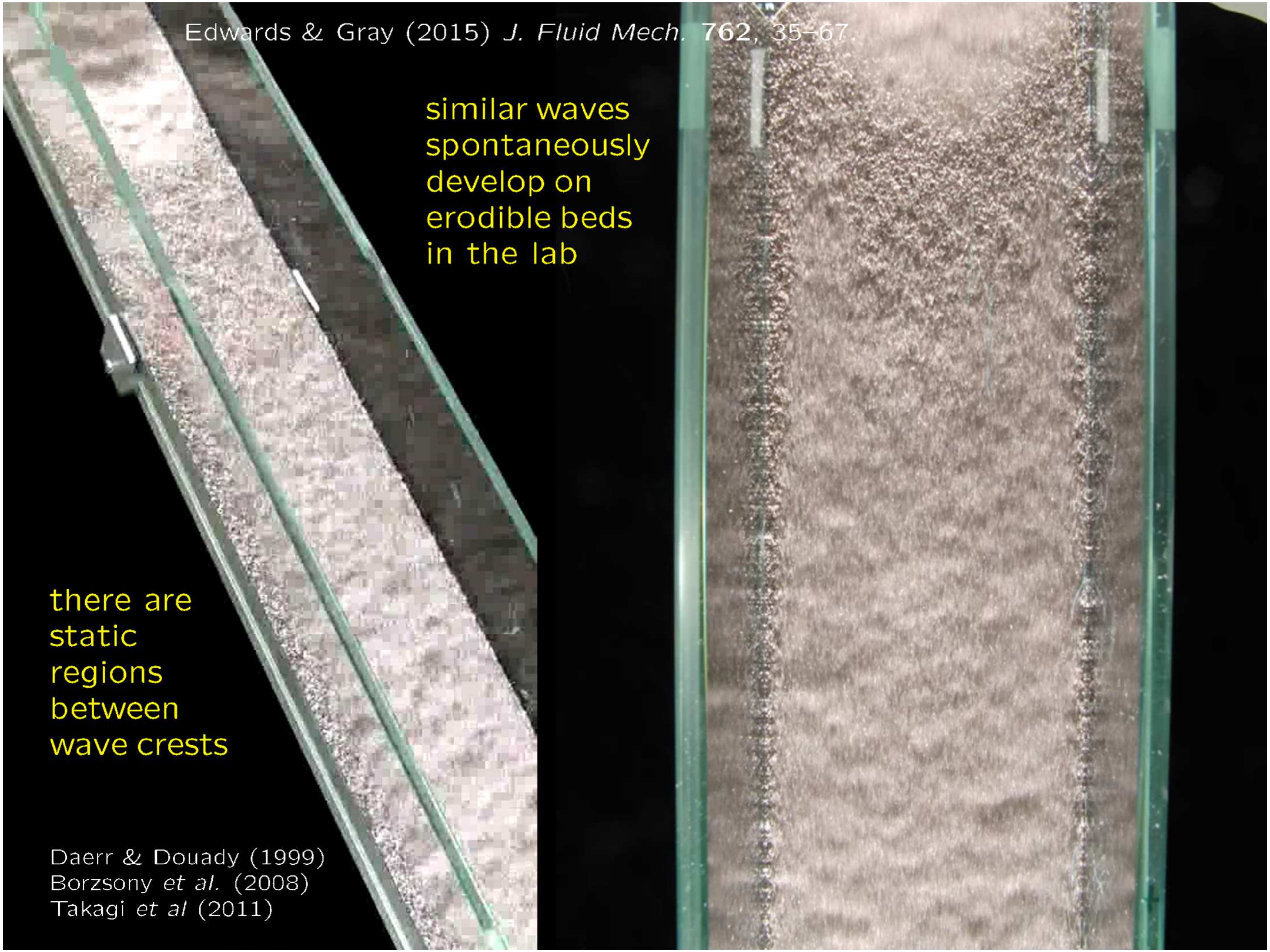
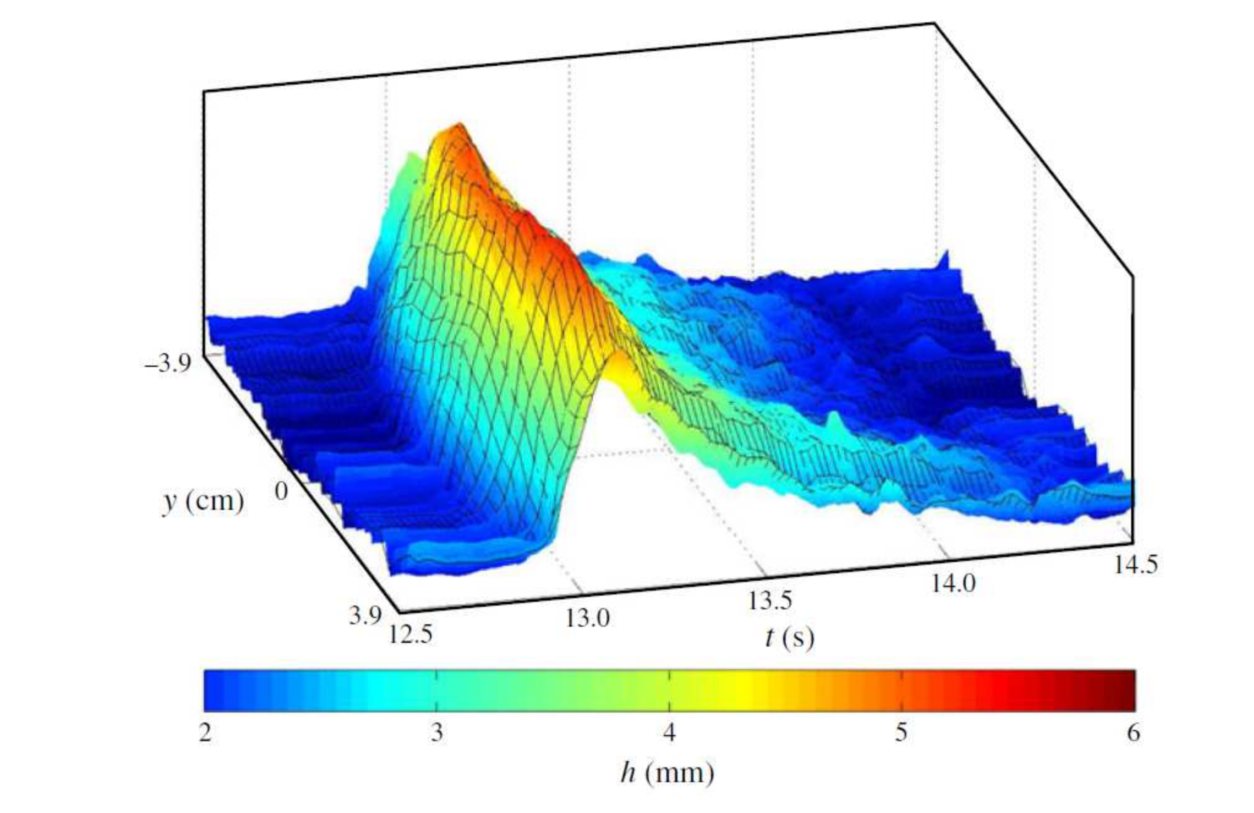
MANCHESTER 1824

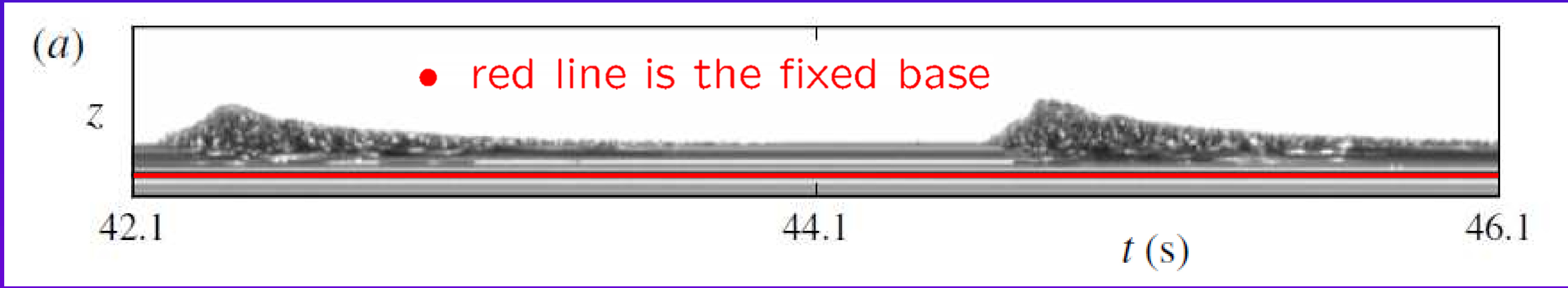
Erosion-deposition waves in shallow granular free-surface flows Nico Gray

- 15^{th} Oct 2000 an unintentional release of 150 000 m^3 water led to a debris flow in Fully Switzerland that had regular surges


Edwards & Gray (2015) *J. Fluid Mech.* **762**, 35–67.
similar waves spontaneously develop on erodible beds in the lab
there are static regions between wave crests
Daerr & Douady (1999)
Borzsony *et al.* (2008)
Takagi *et al* (2011)

