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Exact Functional Renormalization Group for Wetting Transitions in $1+1$ Dimensions.

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Abstract. – Using an *exact* functional renormalization group (RG), we study wetting transitions in $d = 1 + 1$ dimensions. This RG acts in the function space of the interaction potentials for two interfaces. The RG flow exhibits i) a whole line of fixed points and ii), a separatrix which represents an analytic continuation of the fixed-point line. Numerical and analytical work shows that the line of fixed points almost forms a *closed loop* in function space. The separatrix, on the other hand, describes a subregime where the wetting transitions are first-order but have unusual scaling properties. A generalization of the RG indicates that these unusual transitions are replaced by usual first-order transitions in $d \neq 1 + 1$.

Functional renormalization group (RG) methods were originally developed in the context of bulk critical phenomena, *i.e.* critical phenomena arising at phase transitions in bulk systems. The first such method was nonperturbative and led to an approximate functional RG [1] which served, however, as a starting point for perturbative but systematic RG methods [2]. For bulk critical phenomena, exact functional RG transformations have also been derived [3], but they are so complicated that it proved difficult to use them.

More recently, functional RG methods have been applied to unbinding phenomena involving low-dimensional objects or manifolds such as interfaces, membranes and polymers. These manifolds undergo thermally excited shape fluctuations which renormalize their mutual interactions. Quite generally, this renormalization increases the repulsive part of the interactions, and can lead, for sufficiently strong fluctuations, to a phase transition from bound to unbound states of the manifolds. For interfaces, membranes and polymers, these unbinding transitions are wetting, adhesion and adsorption transitions, respectively [4].

So far, two approximate RGs have been studied, namely a linear RG for wetting in $d = 2 + 1$ dimensions [5-7] and a nonlinear RG for wetting and adhesion in general d [8, 9]. This approximate RG acts in the space of interaction potentials for the interfaces bounding

(1) It is amusing to note that our RG scheme also gives a classification for the ground-state properties of *all* 1-dimensional Schrödinger equations with a hard wall at $z = 0$.

the wetting layer, and in $d = 1 + 1$ exhibits *two* different subregimes for the critical behaviour. On the other hand, the exact solution for *one particular* form of the interaction potential in $d = 1 + 1$ shows that there are, in fact, *three* different subregimes with very different scaling properties [10] ([1]). Therefore the previous RG approaches do not give a complete description of the *global* structure within the function space of interaction potentials.

In this paper, we use a decimation-type RG which is *exact* for wetting transitions in $1 + 1$ dimensions [11]. As shown below, this exact RG has the following properties: i) it acts in the function space of potentials, $U(z, z')$, where z and z' represent the separation of two interacting interfaces; ii) it has a line of fixed points, $U^*(z, z')$. These fixed points exhibit an unexpected symmetry since they depend only on the product $z \cdot z' : U^*(z, z') = \bar{U}(z \cdot z')$; iii) the line of fixed points exhibits two branches, and the RG flow has a parabolic character describing two subregimes (A) and (B). iv) In addition, the RG flow leads to a separatrix which represents an analytic continuation of the fixed-point line. This separatrix belongs to a third subregime (C) where the wetting transitions are first-order but have unusual scaling properties. As one approaches the boundary between the line of fixed points and this separatrix, the fixed points on the two branches are found to be identical apart from a singular piece at $z \cdot z' = 0$. In this way, the line of fixed points contains a *closed loop* in function space.

The decimation-type RG can be generalized to wetting in $d \neq 1 + 1$ via a Migdal-Kadanoff bond moving scheme. This generalized scheme (which is no longer exact) indicates that the presence of subregime C might be a special feature of wetting in $d = 1 + 1$.

