Langevin description of superdiffusive Lévy processes

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(Received 7 November 2011; revised manuscript received 25 April 2012; published 19 October 2012)

The description of diffusion processes is possible in different frameworks such as random walks or Fokker-Planck or Langevin equations. Whereas for classical diffusion the equivalence of these methods is well established, in the case of anomalous diffusion it often remains an open problem. In this paper we aim to bring three approaches describing anomalous superdiffusive behavior to a common footing. While each method clearly has its advantages it is crucial to understand how those methods relate and complement each other. In particular, by using the method of subordination, we show how the Langevin equation can describe anomalous diffusion exhibited by Lévy-walk-type models and further show the equivalence of the random walk models and the generalized Kramers-Fokker-Planck equation. As a result a synergetic and complementary description of anomalous diffusion is obtained which provides a much more flexible tool for applications in real-world systems.

DOI: 10.1103/PhysRevE.86.041134 PACS number(s): 05.40.Fb, 05.10.Gg, 52.65.Ff

I. INTRODUCTION

Much evidence has been gathered that systems exhibiting superdiffusion are ubiquitous in many fields of science [1–9]. It is remarkable that the theoretical advances in various successful models often develop independently and are driven by particular applications or types of the questions pertinent to the corresponding system. Random-walk models, such as Lévy walks [10,11] or Lévy flights [12], fractional Brownian motion [13], and fractional Kramers equations [14,15], are the archetype models leading to the superdiffusive behavior and it is not *a priori* clear which model is a suitable choice for a specific problem. Just from the behavior of the mean-squared displacement (MSD), those models cannot be distinguished since all of them can generate anomalous scaling of MSD $\langle x^2(t)\rangle \sim t^\alpha$ with $\alpha > 1$. A way to possibly discriminate which model is a proper candidate to quantitatively describe a real system is to look at sample trajectories of the respective stochastic models. This implies that the appropriate stochastic differential equation which generates single trajectories needs to be known.

While random-walk models proved to be successful in explaining the origin of anomalous transport in many systems on a qualitative level [2], they appear to be too coarse for most real-world problems. A good example of that is given by the the studies devoted to the motility of living organisms. Starting with albatrosses [16] and spider monkeys [17] and followed by a series of papers discovering anomalous behavior of animals that differ in size and habitats [8,9], this field has expanded the standard paradigm of random walks in biology [18]. However, it is the continuing analysis of the trajectories of swimming microorganisms that revealed further complexity of the problem, demanding a more detailed modeling. The small size of the swimming objects makes them susceptible to Brownian motion and altering of their swimming direction—a process of rotational diffusion that is best described in terms of Langevin dynamics [19]. Recent analysis of the migrating epithelial cells [20] employed the fractional Klein-Kramers equation to describe their spreading. Another series of very detailed work advocated the phenomenological Langevin approach, including a model with multiplicative noise, to describe the motion of amoebae [21]. Modern measurement techniques allow us to track the motion of cells with fractions-of-a-second resolution, thus giving access to previously hidden information. Again, with the help of Langevin dynamics, it has become possible to extract the characteristic features of swimming parasites from high-resolution data [22]. The strength of the Langevin description lies in its ability to include naturally fluctuations appearing on different scales and produce continuous trajectories that can be easily confronted against experimental results. Moreover, algorithms were developed to reconstruct the set of Langevin equations directly from the acquired data [23]. These recent developments motivated us to unify the models of anomalous superdiffusion on the basis of the Langevin approach, thus giving a practitioner an easy-to-use and -interpret toolbox to analyze stochastic transport phenomena that occur in complex biological and dynamical systems.

In this paper we present a very general system of Langevin equations and show how various established models of superdiffusion emerge as special cases. As a result, the proposed system elucidates how the different concepts describing superdiffusion are related and serve as a unifying approach. Our consideration is restricted to Langevin equations with uncorrelated noise sources and covers the majority of the most prominent superdiffusive models: Lévy flights [1], Lévy walks [10,11], a weakly damped kinetic model of Lévy walks leading to a generalized Klein-Kramers equation [14,15], and random walks with random velocities [24]. Note that fractional Brownian motion, as processes with correlated noise, lies beyond the scope of this paper. In our approach, we will utilize the method of subordination that previously was successfully applied for the description of the subdiffusion (MSD grows slower than linearly in time, $\alpha < 1$) in the model of continuous-time random walks (CTRWs) [25–28].