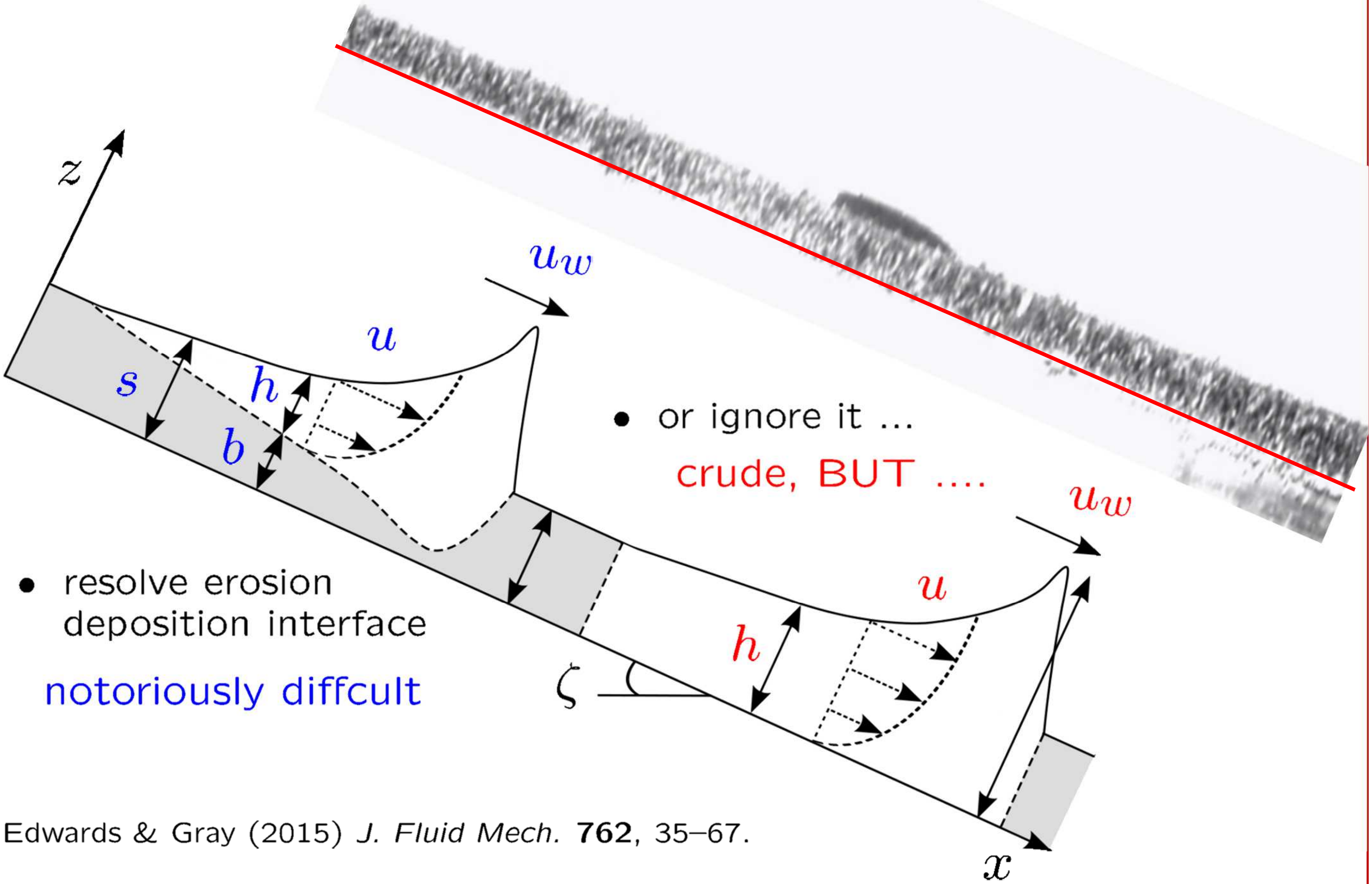


- waves are typically 5mm peak height and have 2mm stationary layer



- side-on "photo-finish" shows basal erosion and deposition

Granular solid-fluid phase transition in depth-averaged framework



- resolve erosion deposition interface

notoriously diffcult

- or ignore it ...

crude, BUT ....

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Use shallow water avalanche model …

- Uses shallow water avalanche model (e.g. Grigorian *et al.* 1967, Gray *et al.* 1999, 2003) for the thickness h and the depth-averaged velocity $\bar{u}$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) = 0$$

$$\frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}(\chi h\bar{u}^2) + \frac{\partial}{\partial x}\left(\frac{1}{2}h^2 g \cos\zeta\right) = hgS$$

- where $\chi = \overline{u^2}/\bar{u}^2$ is the shape factor, g is the constant of gravitational acceleration and the source term

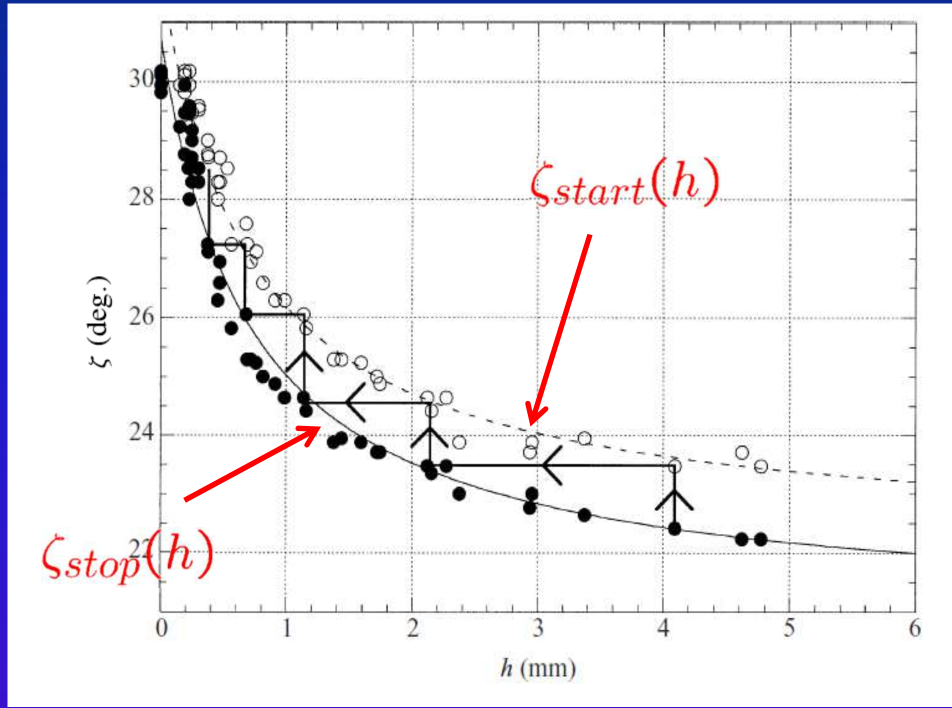
$$S = \sin\zeta - \mu\frac{\bar{u}}{|\bar{u}|}\cos\zeta$$

- consists of gravitational acceleration and basal fiction μ

Grigorian, Eglit & Iakimov (1967), *Tr. Vysokokogornogo Geofizich Inst.* **12**, 104-113.
Gray, Wieland & Hutter (1999) *Proc. Roy. Soc. A* **455**, 1841-1874
Gray, Tai & Noelle (2003) *J. Fluid Mech.* **491**, 161-181

Pouliquen & Forterre (2002)

- Measured basal friction by determining the thickness as which the grains
  - came to rest
  - when they started moving again from a static state



- gave effective basal friction law

$$\mu(h,\mathrm{Fr}) = \begin{cases} \mu_1 + \dfrac{\mu_2-\mu_1}{1+h\beta/(L\mathrm{Fr})}, & \mathrm{Fr} \geq \beta, \quad \text{dynamic} \\ \left(\dfrac{\mathrm{Fr}}{\beta}\right)^{\kappa}(\mu_1-\mu_3)+\mu_3+\dfrac{\mu_2-\mu_1}{1+h/L}, & 0<\mathrm{Fr}<\beta, \quad \text{intermediate} \\ \mu_3 + \dfrac{\mu_2-\mu_1}{1+h/L}, & \mathrm{Fr}=0, \quad \text{static} \end{cases}$$

- where Fr is the Froude number, $\kappa = 10^{-3}$ and $\mu_1 = \tan\zeta_1$, $\mu_2 = \tan\zeta_2$ and $\mu_3 = \tan\zeta_3$ are the tangents of the angles, ζ_1, ζ_2 and ζ_3.

Pouliquen & Forterre (2003) *J. Fluid Mech.* **453**, 133-151.

Travelling-wave solutions in the absence of viscosity

- In a frame travelling at speed u_w with coordinates

$$\xi = x - u_w t, \qquad \tau = t.$$

- Assuming $\partial/\partial\tau = 0$ and $\chi = 1$ the system is reduces to

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(h(\bar{u} - u_w)\right) = 0,$$

$$h(\bar{u} - u_w)\frac{\mathrm{d}\bar{u}}{\mathrm{d}\xi} + hg\cos\zeta\frac{\mathrm{d}h}{\mathrm{d}\xi} = hg\cos\zeta(\tan\zeta - \mu)$$

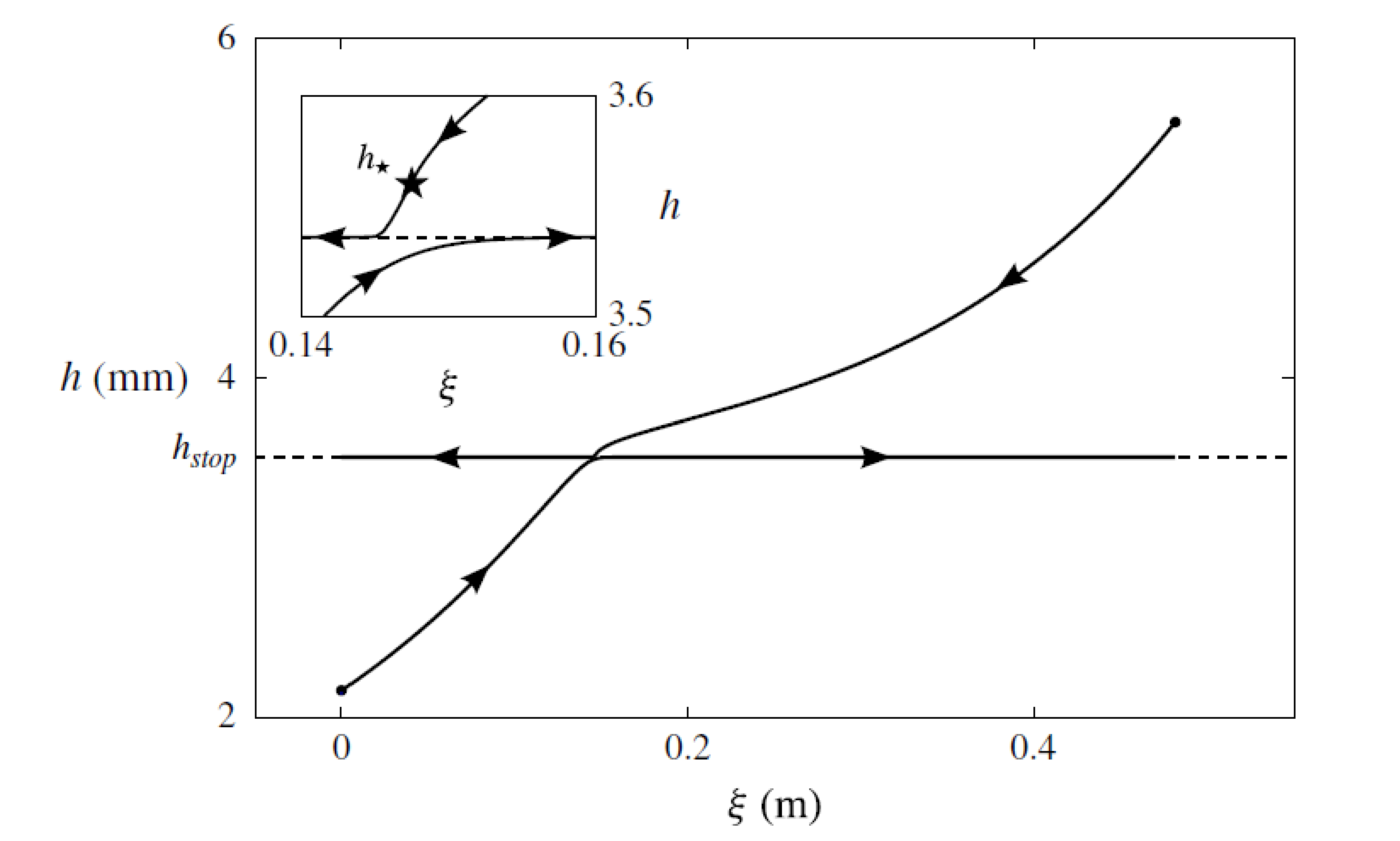
- Since $\bar{u} = 0$ in a stationary layer of thickness $h = h_+$