We start with the Gaussian interface Hamiltonian

$$H\{l_i\} = \sum_{i=1}^{N} \frac{K}{2} (l_i - l_{i+1})^2 + V(l_i), \tag{1}$$

where K is the interfacial stiffness. The interface feels an interaction potential, $V(l)$, which contains a hard wall with $V(l) = \infty$ for $l < 0$, and decays to zero for large l. The continuous variable l represents the separation of the interface from the wall. If, in a coarse-graining procedure, one goes to larger and larger length scales, this interaction potential $V(l)$ is renormalized by the thermal shape fluctuations. This renormalization can be studied in a simple way by use of the transfer matrix T. With the dimensionless variables $z \equiv \sqrt{K/T}\, l$ and $U(z) \equiv 1/T\, V(\sqrt{T/K}\, z)$, the transfer matrix has the form

$$T(z, z') = \exp\left[-\frac{1}{2}(z - z')^2 - \frac{1}{2} U(z) - \frac{1}{2} U(z')\right] \tag{2}$$

with the partition function Z given by

$$Z = \mathrm{Tr}\{T^N\}. \tag{3}$$

Integration over every second degree of freedom, which corresponds to a rescaling factor $b = 2$, implies that T is replaced by $T \cdot T$. In order to obtain a fixed point appropriate for wetting transitions in $d = 1 + 1$, the variable z is rescaled according to $z' = b^{\zeta} z$ with

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roughness exponent $\zeta = 1/2$. The RG transformation defined in this way has the form

$$T'(z, z'') = \int_{-\infty}^{+\infty} dz' \, T\left(\sqrt{2}\, z, z'\right) T\left(z', \sqrt{2}\, z''\right). \tag{4}$$

Thus, this transformation which has the trivial fixed point $T^* = \exp[-(1/2)(z - z')^2]$ acts in the enlarged function space of all potentials $U(z, z')$, with initial value $U(z, z') = (1/2)[U(z) + U(z')]$. This potential is renormalized according to

$$U^{(N+1)}(z, z'') = -\ln \int_0^{\infty} \frac{dz'}{\sqrt{\pi}} \exp\left[-\left(z' - (z + z'')/\sqrt{2}\right)^2 - G(z'|z, z'')\right], \tag{5a}$$

$$G(z'|z, z'') = U^{(N)}\left(z', \sqrt{2}z\right) + U^{(N)}\left(z', \sqrt{2}z''\right), \tag{5b}$$

where the hard wall condition leads to an integration cut-off at zero.

First we will determine all nontrivial fixed points of (5) that decay to zero for large z. Two such fixed points, namely $U_{\pm}^*(z, z') = -\ln(1 \pm \exp[-2zz'])$, have been previously found by Huse [11]. They decay exponentially and thus describe wetting with short-ranged potentials. In addition, we find a whole line of fixed points. Their asymptotic behaviour can be determined analytically from (5) and is given by

$$U^*(z, z') \approx \rho/(z \cdot z'), \qquad \text{for large } z, z', \tag{6}$$

$$U^*(z, z') \approx -(\sigma/2) \ln(z \cdot z'), \qquad \text{for small } z, z'. \tag{7}$$

Numerically we find that the lines with $U^*(z, z') = \text{const}$ in the (z, z')-plane represent hyperbolas. This implies that the fixed points U^* depend only on the product $z \cdot z'$, *i.e.* $U^*(z, z') = \bar{U}(z \cdot z')$, which is an unexpected symmetry. On the other hand, if one assumes this symmetry, the fixed-point equation (5) simplifies and becomes

$$\bar{U}(z^2) = -\ln \int_0^{\infty} \frac{dz'}{\sqrt{\pi}} \exp\left[-\left(z' - \sqrt{2}z\right)^2 - 2\bar{U}\left(\sqrt{2}\, z' z\right)\right]. \tag{8}$$

One can then check numerically that the solutions of (8) also satisfy (5) with $U(z, z') = \bar{U}(z \cdot z')$. In this way, we obtained very precise numerical solutions of the fixed-point equation (5).