In the following, we start with a brief introduction to the continuous-time random-walks and subordination procedure. We then construct a general system of Langevin equations and show how several superdiffusive models emerge as special cases of this system. Thereby we demonstrate the relation between the generalized Klein-Kramers equation and random walks with random velocities. We conclude with a short discussion of the main results of the paper.

II. DESCRIBING THE LIMIT PROCESS OF CTRW AND SUBORDINATION

As is well known, the limit process of the one-dimensional random walk with zero-mean and finite variance is a Brownian motion B_t which is governed by the classical diffusion equation

$$\frac{\partial}{\partial t} f(x,t) = D \frac{\partial^2}{\partial x^2} f(x,t), \tag{1}$$

where $f(x,t)$ is the probability density function (PDF) of the process B_t to find a random walker in the point x at the time t and D is the diffusion constant. In the Langevin picture, Brownian motion can be described as the solution of the Itô stochastic differential equation [29] or, more commonly in the physics literature, in the form of the Langevin equation that defines the position of the particle, $x(t)$, as a function of time,

$$\frac{d}{dt} x(t) = \Gamma(t), \tag{2}$$

where the noise term $\Gamma(t)$ is defined as a sequence of uncorrelated random numbers with zero mean and finite variance. The diffusion equation (1) and the Langevin equation (2) are completely equivalent descriptions of Brownian motion, with the diffusion constant being related to the correlation function of the noise according to $\langle \Gamma(t)\Gamma(t') \rangle = 2D\delta(t-t')$.

If a process deviates from the linear time dependence of MSD characteristic for classical diffusion, one speaks of anomalous diffusion. A widely used approach to describe anomalous (sub-) diffusion is the continuous-time random walk [30]. In CTRW a random walker waits for a random time τ before making an instantaneous jump to another location. The distribution of waiting times is governed by PDF $W(\tau)$. In the case of an infinite mean waiting time, $\langle \tau \rangle = \int_0^\infty \tau W(\tau) d\tau$, the diffusion process is anomalously slow and requires a special treatment. If the displacements of the random walker are independent of the waiting times, the random walk can be separated into two disjoined processes. The first one describes the location of the walker after n displacements, $X(n)$. The second process, N_t, counts the number of displacements after time t. Here we also assume that the variance of displacements is finite. The CTRW is then described by the combined process $X(N_t)$, thus giving the first example of subordination [31].

The limit process of the CTRW can be described by a generalized diffusion equation that accounts for an arbitrary waiting-time distribution (see Ref. [32]),

$$\frac{\partial}{\partial t} f(x,t) = \tilde{D} \int_0^t \phi(t-t') \frac{\partial^2}{\partial x^2} f(x,t') dt', \tag{3}$$

where $\tilde{D}$ is a generalized diffusion coefficient and the time evolution kernel, $\phi(t)$, is determined by the waiting-time PDF $W(t)$. Namely the Laplace transform of $\phi(t)$ ($\mathcal{L}\{\phi(t)\} = \hat{\phi}(\lambda)$) is given by

$$\hat{\phi}(\lambda) = \frac{\lambda \hat{W}(\lambda)}{1-\hat{W}(\lambda)}, \tag{4}$$

where we used a hat to denote the transform and λ as a coordinate in Laplace space. A rigorous derivation of the generalized diffusion equation (3) and relation (4) can be found in Ref. [32]. Note that exponentially distributed [33] waiting times, i.e., $W(\tau) = \langle \tau \rangle^{-1} \exp(-t/\langle \tau \rangle)$ lead to $\phi(\tau) = \delta(\tau)/\langle \tau \rangle$ and, thus, recover the classical diffusion limit with the relationship $D = \tilde{D}/\langle \tau \rangle$. When the waiting-time PDF asymptotically behaves as a power-law function $W(t) \sim (t/t_0)^{-\alpha-1}$, where $\alpha > 0$ and t_0 is some time constant, the generalized diffusion equation (3) can be written in the asymptotic form of the fractional diffusion equation with fractional time derivative [2].