$$h(\bar{u} - u_w) = -h_+ u_w \qquad \Rightarrow \qquad \bar{u} = u_w\left(1 - \frac{h_+}{h}\right).$$

- The flow thickness for which $\mathrm{Fr} = \beta$ is now defined as $h = h_\star$

$$\Rightarrow \qquad u_w = \frac{\beta h_\star^{3/2}\sqrt{g\cos\zeta}}{h_\star - h_+}.$$

Edwards & Gray (2015) *J. Fluid Mech.* **762**, 35-67.



- Integration of the first order ODE indicates a problem
- solution asymptotes to a critical thickness $h_* \gg h_{crit} > h_{stop}$
- To get through this point, ones needs a little bit of viscosity

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The μ(I)-rheology for liquid-like granular flows

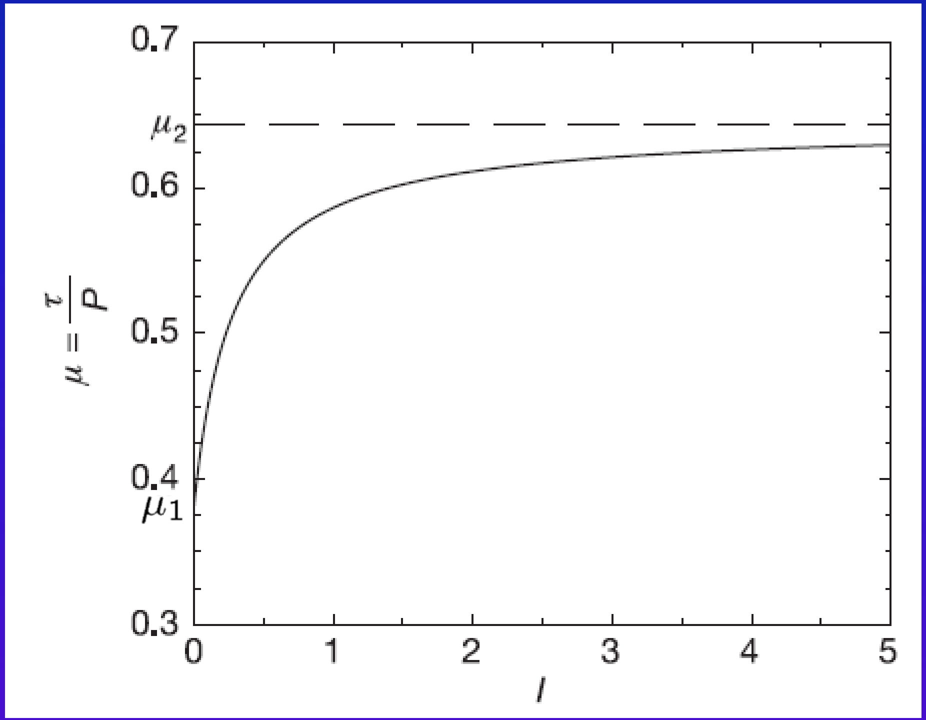
- GDR MIDI (2004) and Jop *et al.* (2006): proposed constitutive law

$$\boldsymbol{\tau} = \mu(I) p \frac{\boldsymbol{D}}{||\boldsymbol{D}||}$$

- where 2nd invariant

$$||\boldsymbol{D}|| = \sqrt{\frac{1}{2}\mathrm{tr}\boldsymbol{D}^2}$$

- If $\mu =$ const this reduces to Mohr-Coulomb law



- BUT, friction μ is a function of the inertial number I

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{I_0/I + 1}, \qquad I = \frac{2||\boldsymbol{D}||d}{\sqrt{p/\rho^*}},$$

- where d is the particle diameter and ρ^* is the intrinsic density.

The Bagnold solution

- For steady-uniform flow $\boldsymbol{u} = (u(z), 0, 0)$ the normal and downslope momentum balances imply that

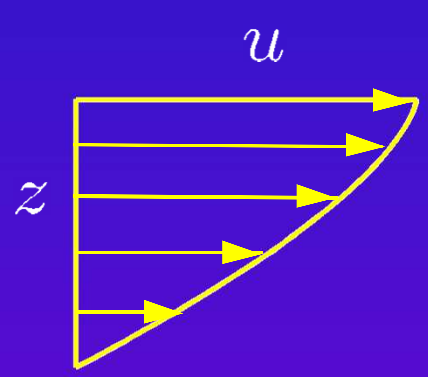
$$p = \rho g(h-z)\cos\zeta, \qquad \tau_{xz} = \rho g(h-z)\sin\zeta$$

- Rheology then implies $\mu(I) = \tan\zeta$ and hence I is equal to a constant

$$I_\zeta = I_0\left(\frac{\tan\zeta - \tan\zeta_1}{\tan\zeta_2 - \tan\zeta}\right)$$

- Solve I equation for the downslope velocity

$$u = \frac{2I_\zeta}{3d}\sqrt{\Phi g\cos\zeta}\left(h^{3/2} - (h-z)^{3/2}\right).$$



- The depth-averaged Bagnold velocity satisfies

$$\bar{u} = \frac{2I_\zeta}{5d}\sqrt{\Phi g\cos\zeta}\, h^{3/2}$$

The depth-averaged μ(I)-rheology for granular flows

- To first order the inviscid avalanche equations emerge naturally with the dynamic basal friction law

$$\mu_b(h, \mathsf{Fr}) = \mu_1 + \frac{\mu_2 - \mu_1}{\beta h/(L\mathsf{Fr}) + 1}, \qquad \mathsf{Fr} > \beta,$$

- This is just Pouliquen & Forterre's (2002) law, where

$$\mathsf{Fr} = \frac{|\bar{\boldsymbol{u}}|}{\sqrt{gh\cos\zeta}}$$

- Now add in the in-plane deviatoric stress

$$\tau_{xx} = \mu(I)p\frac{D_{xx}}{||\boldsymbol{D}||}$$

- Assume shallow and use Bagnold solution to evaluate

$$D_{xx} = \frac{\partial u}{\partial x}, \qquad ||\boldsymbol{D}|| = \frac{1}{2}\left|\frac{\partial u}{\partial z}\right|$$

- It follows that the in-plane deviatoric stress is

$$\tau_{xx} = 2\rho g \sin\zeta \left(h^{1/2}(h-z)^{1/2} - (h-z) \right) \frac{\partial h}{\partial x}.$$

- formal depth-integration gives

$$h\bar{\tau}_{xx} = \frac{1}{3}\rho g \sin\zeta \, h^2 \frac{\partial h}{\partial x}.$$

- Use depth-averaged Bagnold velocity to reformulate

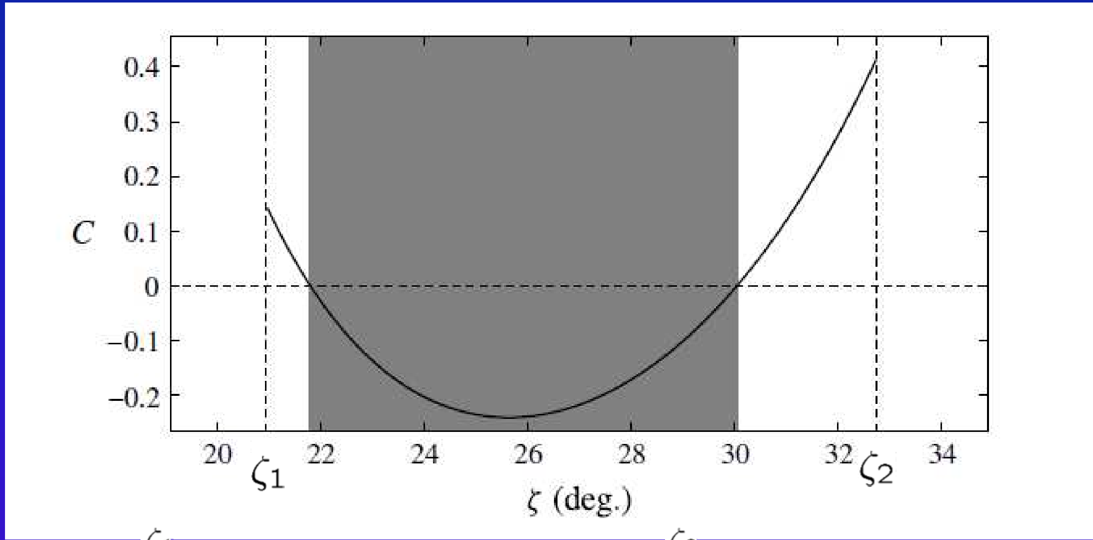
$$h\bar{\tau}_{xx} = \rho \nu h^{3/2} \frac{\partial \bar{u}}{\partial x}$$

- where the angle dependent coefficient ν is determined

$$\nu = \frac{2L\sqrt{g}}{9\beta} \frac{\sin\zeta}{\sqrt{\cos\zeta}} \left(\frac{\tan\zeta_2 - \tan\zeta}{\tan\zeta - \tan\zeta_1} \right).$$

Gray & Edwards (2014) *J. Fluid Mech.* **755**, 503-534.