The parameters ρ and σ introduced in (6) and (7) parametrize the long-ranged and the short-ranged part of U^*, respectively. Every fixed point corresponds to a unique value of σ, while ρ is a unique function of σ given by

$$\rho = \rho^* \sigma (2 - \sigma) \tag{9}$$

with $\rho^* = -1/8$. This function, together with the values of ρ and σ for the numerically determined fixed points, is displayed in fig. 1.

Two special cases are the short-ranged fixed points $U_{\pm}^*$ with $(\rho, \sigma) = (0, 0)$ and $(0, 2)$. This parabolic structure is similar to the line of fixed points obtained by the approximate nonlinear RG [9]. For infinitesimal rescaling factor, the latter fixed points exhibit a power law singularity $\sim 1/z^2$, while the exact RG considered here leads to a logarithmic divergence for small z. This logarithmic singularity seems to be essential in order to get a complete

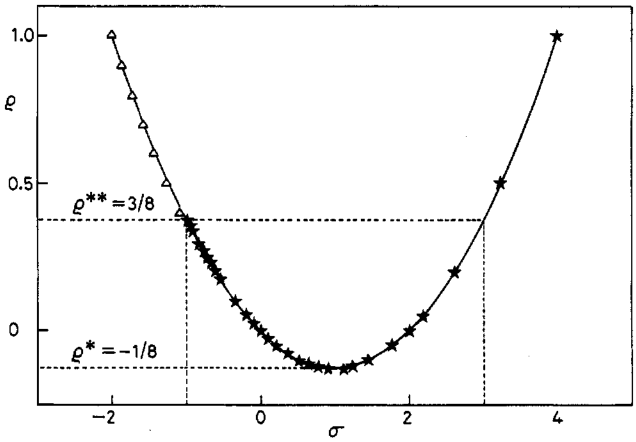

Fig. 1. – Numerically determined fixpoints (★) and the separatrix in subregime C (△). The line represents the function $\rho = \rho^* \sigma(2-\sigma)$ as in (9).

description of the critical behaviour which depends on the parameter ρ in a complicated way.

The upper branch of the parabola with $\sigma > 1$ consists of a line of stable fixed points $U_0^*(z, z')$ which describe unbound states of the interfaces. This branch ends at the point $(\rho, \sigma) = (\rho^*, 1)$, which corresponds to wetting transitions of infinite order (subregime A). A lower branch with $-1 < \sigma < 1$ consists of unstable fixed points $U_c^*(z, z')$ corresponding to second-order transitions with a parameter-dependent critical exponent $\nu_\parallel$ (subregime B). In this case, the parallel correlation length $\xi_\parallel$ diverges as $\xi_\parallel \sim (T_u - T)^{-\nu_\parallel}$. The critical exponent $\nu_\parallel$ is determined by $\nu_\parallel = 1/y$, with y being the relevant scaling index at the fixed point. Numerically, we find that the relation

$$y = (1-\sigma)/2 \quad \text{for } -1 < \sigma < 1 \tag{10}$$

is satisfied within an accuracy of about one percent. A combination of (9) and (10) now leads to the critical exponent $\nu_\parallel = 2/(1-\sigma) = 2/[1+8\rho]^{1/2}$ for $-1 \leqslant \sigma \leqslant 1$ which is the exact result as obtained by transfer matrix methods for a special form of the interaction potential [10].

At $(\rho, \sigma) = (\rho^{**}, -1)$ with $\rho^{**} = -3\rho^*$ the line of fixed points ends. Let us focus on this endpoint $(\rho^{**}, -1)$ and consider the weight function $P_\sigma^*(z, z') = \exp[-U_\sigma^*(z, z')]$ at a fixed point U_σ^* with $\sigma > -1$. Remember that $P_\sigma^*(z, z') = \bar{P}_\sigma(z \cdot z')$. If one now inserts the form $\bar{P}_\sigma(z^2) = z^\sigma f(z^2)$ with $\sigma = -1 + \varepsilon$ into the integral equation (8), one obtains the expansion