An intuitive formulation of the Langevin equation for the limit process of a CTRW with long waiting times was introduced by Fogedby [25]. As the CTRW includes an additional random process for the waiting times Fogedby considered a coupled set of stochastic differential equation for the position x of the random walker at time t,

$$\frac{d}{ds} x(s) = \Gamma(s), \quad \frac{d}{ds} t(s) = \eta(s), \tag{5}$$

where $\Gamma(s)$ and $\eta(s)$ are two noise sources which are assumed to be independent for the decoupled case. The process $s(t)$, being the inverse of the solution of the second equation $t(s)$, can be considered as the limit process of the number of steps N_t of the corresponding CTRW. Note that, due to causality, the $\eta(s)$ have to be strictly non-negative. The limit process of the CTRW is then obtained as $x[s(t)]$. The idea of Fogedby is identical to the mathematical concept of subordination. In this context $x(s)$ is called a parent process and s its operational time. The process $t(s)$, called subordinator, has to be nondecreasing and right continuous so its inverse, $s(t)$, called the inverse subordinator, can exist. The process $x(t) = x[s(t)]$ is referred to as subordinated to the parent process.

In practice, when applied to the problem of the subdiffusion with power-law-distributed waiting times, the method of subordination works as follows. The random numbers $\eta(s)$ are drawn from the one-sided Lévy stable distribution that leads to large "jumps" in $t(s)$ [34]. Correspondingly, the inverse function, $s(t)$ will contain plateaus that are responsible for long waiting times. If $\Gamma(t)$ is a standard uncorrelated Gaussian noise, the coordinate $x(s)$ will perform a standard Brownian motion in operational time s. Due to the traps in $s(t)$, however, the steps of standard Brownian motion will occur after long waiting times, which would lead to anomalously slow diffusion.

A solution for the probability density of the subordinated process can be readily obtained by an integral transform,

$$f(x,t) = \int_0^\infty ds p(s,t) f_0(x,s), \tag{6}$$

where $f_0(x,s)$ is the probability distribution of the parent process $x(s)$ and $p(s,t)$ is the probability density of the inverse subordinator $s(t)$. An important example is subordinated Brownian motion $B[s(t)]$, sketched above. For the widely applied class of fully skewed Lévy stable subordinators, the time evolution of the PDF of the process is governed by the time-fractional diffusion equation, which is a special case of Eq. (3) [27]. In this case, the distribution $p(s,t)$ of the inverse subordinator is well known [35,36]. The behavior of the MSD is given by $\langle x^2(t)\rangle \propto t^\alpha$, where the exponent α is related to the exponent describing the tail of the waiting-time PDF and $0 < \alpha < 1$.

While we have seen how subordination works for subbiffusive processes and how it is related to the generalized diffusion equation, our present goal are superdiffusive processes, which we consider in the next section.

III. A GENERAL LANGEVIN EQUATION FOR SUPERDIFFUSION

As we have seen, the method of subordination can be applied to describe anomalous subdiffusion processes. However, subordination is not limited to subdiffusion but can also be applied to obtain superdiffusive sample paths. In this manuscript, we show that superdiffusive processes can be described by considering subordination of a random velocity process. While subdiffusion arises from long waiting times, when a particles spends a long time at the same point, the superdiffusive processes are those with anomalously long displacements of the random walker. There are three standard pathways leading to superdiffusion. First, one allows for instantaneous long jumps of a walker—Lévy flights [11]. The second model assumes constant velocity of random walkers, v, but the moving times, τ, can be anomalously long—the Lévy walk model. Here the simple coupling $|x| = v\tau$ leads to very long flights of the particle. Finally, the third model allows the velocity of random walkers, v, itself to be a random variable with a PDF $g(v)$. A broad distribution of those velocities can lead to long displacements as well. We will call this model a random walk with random velocities. In the following, we will apply the subordination method to provide a Langevin description of those random-walk processes.

A very general case of Brownian motion in position-velocity space can be described by the following system of Langevin equations:

$$\frac{d}{dt}x(t) = v(t), \quad m\frac{d}{dt}v(t) = -\gamma v(t) - U'(x) + \Gamma_v(t), \tag{7}$$

where m is the mass of the particle and γ is a friction constant and an external force arises from a potential $U(x)$. The correlation function of the noise is $\langle \Gamma_v(t)\Gamma_v(t')\rangle = 2D_v\delta(t-t')$. The corresponding Fokker-Planck equation is the celebrated Klein-Kramers equation describing the diffusion of Brownian particles where inertial effects have to be taken into account.