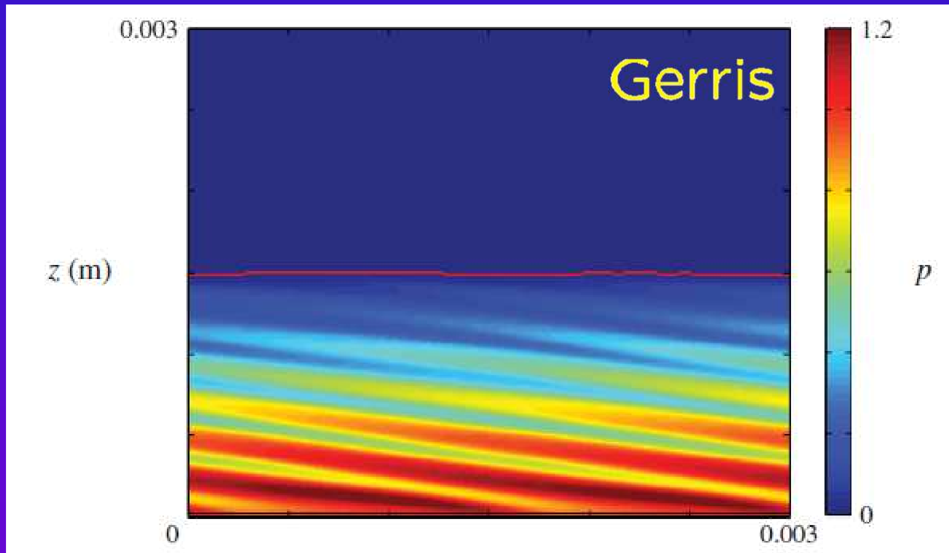
Well-posed and ill-posed behaviour of the μ(I)-rheology



- Coefficient ν is negative outside range of steady-uniform flow
- Depth-averaged $\mu(I)$-rheology is ill-posed in this region.

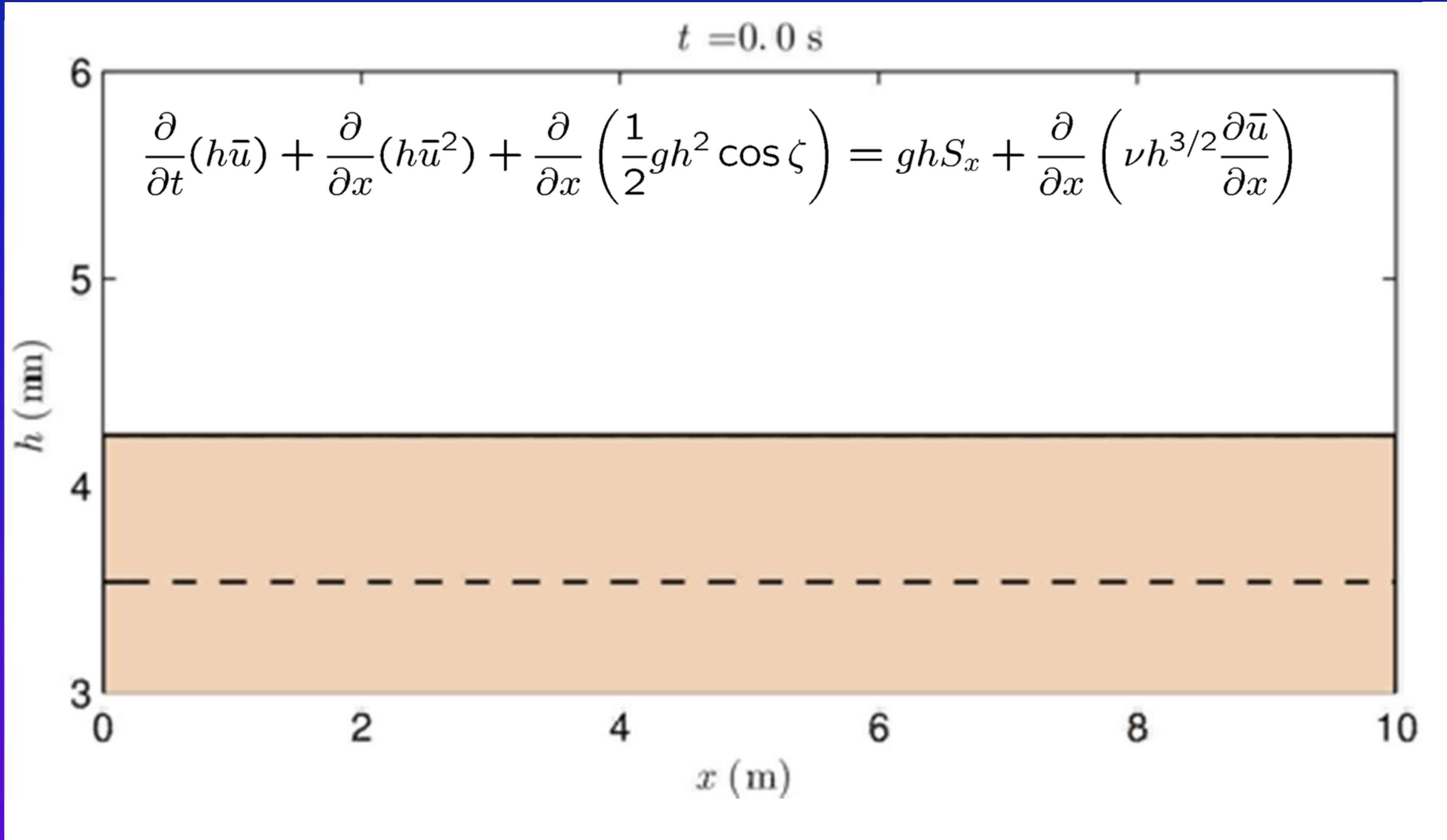
ζ_1 ζ_2

- consistent with full $\mu(I)$-rheology, which is well-posed for angles in the grey region (above)
- 2D transient simulations of Bagnold flow blow-up via oblique pressure perturbations for angles in white region (above)



Barker *et al.* (2015) *J. Fluid Mech.* **779**, 794–818.

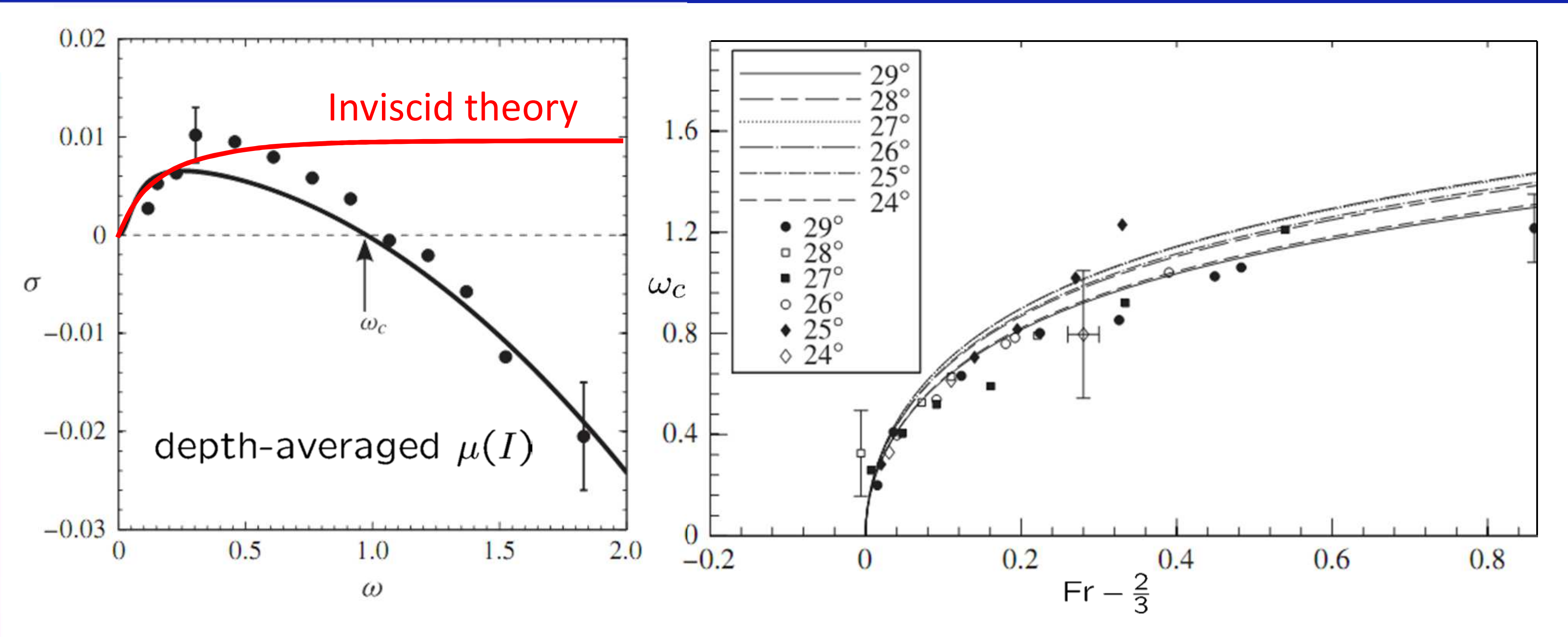
Application to granular roll-waves



- Adds a singular perturbation to the momentum equation
- This is the only form that is not singular in h or $\bar{u}$