$$\bar{P}_{-1+\varepsilon}(z^2) = z^{-1+\varepsilon}\left(\varepsilon\sqrt{2\pi}(1-z^2) + \sqrt{\pi/2}(1+K_1\varepsilon)z^4 - \sqrt{\pi/2}(1+K_2\varepsilon)z^6\right) \tag{11}$$

for small z and small ε with two constants K_1 and K_2. The nonsingular terms can be shown to sum up to the stable fixed point $\bar{P}_3(z^2)$ which implies

$$\bar{P}_{-1+\varepsilon}(z^2) \approx \sqrt{2\pi} \cdot \varepsilon z^{-1+\varepsilon} + \bar{P}_3(z^2). \tag{12}$$

Thus, $\bar{P}_{-1}$ and $\bar{P}_3$ are identical apart from the distributionlike singularity at the wall. This identity of $\bar{P}_{-1}$ and $\bar{P}_3$ for $z \neq 0$ truncates the line of fixed points, since the two branches meet for $z \neq 0$ in function space. In this sense, the line of fixed points forms a closed loop. The weight function $\bar{P}_{-1+\varepsilon}(z^2)$ has a unique minimum at $z = z_0$ with $z_0 \sim \varepsilon^{1/4}$ for small ε. This

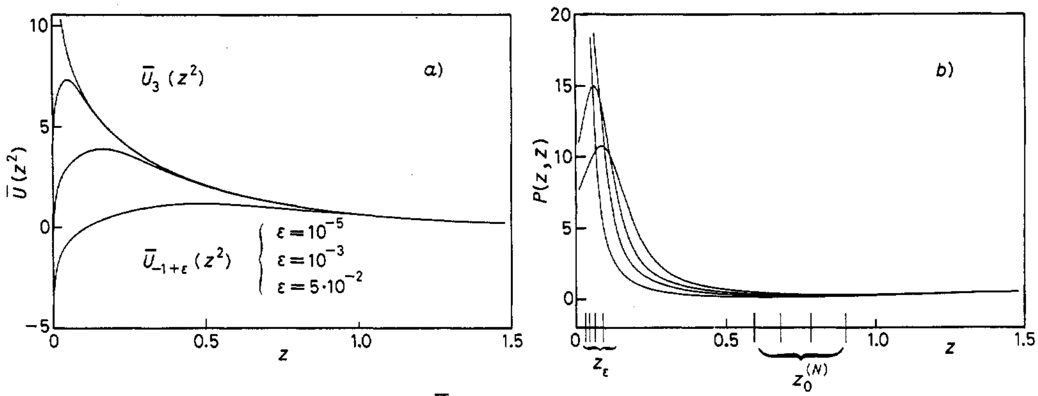

Fig. 2. – *a*) Three critical fixed points $\bar{U}_\sigma$ with $\sigma = -1 + \varepsilon$ and small ε, and the fixed point $\bar{U}_3$ corresponding to an unbound state. *b*) Four iterations, $P^{(N)}(z, z)$, of the weight function in subregime C with $N = 5$, 6, 7 and 8. The cut-off z_ε and the position of the minimum z_0 are labelled.

approach to $\sigma = -1$ is displayed in fig. 2*a*) where three different fixed points for small ε are plotted together with $\bar{U}_3(z^2)$.

For $\rho > \rho^{**}$ the fixed-point line is followed by subregime C. The RG flow still reveals the transitions in terms of a separatrix, but this separatrix no longer maps onto a fixed point. In fact, the RG flow tries to build up a weight function which behaves as $\sim z^\sigma$ with $\sigma < -1$ at the wall. As displayed in fig. 1, $\sigma = \sigma(\rho)$ still satisfies the parabolic relation (9). However, in contrast to subregime B, the flow cannot reach any fixed point because of the non-integrability of z^σ for $\sigma < -1$.

