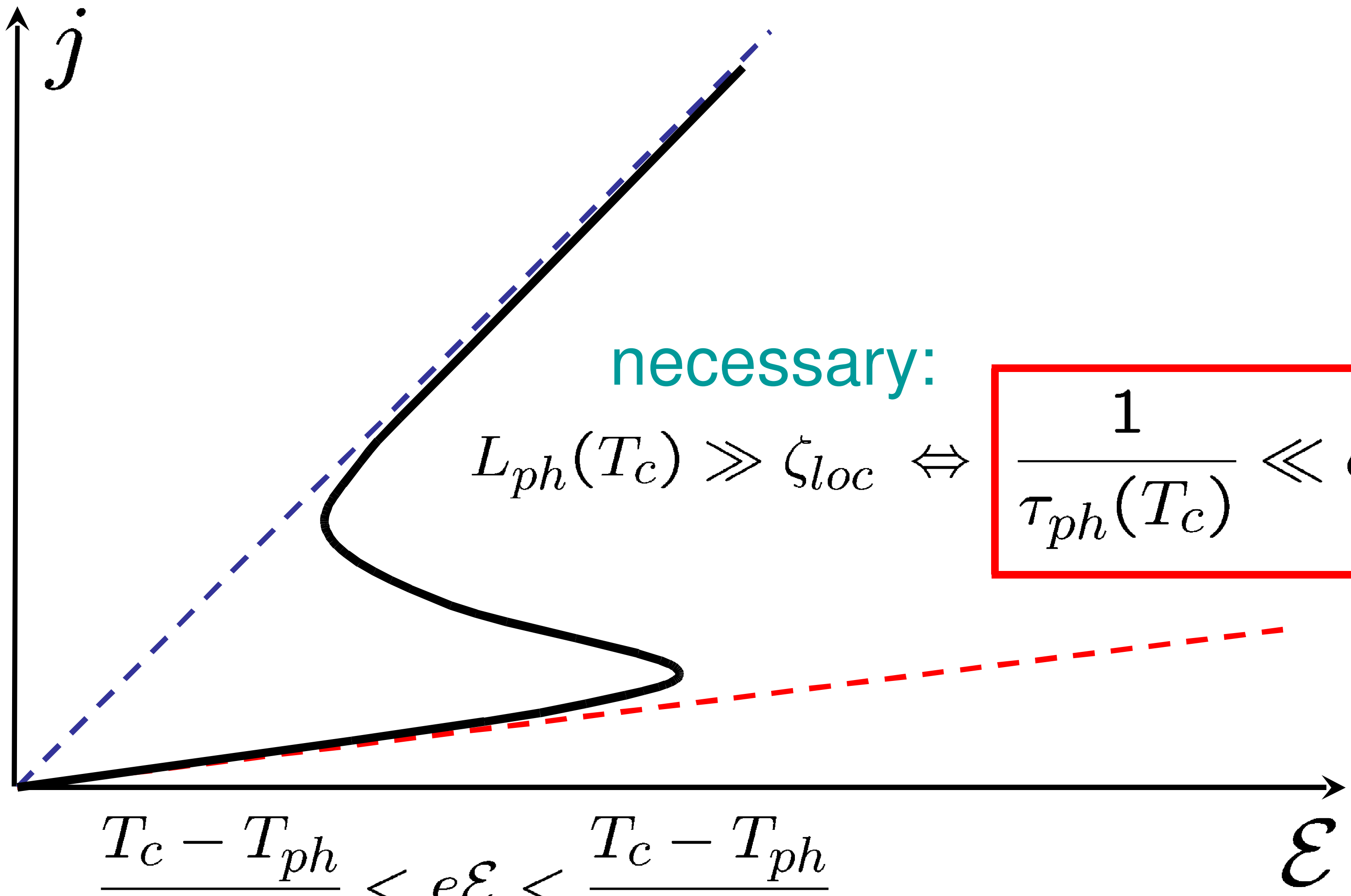


Bistable *I-V* curve:



necessary:

$$L_{ph}(T_c) \gg \zeta_{loc} \;\Leftrightarrow\; \boxed{\frac{1}{\tau_{ph}(T_c)} \ll \delta_\zeta}$$

$$\frac{T_c - T_{ph}}{L_{ph}(T_c)} < e\mathcal{E} < \frac{T_c - T_{ph}}{\zeta_{loc}}$$