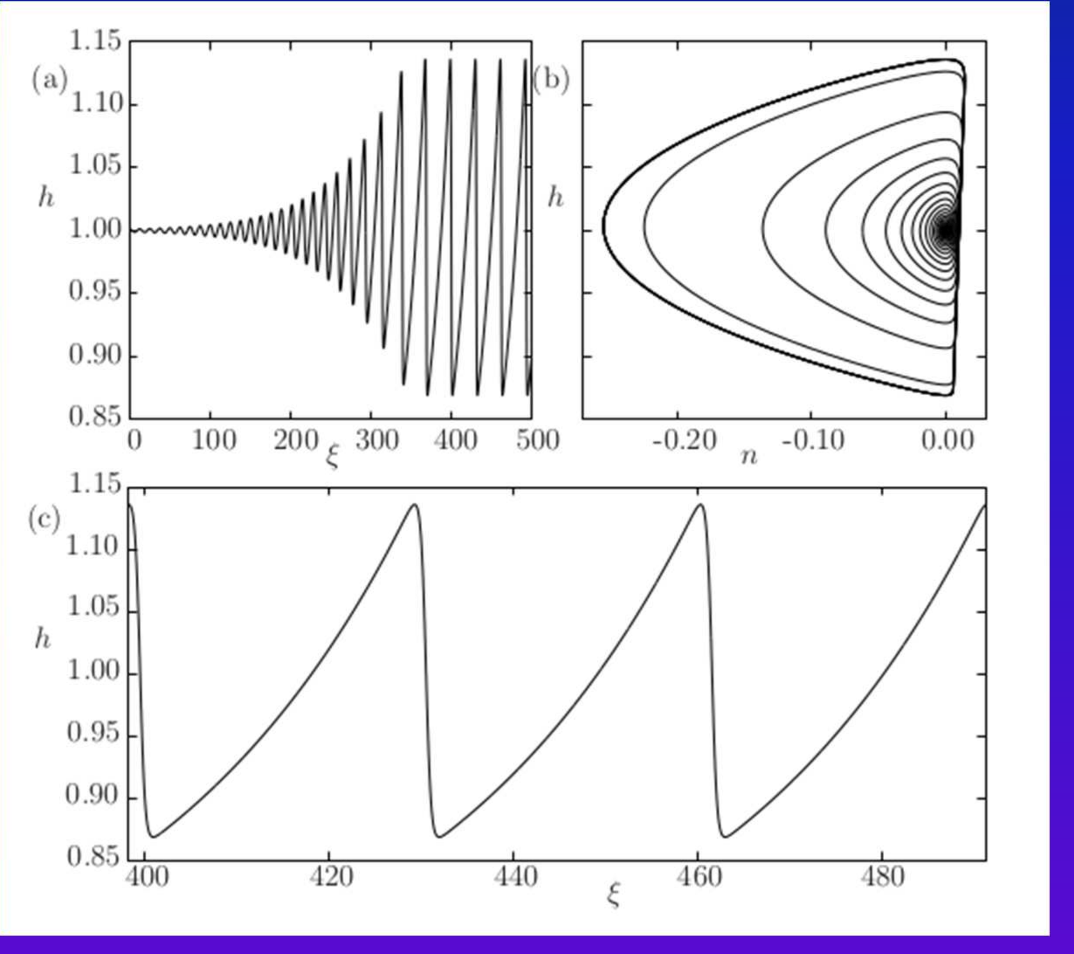
Edwards & Gray (2015) *J. Fluid Mech.* **762**, 35-67.

Measurements of the spatial growth rate of granular roll-waves



- Forterre & Pouliquen (2003) used loudspeaker to initiate roll waves of a given frequency
- Inviscid theory predicts critical Froude $Fr_c = 2/3$, but growth occurs at all frequencies ω
- Depth-averaged rheology predicts the cut-off frequency ω_c
- MATCHES WITHOUT ANY FITTING PARAMETERS

Exact travelling wave solutions for roll waves

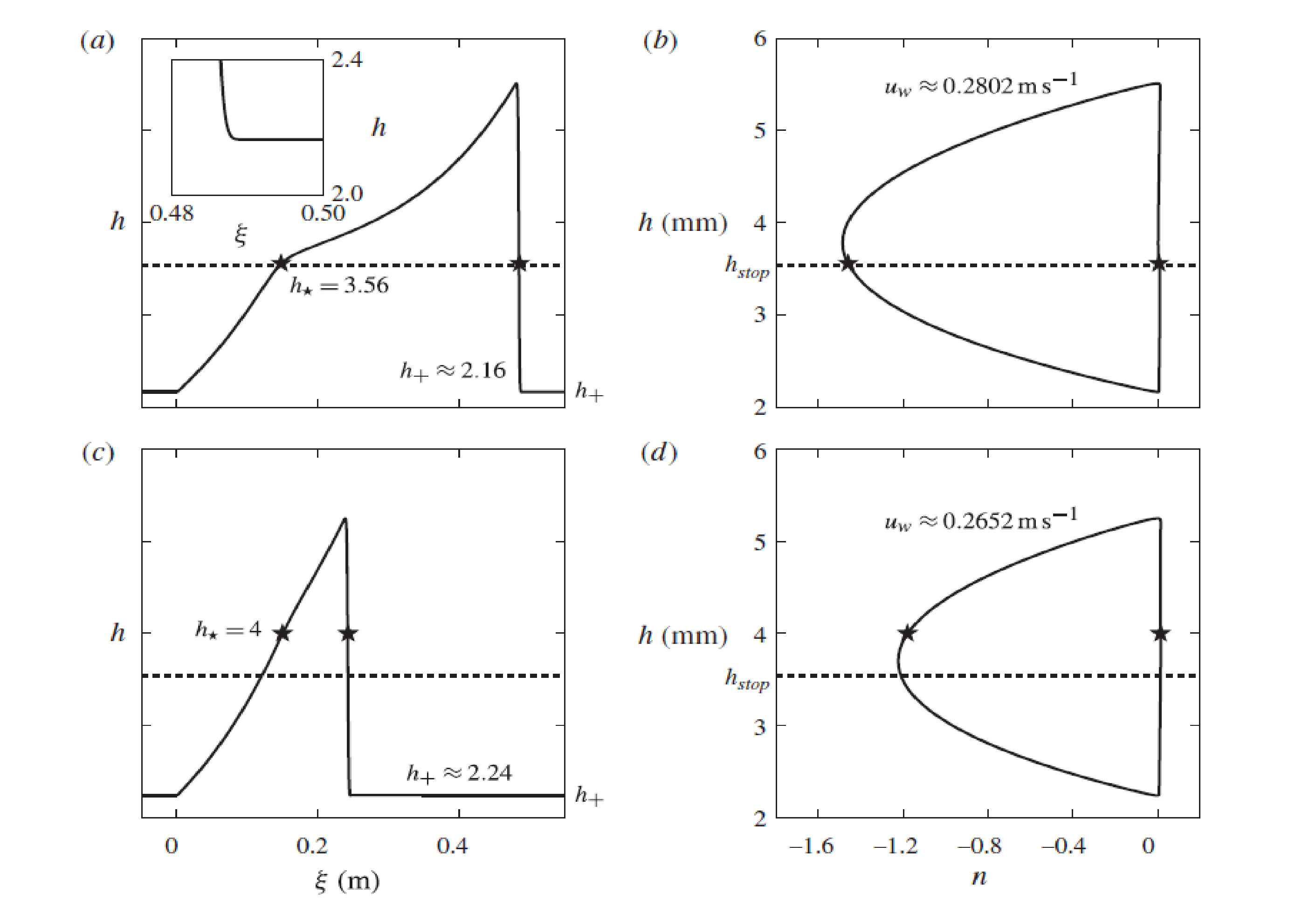


$$n = \frac{dh}{d\xi}$$

- computed by numerically integrating 2nd order ODE with prescribed Fr and u_w until a limit cycle is formed

Gray & Edwards (2014) *J. Fluid Mech.* **755**, 503-534.

Exact travelling wave solutions for erosion-deposition waves

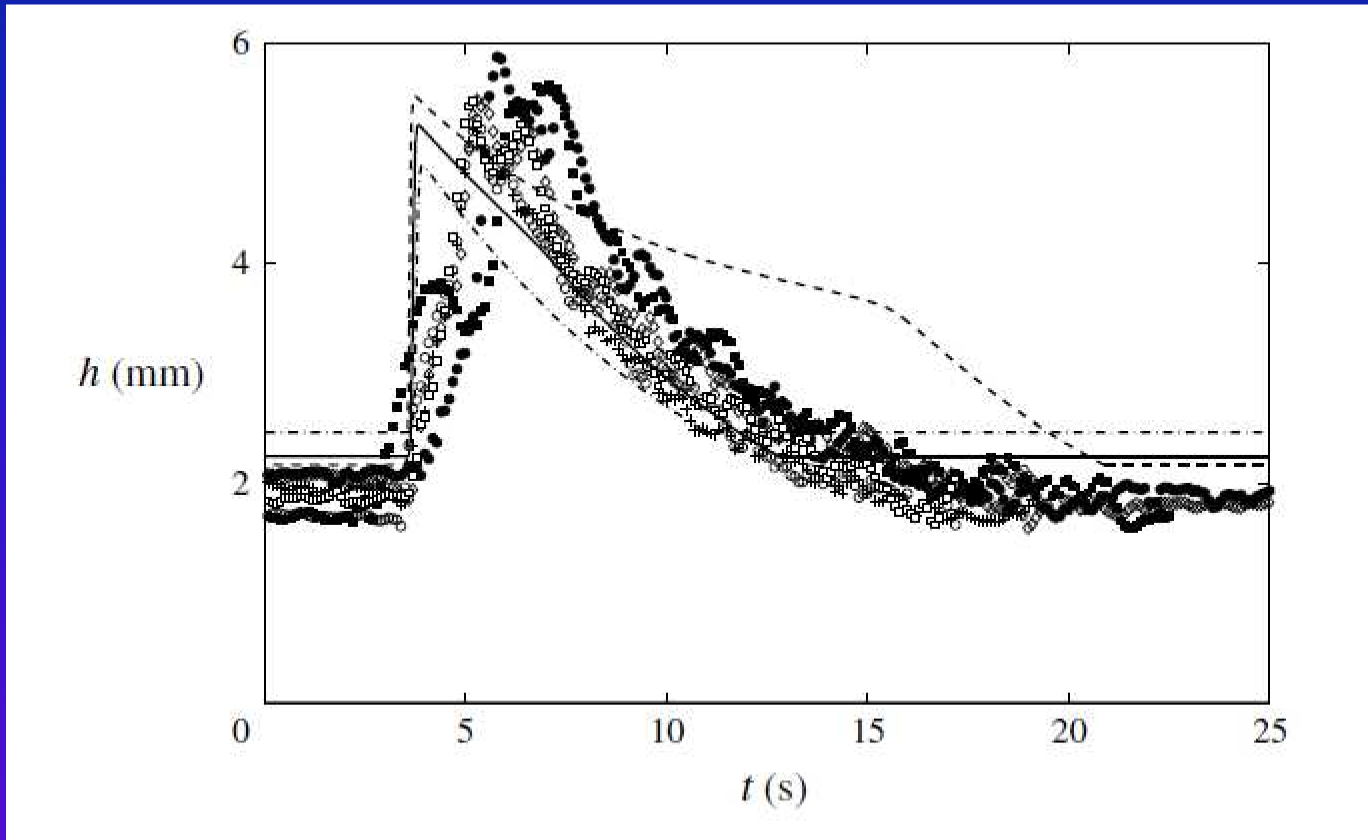


$$n = \frac{dh}{d\xi}$$

- For each solution h_+ and h_* must be prescribed.
- viscosity allows solution to cross the critical line!