In order to understand how the RG flow describes the anomalous transitions in subregime C, it is useful to study how finite-size effects are embodied in the RG transformation: the N-th RG step can be interpreted as doubling the system size, namely going from T^n to T^{2n} with $n = 2^N$. In a finite system, there exists no bound state of the interface because it «tunnels» away from the wall with a finite probability [12]. As a consequence, the weight function $P(z, z)$ reaches a constant value for large z.

Some iterations, $P^{(N)}(z, z)$, of the RG in subregime C are shown in fig. 2*b*). These functions have a minimum at $z = z_0^{(N)}$. This minimum divides the weight function in a wall region that contains information about the probability distribution in the thermodynamic limit, and into a tail region. The tail approaches a stable fixed point $\exp[-U_0^*]$ and describes a state which is tunnelled off from the wall. The location $z_0^{(N)}$ of the minimum defines a length scale $L_\perp \sim z_0^{(N)} b^{N\zeta}$ with $\zeta = 1/2$ for the finite system. It is pushed to the wall by the flow as $z_0^{(N)} \sim b^{-\delta N}$ for large N. Since the size of the system grows like $L_\parallel \sim b^N$, the exponent $\zeta = 1/2 - \delta$ plays the role of the roughness exponent ζ at the transition in subregime C.

Numerical iterations of the RG indicate that the weight function P_σ with $\sigma < -1$ has the asymptotic behaviour $P_\sigma^{(N)}(z, z) \approx a_N z^\sigma + P_{2-\sigma}^*(z, z)$, for z bigger than a cut-off $z_\varepsilon \sim b^{-N/2}$. Since $P_{2-\sigma}^*(z, z) \approx z^{2-\sigma}$ for small z as follows from (7), $P_\sigma^{(N)}(z, z)$ has a minimum at $z_0^{(N)} \sim a_N^\varkappa$ with $\varkappa = 1/(2 - 2\sigma)$. In order to iterate the RG transformation, the integral in (5) must be convergent which requires that $a_N \sim b^{N(\sigma+1)/2}$. These two conditions suffice to evaluate δ as $\delta = (\sigma + 1)/4(\sigma - 1)$ which is consistent with δ-values obtained numerically.

We showed how our exact RG describes the whole structure of wetting transitions in $d = 1 + 1$ dimensions. It is possible to generalize this method to arbitrary dimension d by a Migdal-Kadanoff RG scheme. In this method, bond shifting leads to an additional factor 2^{d-2} in the transfer matrix. To recover the trivial fixed point, we have to rescale with roughness exponent $\zeta = (3-d)/2$. This leads to the generalized recursion relation

$$U^{(N+1)}(z, z'') = -\ln\left\{2^{1/2-\zeta}\int_0^\infty \frac{dz'}{\sqrt{\pi}}\exp\left[-\left(2^{1/2-\zeta}z' - \frac{z+z''}{\sqrt{2}}\right)^2 - G(z'|z, z'')\right]\right\}$$

with

$$G(z'|z, z'') = 2^{1-2\zeta}(U^{(N)}(z', 2^\zeta z) + U^{(N)}(z', 2^\zeta z'')) . \tag{13}$$

For $\zeta = 1/2$, one recovers the exact RG used above.

The long- and the short-range behaviour of the fixed points for $z = z'$ are now given by $U^*(z, z) \approx \rho/z^\tau$ with $\tau = 2(d-1)/(3-d)$ for large z, and by $U^*(z, z) \approx pz^\beta$ with $\beta = 2(2-d)/(3-d)$ for small z. This indicates that $d = 2$ (*i.e.* $\beta = 0$) is a singular case, since the fixed points there have a logarithmic singularity at the wall. The weight function P then has a power law behaviour leading to the existence of the special subregime C. For $d \neq 2$, P has an exponential behaviour which could not lead to such a subregime. Therefore, this RG scheme indicates that subregime C is replaced by usual first-order transitions for $d \neq 2$. In fact, recent studies using approximate functional RGs indicate that additional subregimes appear for large τ via an infinite sequence of bifurcations which describe first-order transitions and higher-order multicritical points [13, 14].

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