Based on Eq. (7), we propose to consider the following Langevin set of equations:

$$\frac{d}{dt}x(t) = v(t), \tag{8}$$

$$m\frac{d}{ds}v(s) = -\gamma v(s) - U'(x) + \xi(s), \tag{9}$$

$$\frac{d}{ds}t(s) = \eta(s), \tag{10}$$

where $\xi(s)$ are random numbers drawn from an α-stable distribution (see Ref. [37]) and $\eta(s)$ are random numbers drawn from a probability distribution with positive support. Disregarding for a moment Eq. (8)–(10) define a process $v(t) = v[s(t)]$ where the process defined by Eq. (9) is subordinated by the solution of Eq. (10). For the special case where the noise in Eq. (9) is drawn from a Gaussian distribution and $U(x)$ is constant, $v[s(t)]$ is a subordinated Brownian motion in a harmonic potential (in *velocity space*) or a subordinated Ornstein-Uhlenbeck process [38]. In the following, we show how several superdiffusive processes emerge as special cases from the Langevin system [Eqs. (8)–(10)].

A. The fractional Klein-Kramers equation

We discuss the application of the general system [Eqs. (8)–(10)] based on a problem of anomalous diffusion of inertial weakly damped particles that was considered in Ref. [14] and which led to a new type of fractional Klein-Kramers equation. The underlying microscopic model assumes particles moving with a given velocity which are, from time to time, subjected to random velocity changes. Let $\eta(x,v,t)dxdvdt$ be the probability to make a transition to the velocity v at position x at time t. Introducing the waiting-time distribution $W(t)$ between the velocity transitions and the transition amplitude $F(v;v')$ to perform a velocity jump from v' to v, the governing equation for $\eta(x,v,t)$ is [14]

$$\eta(x,v,t)=\int_0^t dt' W(t-t')\int dv' F(v;v')\eta(x-v'(t-t'),v',t'). \tag{11}$$

In order to obtain an equation for the probability $f(x,v,t)dxdvdt$ to find a particle in the phase space volume $(x,v)dxdv$ at time t we introduce the probability $w(\tau) = 1 - \int_0^\tau W(\tau')d\tau'$ that no transition occurs during τ, leading to

$$f(x,v,t) = \int_0^t dt' w(t-t')\eta(x-v(t-t'),v,t'). \tag{12}$$

Eliminating the density $\eta(x,v,t)$, one obtains the master equation

$$\begin{aligned}\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right] & f(x,v,t)\\ &= \int_0^t dt'\phi(t-t')\int dv'[F(v;v')-\delta(v-v')]\\ &\quad\times f(x-v'(t-t'),v',t'). \end{aligned}\tag{13}$$

To relate this equation to the Langevin description, we consider the case where the velocity transitions are governed by the Gaussian kernel,

$$F(v;v') = \sqrt{\frac{\Lambda}{4\pi\tilde{Q}}}e^{-\frac{(v-v'-\tilde{\gamma}v'/\Lambda)^2}{4\tilde{Q}/\Lambda}}, \tag{14}$$

where Λ is a parameter with units $[\Lambda] = s^{-1}$, $\tilde{\gamma} = \gamma/m$, and $\tilde{Q} = Q/m^2$ with Q being proportional to the variance of the noise $\xi(s)$ (see below). For large values of this parameter, one obtains, for an arbitrary function $h(v)$,

$$\int dv' F(v;v')h(v') = h(v) + \Lambda^{-1}\left[\tilde{\gamma}\frac{\partial}{\partial v}v + \tilde{Q}\frac{\partial^2}{\partial v^2}\right]h(v) \tag{15}$$

to order Λ^{-1} [15]. If one now further assumes the time evolution kernel in Laplace space to be of the form $\hat{\phi}(\lambda) = \Lambda_\alpha \lambda^{1-\alpha}$, corresponding to waiting-time distributions of the asymptotic form $\hat{W}(\lambda) \approx 1 - \lambda^\alpha/\Lambda_\alpha$, it was shown in Ref. [15] that, from Eq. (13), a fractional Klein-Kramers equation can be derived,

$$\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right]f(x,v,t) = \mathcal{D}_t^{1-\alpha}\left[\gamma_\alpha\frac{\partial}{\partial v}v + Q_\alpha\frac{\partial^2}{\partial v^2}\right]f(x,v,t), \tag{16}$$

where $Q_\alpha = \tilde{Q}\Lambda_\alpha/\Lambda$ is a generalized diffusion constant and $\gamma_\alpha = \tilde{\gamma}\Lambda_\alpha/\Lambda$ is a generalized damping constant. Observe that $[\Lambda_\alpha] = s^{-\alpha}$. The operator $\mathcal{D}_t^{1-\alpha}$ is the so-called fractional substantial derivative introduced in Ref. [15] and given in Laplace space by

$$\mathcal{L}\{\mathcal{D}_t^{1-\alpha} f(t)\} = (\lambda + v\partial_x)^{1-\alpha}\,\hat{f}(\lambda). \tag{17}$$