Edwards & Gray (2015) *J. Fluid Mech.* **762**, 35-67.

Exact travelling wave solutions for erosion-deposition waves



- Exact solution picks off the correct amplitude and wavelength
- ALTHOUGH its shape is a little different
- MAJOR STEP FORWARD in modelling erosion-deposition problems with shallow erodible layers

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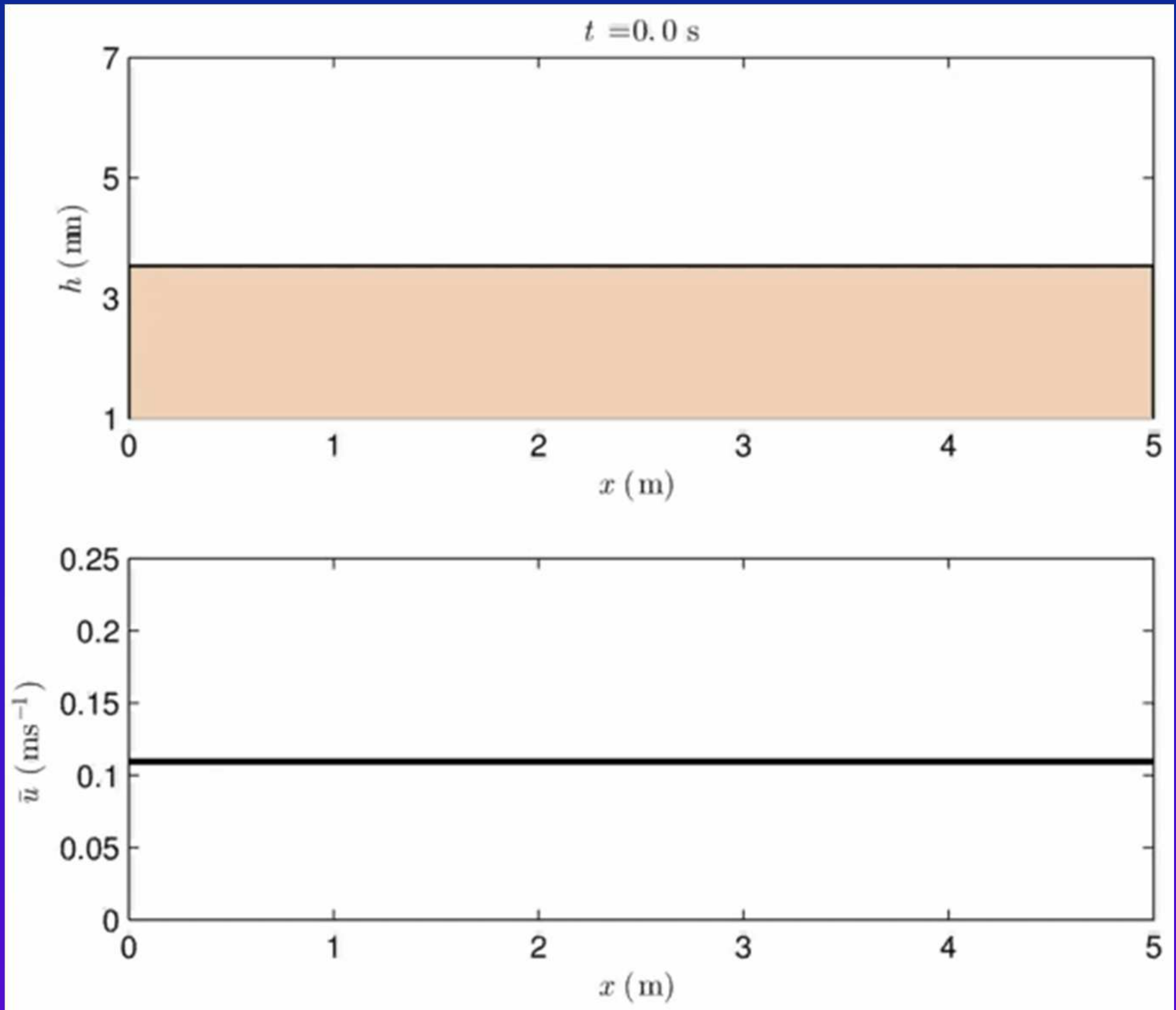
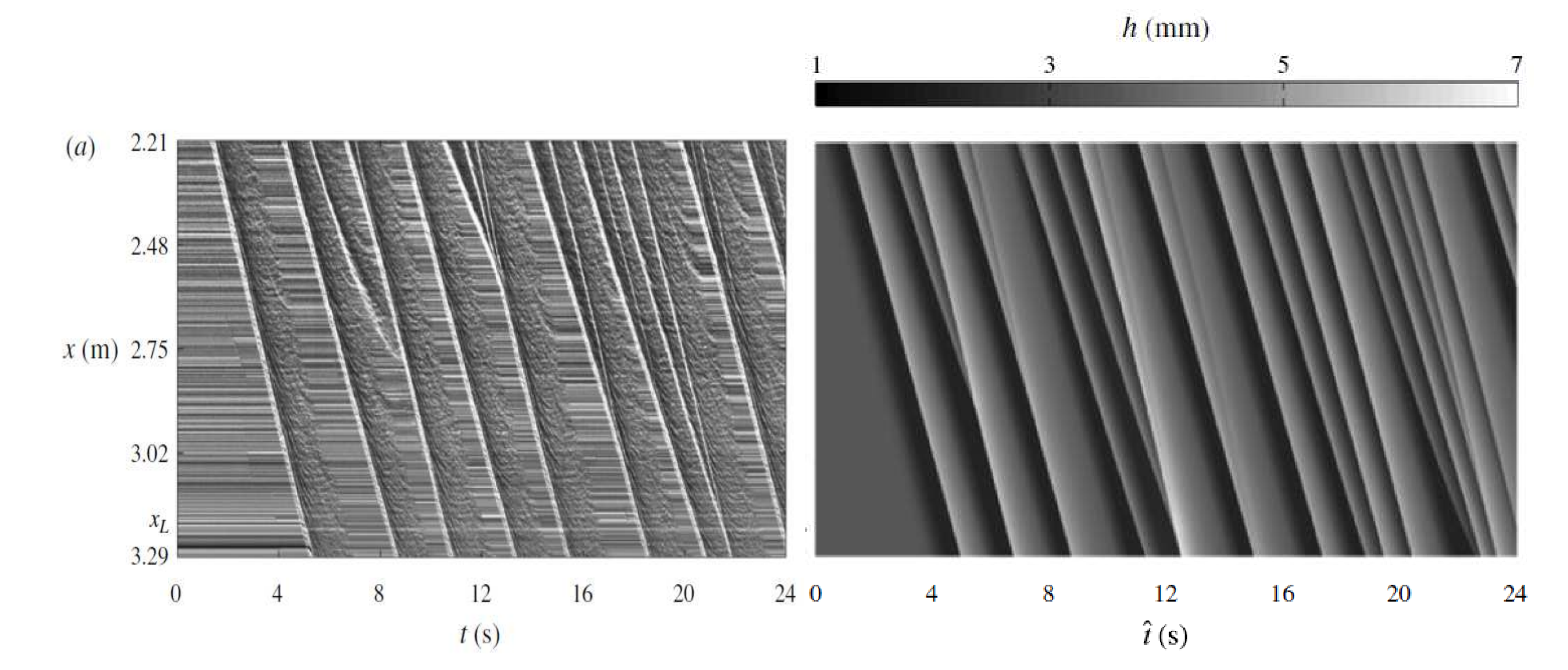


- Numerical solutions with random noise rapidly coarsen into large amplitude waves
- Close to stopping very destructive waves are formed!

Edwards & Gray (2015) *J. Fluid Mech.* **762**, 35-67.

Complex coarsening dynamics is qualitatively reproduced



- Experimental space-time plot shows:-
  - regions of stationary material as horizontal straight lines
  - the wave-fronts as white lines
- very similar in computations (right)