Instead of conclusion: FAQ

Q: What if one takes Coulomb interaction?

A: $\lambda \propto \int V(\vec{r})\, d^d\vec{r}$

$V_{\text{Coulomb}}(r) \propto 1/r^3$ (dipole transitions!)

$d = 1, 2$ – nothing changes,

$d = 3$ – ask Levitov

Q: Can we have $\lambda \gg 1,\ T_c \ll \delta_\zeta$?

A: Yes, if the range of $V(r)$ is large.

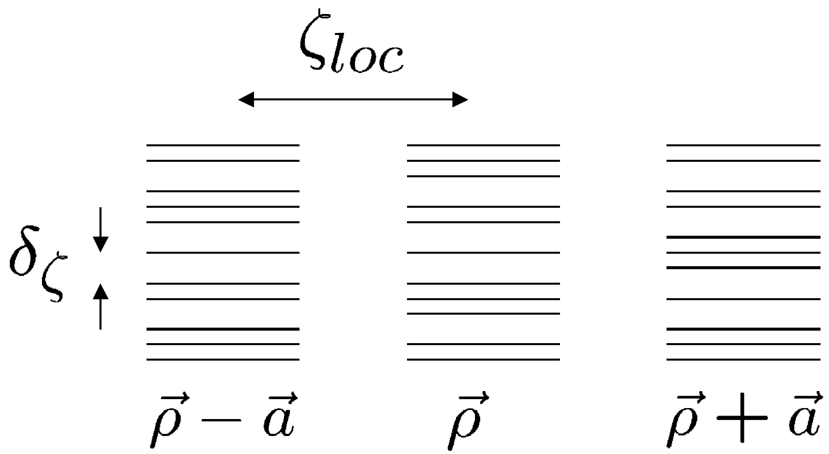
$T_c \ll T \ll \delta_\zeta$ – e-e VRH

Our model: localization volumes → discrete grains

$$\hat{H}_0 = \underset{\vec{\rho},l}{\ } \hat{c}_l^{\dagger}(\vec{\rho}) \quad \xi_l(\vec{\rho})\, \hat{c}_l(\vec{\rho}) + I\delta_{\zeta} \underset{\vec{a},m}{\ } \hat{c}_m(\vec{\rho}+\vec{a})$$

$$\hat{V}_{\text{int}} = \frac{1}{2} \underset{\vec{\rho};l_1,l_2;j_1,j_2}{\ } V_{l_1,l_2}^{j_1,j_2}(\vec{\rho})\, \hat{c}_{l_1}^{\dagger}(\vec{\rho})\, \hat{c}_{l_2}^{\dagger}(\vec{\rho})\hat{c}_{j_1}(\vec{\rho})\, \hat{c}_{j_2}(\vec{\rho})$$

$$V_{l_1,l_2}^{j_1,j_2} = \lambda\delta_{\zeta} \frac{\sigma_{l_1}^{j_1}\sigma_{l_2}^{j_2}}{2} \quad \frac{\xi_{j_1}-\xi_{l_1}}{\delta_{\zeta}} \quad \frac{\xi_{j_2}-\xi_{l_2}}{\delta_{\zeta}} - (l_1 \quad l_2)$$



$$\Upsilon(x) = \theta\left(\frac{M}{2} - |x|\right)$$

$$1 \ll \frac{1}{M^2} \ll \lambda \leq I$$