Equation (16) was discussed in detail in Ref. [14]. The inclusion of external potentials into this concept is straightforward [14] and some analytical results have been obtained for harmonic potentials [41]. The fractional Klein-Kramers equation derived from a well-defined stochastic process differs from the proposed fractional generalizations of the Klein-Kramers equations introduced in Refs. [38] and [39] in the introduction of the fractional substantial derivative Eq. (17). As only the fractional substantial derivative considers the retardation of the PDF during the flight times, there is a subtle but very important difference between these equations. A detailed comparison of the different fractional generalizations of the Klein-Kramers equation and sample trajectories of their corresponding stochastic processes can be found in Ref. [40].

The stochastic process leading to the fractional Klein-Kramers equation can be considered as a subdiffusive CTRW of the velocity coordinate in a harmonic potential. Hence, the limit process for the velocity is a subordinated Brownian motion in a harmonic potential with a fully skewed Lévy-stable subordinator. This implies that the Langevin equation for the velocity is Eq. (9) with $U'(x) = 0$ and $\xi(s)$ is Gaussian noise with the correlation function $\langle\xi(s)\xi(s')\rangle = 2Q\delta(s-s')$. The noise term in Eq. (10), $\eta(s)$, has to be chosen to be a one-sided Lévy-stable noise with parameter α whose distribution is given in Laplace space by

$$\hat{L}_\alpha(z) = e^{-z^\alpha} \tag{18}$$

with

$$z = \left(\frac{s}{\mu_\alpha}\right)^{\frac{1}{\alpha}}\lambda, \tag{19}$$

where μ_α is scale parameter being determined by the ratio of physical time scale and the internal time scale. The function $L_\alpha(\cdot)$ is the density of the fully skewed Lévy distribution of order α, which can be obtained from the general α-stable distribution by setting the skewness parameter $\beta = 1$ together with suitable parametrization of the shape parameter. For the details and simulations of this distribution, we refer to Ref. [34]. The distribution of the inverse subordinator $s(t)$ for the noise distribution [Eqs. (18) and (19)] is given by

$$p(s,t) = \frac{1}{\alpha}\frac{(\mu_\alpha)^{\frac{1}{\alpha}}t}{s^{1+\frac{1}{\alpha}}}L_\alpha\left[\frac{(\mu_\alpha)^{\frac{1}{\alpha}}t}{s^{\frac{1}{\alpha}}}\right]. \tag{20}$$

Finally, the relation between the generalized drift and diffusion coefficients of Eq. (16) and the parameters of the corresponding Langevin equation can be obtained by observing that $\Lambda_\alpha = \mu_\alpha\Lambda$. One then obtains $Q_\alpha = \mu_\alpha Q/m^2$ and $\gamma_\alpha = \mu_\alpha\gamma/m$. The position of the particle is obtained from the velocity $v[s(t)]$ by direct integration,

$$x(t) = \int_0^t v[s(t')]dt', \tag{21}$$

and the time evolution of the MSD in terms of the parameters of the Langevin equation is then obtained as

$$\langle x^2(t)\rangle = \frac{2}{\Gamma(3-\alpha)}\frac{Q}{\mu_\alpha\gamma^2}t^{2-\alpha} + (1-\alpha)\frac{Q}{m\gamma}t^2, \tag{22}$$

leading to a transition from a superdiffusive behavior at short times to a ballistic motion at long times. A detailed discussion of the subordinated Ornstein-Uhlenbeck process for the velocity process is provided in Ref. [42].

If we furthermore assume that $\gamma = 0$, a generalized Obukhov model is obtained which can be solved analytically [15]. Observe that this generalized Obukhov model differs from the one discussed in Ref. [43]. For a nonsubordinated velocity process, i.e., $\eta(s) = 1$, the standard Kramers equation is recovered.