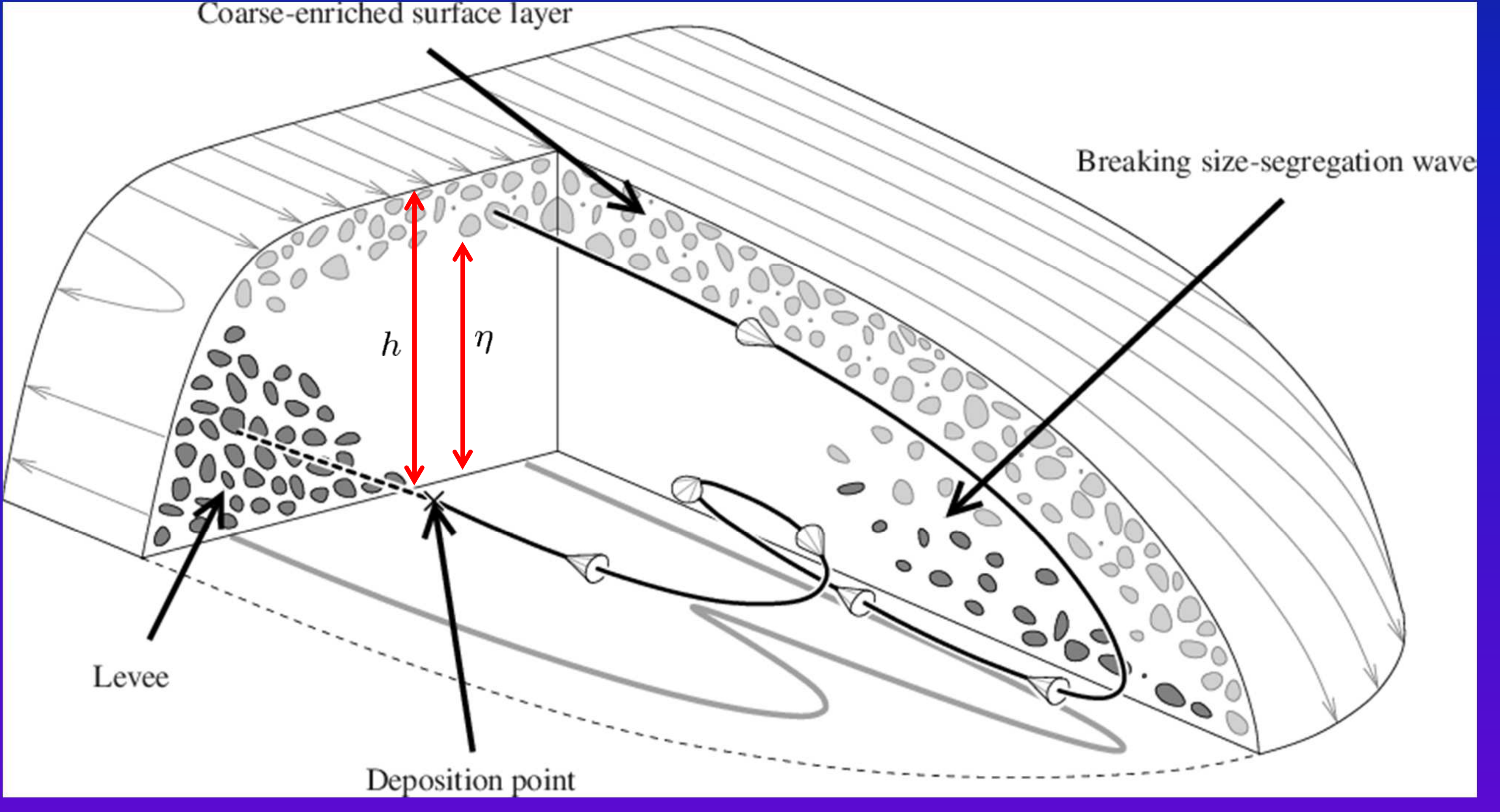
Edwards & Gray (2015) *J. Fluid Mech.* **762**, 35-67.

The depth-averaged μ(I)-rheology also plays critical role in fingering instabilities

Pouliquen, Delours & Savage (1997), Nature. **386**, 816-817.
Woodhouse *et al.* (2012), J. Fluid Mech. **709**, 543-580.

Kokelaar *et al* (2014) Earth Planet. Sci. Lett. 385, 172-180.

Schematic diagram for the levee formation process



- larger particles are shouldered to the sides to create levees
- this is an example of a segregation-mobility feedback effect

Inviscid avalanche model for segregation-mobility induced fingers

- For avalanche thickness h, small particle thickness η and depth-averaged velocity $\overline{\boldsymbol{u}}$ the 2D coupled model is

$$\begin{aligned}
\frac{\partial h}{\partial t} + \mathrm{div}(h\overline{\boldsymbol{u}}) &= 0, \\
\frac{\partial \eta}{\partial t} + \mathrm{div}\left(\eta\overline{\boldsymbol{u}} - (1-\alpha)\eta\left(1-\frac{\eta}{h}\right)\overline{\boldsymbol{u}}\right) &= 0, \\
\frac{\partial}{\partial t}(h\bar{u}) + \mathrm{div}(h\overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}) + \mathrm{grad}\left(\frac{1}{2}gh^2\cos\zeta\right) &= hg\boldsymbol{S},
\end{aligned}$$

- source terms composed of gravity and basal friction

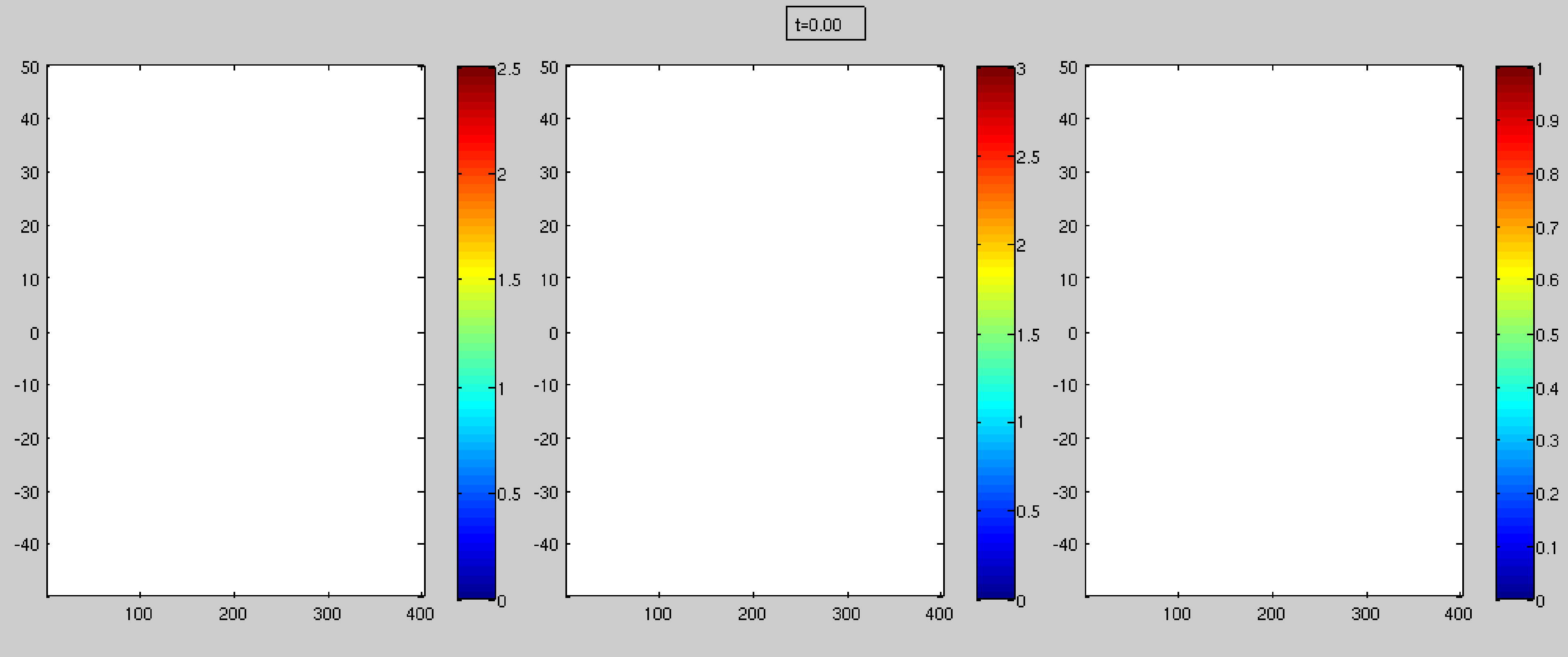
$$\boldsymbol{S} = \begin{pmatrix} \sin\zeta - \mu(\overline{u}/|\overline{\boldsymbol{u}}|)\cos\zeta, \\ -\mu(\overline{v}/|\overline{\boldsymbol{u}}|)\cos\zeta, \end{pmatrix}$$

- coupling through $\bar{\phi} = \eta/h$ dependent friction coefficient

$$\mu = \left(1-\bar{\phi}\right)\mu^L + \bar{\phi}\mu^S, \qquad \mu^L > \mu^S$$

Gray & Kokelaar (2010) *J. Fluid Mech.* **652**, 105–137

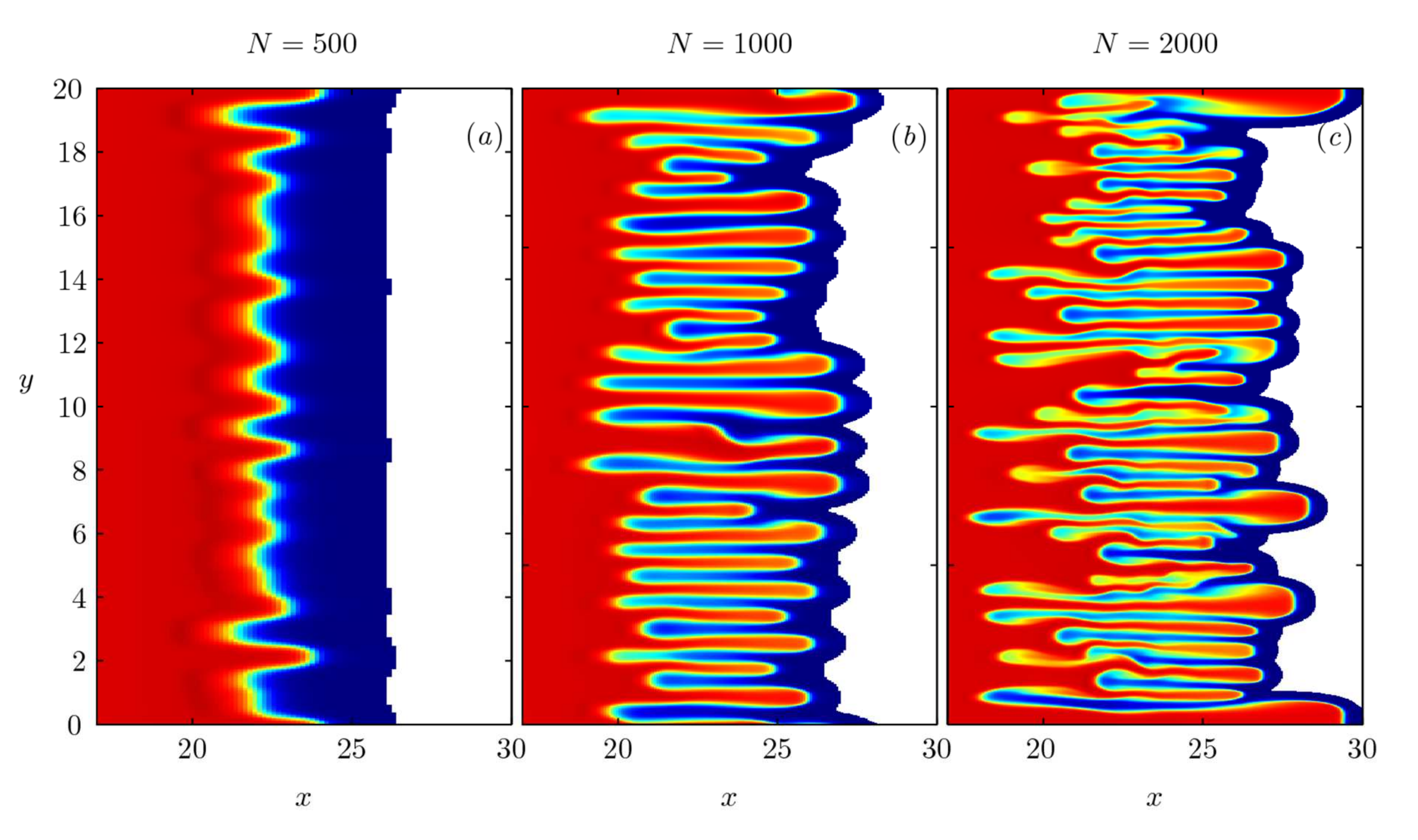
Woodhouse *et al.* (2012), J. Fluid Mech. **709**, 543-580.



h $|\overline{\boldsymbol{u}}|$ $\overline{\phi}$

- The model is hyperbolic
- captures the instability mechanism
- and forms large rich lateral levees, BUT ....

Numerical solutions are grid dependent …!



- Such ill-posed behaviour is an indication that some important physics is missing – in this case viscosity.

Woodhouse *et al* (2012), J. Fluid Mech. **709**, 543-580.

A two-dimensional fully coupled model including rheology

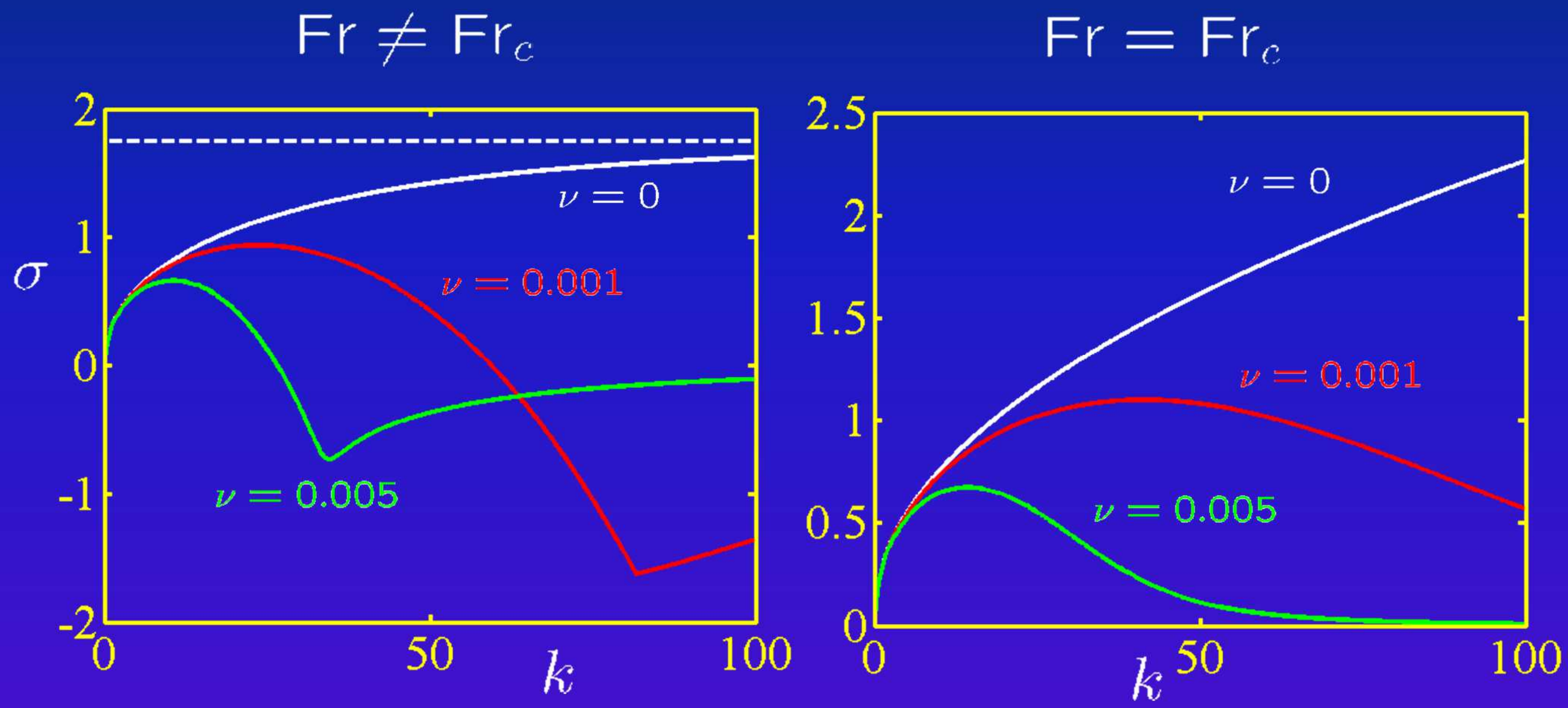
- When the depth-averaged $\mu(I)$-rheology is generalized to 2D it suggests a system of conservation laws of the form

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \mathrm{div}(h\overline{\boldsymbol{u}}) &= 0,\\
\frac{\partial \eta}{\partial t} + \mathrm{div}\left(\eta\overline{\boldsymbol{u}} - (1-\alpha)\eta\left(1-\frac{\eta}{h}\right)\overline{\boldsymbol{u}}\right) &= 0,\\
\frac{\partial}{\partial t}(h\bar{u}) + \mathrm{div}(h\overline{\boldsymbol{u}}\otimes\overline{\boldsymbol{u}}) + \mathrm{grad}\left(\frac{1}{2}h^2\cos\zeta\right) &= h\boldsymbol{S} + \mathrm{div}\left(\nu h^{\frac{3}{2}}\overline{\boldsymbol{D}}\right),
\end{aligned}
$$

- where the two-dimensional strain-rate tensor is

$$\overline{\boldsymbol{D}} = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^T\right)$$

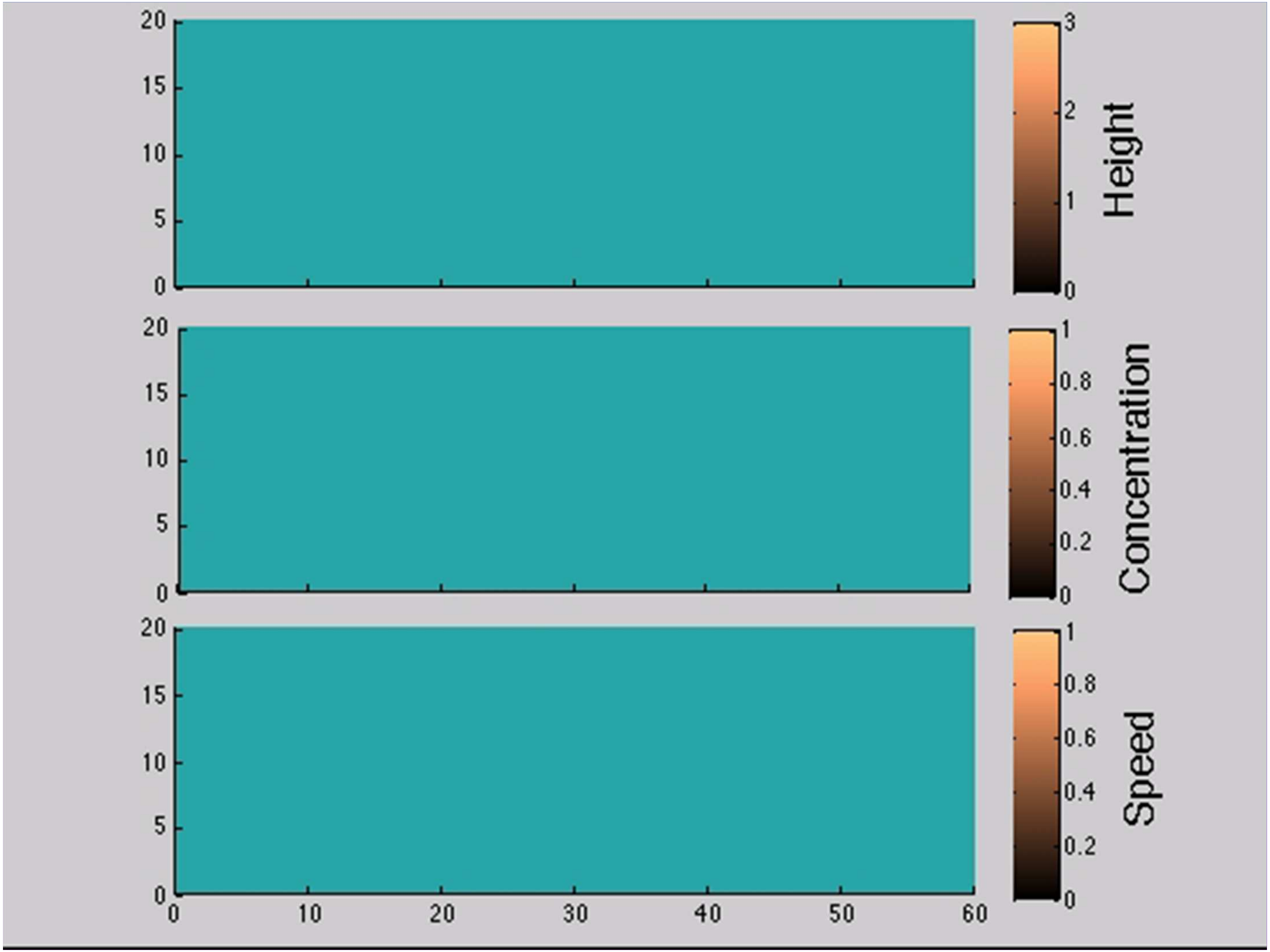
- and $\boldsymbol{L} = \mathrm{grad}(\bar{\boldsymbol{u}})$ is the depth-averaged velocity gradient
- Numerics converges ... (Baker, Johnson & Gray in prep)



- characteristics coincide when

$$\mathsf{Fr} = \mathsf{Fr}_c = \frac{1}{(1-\alpha)|2\eta_0 - 1|}$$

- produces unbounded growth in inviscid case $\nu = 0$.
- The depth-averaged $\mu(I)$-rheology regularizes the equations


Height
Concentration
Speed
0
10
20
30
40
50
60

Important for geophysical mass flows which often form levees which enhance run-out!