B. Lévy walks and random walks with random velocities

The model in the previous section considered a process where the new velocity after each random transition is related to the old velocity before the transition. In other words, the velocity is described by a semi-Markov process. However, in some applications the velocity after the transition can be assumed to be independent of the velocity before the transition. A stochastic process describing such a situation is the famous Lévy walk where a random walker moves ballistically for a random time and then randomly changes direction but keeps the same magnitude of velocity (speed) [10]. The existence of long flight times in this model leads to the superdiffusive behavior.

An extension of this concept is the model of random walks with random velocities (RWRV) introduced in Ref. [44] and recently discussed in detail in Ref. [24]. In this model, a particle moves for a random time with a constant velocity and then performs a transition to another random velocity. In contrast to the Lévy walk, not only is the direction random at the transition but also the absolute value of the velocity is drawn from a probability distribution. Depending on the properties of the two probability distributions for the flight time, $W(t)$, and the velocity, $g(v)$, different regimes of anomalous diffusion can be described [24].

Before we discuss the Langevin system corresponding to these processes the relationship between the fractional Kramers equation and the RWRV model shall be clarified.

Using the notation of the previous section the corresponding equation of the RWRV for the quantity $\eta(x,v,t)$ reads

$$\eta(x,v,t) = g(v)\int_0^t dt' W(t-t')\int dv'\eta(x-v'(t-t'),v',t'). \tag{23}$$

The difference between the full fractional Kramers equation and the reduced RWRV version, stemming from the independence of the velocities before and after a sudden change in the latter model, is reflected by the absence of the velocity transition amplitude in the RWRV. The equation for the phase-space density [Eq. (11)] is obviously the same in both models. Due to the independence of velocities, the PDF $\eta(x,v,t)$ factorizes, i.e., $\eta(x,v,t) = g(v)\nu(x,t)$, where $\nu(x,t)$ is the PDF of the frequency of velocity changes. Hence, for the RWRV one can directly evaluate the marginal PDF $n(x,t) = \int dv f(x,v,t)$. The original derivation of the RWRV is accomplished in terms of the densities $n(x,t)$ and $\nu(x,t)$ [24]. Observe, furthermore, that the RWRV model describes the Lévy walk if the absolute value of the velocity is conserved. An evolution equation for the probability density of a Lévy walk based on a fractional material derivative was also put forward in a number of papers [45–48].

While the approach of Ref. [48] is based on the fractional Klein-Kramers equation discussed in the previous section, the authors of Ref. [47] follow a slightly different approach. The differences between these two approaches are very subtle and are discussed in Ref. [48].

To set up the Langevin system corresponding to the RWRV model and the Lévy walk, it has to be taken into account that the velocity is not a Markov process in both models and, therefore, could not be described by Langevin equation similar to Eq. (9). Nevertheless, the proper Langevin description of these models is possible in the overdamped (Smoluchowski) limit $m/\gamma \to 0$. Without external potential one obtains from Eq. (9)

$$v_o(s) = \gamma^{-1}\xi(s). \tag{24}$$

For the case where $\xi(s)$ is a dichotomous noise source, i.e., a random sequence of the values -1 and 1, and $\eta(s)$ is again a one-sided Lévy stable distribution, the one-dimensional Lévy walk is obtained. Note that the choice of Lévy stable subordinator restricts the behavior of the Lévy walk to the ballistic regime. This is due to the fact that this choice corresponds to a waiting-time distribution with a vanishing mean of the corresponding renewal process. This can be seen as follows. The Laplace transform $\hat{C}(\lambda)$ of the velocity autocorrelation function $C(t) = \langle v(0)v(t)\rangle$ of a renewal process with a waiting-time distribution $W(t)$ behaves asymptotically as [49]

$$\hat{C}(\lambda) \sim \frac{1}{\lambda} + \frac{1}{\langle t\rangle}\frac{\hat{W}(\lambda)-1}{\lambda^2}. \tag{25}$$

This in turn implies $C(\lambda) = 1/\lambda$ for waiting-time distributions without time scale, entailing $C(t) = 1$. Now using the identity

$$\langle x^2(t)\rangle = 2\int_0^t (t-t')C(t')dt' \tag{26}$$

for the mean-squared displacement, one obtains $\langle x^2(t)\rangle \sim t^2$, i.e., ballistic diffusion. This can be intuitively understood by the fact that the renewal processes with scale-free waiting-time distributions are dominated by the longest waiting time which approximately as long as the whole process. During this time (in principle, the whole process), the particle just walks into one direction and, thus, moves ballistically.

The Lévy walk exhibits a sub-ballistic, superdiffusive behavior for the case when the waiting-time distribution $W(t)$ has a finite mean value but a diverging variance, such as $W(t) \sim t^{-1-\alpha}$ for $1 < \alpha < 2$; see Ref. [49]. The standard fully skewed Lévy distribution is a well defined subordinator only for the values of α in the region $0 < \alpha < 1$, therefore, for the superdiffusive but sub-ballistic regime, a different model is needed. A suited subordinator was considered in Ref. [36]. In their model, convergence of the waiting-time process is enabled by centering the process to the mean waiting time which is impossible for α lying in the range $0 < \alpha < 1$. In this case, however, the representation of the subordinator is more involved and, for details, we refer the reader to the original article.

For the general case when $\xi(s)$ are random numbers drawn from a stable distribution $h(\xi)$ whose characteristic function is given by

$$\hat{h}(k) = \int d\xi e^{-ik\xi}h(\xi) = \exp(-D_\delta|k|^{2\delta}), \tag{27}$$

the distribution of the velocity process $h(v_o)$ given by Eq.(24) has an asymptotic power law according to $h(v_o) \sim \frac{D_\delta}{\gamma^{2\delta}}|v_o|^{-1-2\delta}$. The process then is a subordinated stable motion that describes the limit process of the velocity in the RWRV model. The corresponding solution for a sample trajectory then reads

$$x(t) = \int_0^t v_o[s(t')]dt', \tag{28}$$

where $v_o[s(t)]$ is the subordinated version of the velocity process in the overdamped limit [Eq. (24)], i.e., in this case proportional to a subordinated stable motion. The phase diagram of possible transport regimes in Ref. [24] can be readily identified with the stability parameters δ of the stable motion and the subordinator α, respectively. To obtain the regime of flight time distributions with a finite mean value, one has to resort again to the subordinators discussed in Ref. [36]. If the distribution of ξ is a symmetric stable distribution of order one, i.e., a Cauchy distribution, the velocity process is a Cauchy flight. It was shown in Ref. [24] that, in this case, the density of the particles in real space is also Cauchy distributed for any flight-time distribution. It follows that, in the case of Cauchy distributed ξ, the PDF of the process Eq. (28) is Cauchy, irrespective of the specific of the subordinator. In an interesting recent paper, a Langevin description of Lévy walks and their extension based on coupled CTRWs has been considered [50].

C. Lévy flight

The treatment of different superdiffusive Lévy processes shall be concluded with a short discussion of the Lévy flight. This process is a random walk with instantaneous displacements governed by a Lévy stable distribution (or more generally a distribution with power-law tail asymptotics). Accordingly, the limit process is a Lévy stable motion. If we consider the RWRV model discussed in the previous section with potential and without subordination, we get the Langevin equation describing Lévy flights of particles,

$$\dot{x}(t) = -\frac{U'(x)}{\gamma} + \frac{1}{\gamma}\xi(t), \tag{29}$$

which has been discussed in detail in Ref. [51]. Observe that this model exhibits discontinuous trajectories if the probability distribution of ξ lacks a finite scale. A model which generates continuous trajectories of Lévy flight processes has been discussed recently in Ref. [52]. This model works without subordination and shows how the introduction of a specific form of multiplicative noise in Eq. (9) leads to continuous trajectories with the characteristic properties of Lévy flights (interestingly without Lévy noise).

IV. DISCUSSION AND CONCLUSIONS

Starting with a general coupled set of Langevin equations, we have shown how different stochastic models such as the Lévy walk, the random walk with random velocities, and a generalized Klein-Kramers model emerge as special cases of the considered system. All of these models have in common that they exhibit long persistent periods without velocity changes. The underlying idea of unifying these models is based on the subordination of the random velocity process. The considered system of Langevin equations not only unifies the different models but also elucidates the relationship between these models having continuous sample trajectories. Neglecting the subordination process, we were also able to include the Lévy flight into the same framework and, therefore, clarify the connection of the discussed models to this important class of processes. As more and more examples exhibiting complex anomalous dynamics receive attention in various fields of research across disciplines ranging from biology [9] to many-particle systems [53], the universal and flexible tools of the Langevin description can facilitate the understanding of those systems while helping to identify their common and general features. The presented work unifies different classes of the superdiffusive processes on the basis of their Langevin description, which allows us to pinpoint the best candidate model for quantitative description of experimental data generated by real-world systems.

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