Mean-field approach for string nets and beyond

Julien Vidal
Laboratoire de Physique Théorique de la Matière Condensée
Sorbonne Université, CNRS, Paris

in collaboration with:

- S. Dusuel (Paris)
- J. Haegeman, M. Mariën, A. Schotte, L. Vanderstraeten, F. Verstraete (Ghent)
- N. Schuch (Garching)
- J. Carrasco (Madrid)

Outline

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Topological quantum order and its robustness

Main features

2D gapped quantum systems at $T = 0$ with:

- Ground-state degeneracy depends on the system topology
- Anyonic excitations
- Long-range entanglement, nonvanishing topological entropy
- **Robustness against local perturbations**

Topological quantum order and its robustness

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- **Robustness against local perturbations**

However, as early noticed...

"Of course, the perturbation should be small enough, or else a phase transition may occur."

A. Kitaev, Ann. Phys. **303**, 2 (2003)

Topological quantum order and its robustness

Condensed-matter issues

- Nature of phase transitions
- New universality classes
- Low-energy excitations

Topological quantum order and its robustness

Condensed-matter issues

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Tools

- Effective theories (but no local order parameter)
- Monte-Carlo simulations (but sign problem)
- Exact diagonalizations (but huge Hilbert space)
- High-order perturbative expansions (but resummation)
- **Variational approaches (but variational!)**

Outline

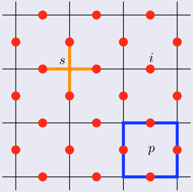
Mean-field ansatz

A simple example: the toric code in a uniform magnetic field

$$H = -J\Big(\sum_s A_s + \sum_p B_p\Big) - h_x \sum_i \sigma_i^x$$

Charge operator: $A_s = \prod_{i \in s} \sigma_i^x$

Flux operator: $B_p = \prod_{i \in p} \sigma_i^z$



A. Kitaev, Ann. Phys. **303**, 2 (2003)

- $J = 0$: 1 ground state + bosonic excitations (trivial phase)
- $h_x = 0$: 4 ground states + anyonic excitations (topological phase)

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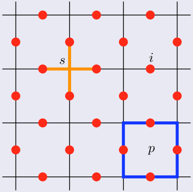

A. Kitaev, Ann. Phys. **303**, 2 (2003)

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Phase diagram

- Mapping onto the transverse-field Ising model (square lattice)
  $\rightarrow 2^{\mathrm{nd}}$ order transition at $h_x/J \simeq 0.328$

S. Trebst, P. Werner, M. Troyer, K. Shtengel, and C. Nayak, Phys. Rev. Lett. 98, 070602 (2007)
A. Hamma and D. A. Lidar, Phys. Rev. Lett. 100, 030502 (2008)

Mean-field ansatz

Mean-field description of the ground state

- Simple ansatz: $\boxed{|\alpha\rangle = \mathcal{N}_\alpha \prod_p \left(\frac{\mathbb{1} + \alpha B_p}{2}\right) |+X\rangle}$
- $\alpha = 0$: exact ground state for $J = 0$
- $\alpha = 1$: exact ground state for $h_x = 0$
- Nonvanishing topological entropy only for $\alpha = 1$

  $\rightarrow$ α **is an "order parameter"**
- Wilson loops: $W_n = \langle\alpha| \prod_{p=1}^{n} B_p |\alpha\rangle = \langle\alpha|B_p|\alpha\rangle^n$

  $\rightarrow$ Trivial perimeter-law (zero-law) for $\alpha = 1$: $W_n = 1$

  $\rightarrow$ Nontrivial area-law for $\alpha \neq 1$: $W_n = \left(\frac{2\alpha}{1+\alpha^2}\right)^n$

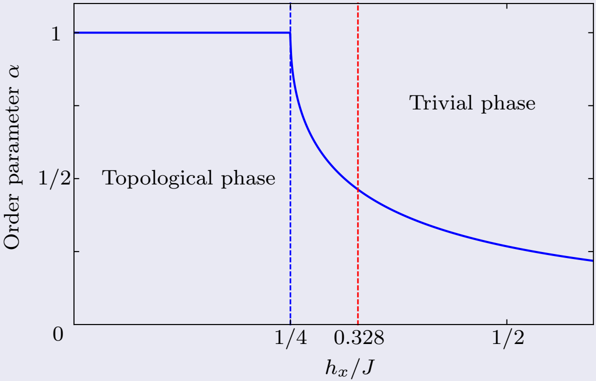
  $\rightarrow$ Factorization property: **uncorrelated-flux approximation**

M. B. Hastings and X.-G. Wen, Phys. Rev. B **72**, 045141 (2005)

S. Dusuel and J. Vidal, Phys. Rev. B **92**, 125150 (2015)

Mean-field ansatz

Mean-field phase diagram (Toric code + X-field)

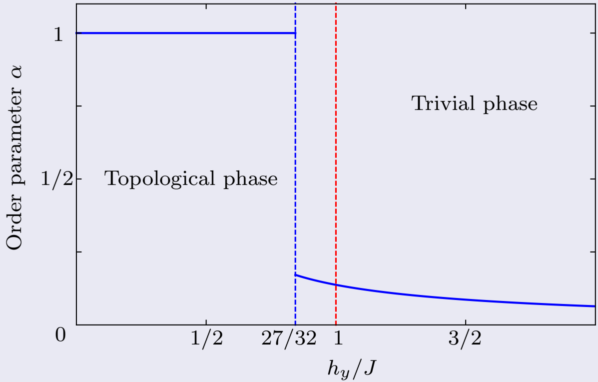


- 2^{nd} order transition (α is a continuous function of h_x/J)
- Critical point at $h_x/J = 1/4$ ($\sim 24\,\%$ off)

S. Dusuel and J. Vidal, Phys. Rev. B **92**, 125150 (2015)

Mean-field ansatz

Mean-field phase diagram (Toric code + Y–field)



- Exact mapping onto the quantum compass model (1st-order QPT at $h_y = J$)
- Transition point at $h/J = 27/32$ ($\sim 16\,\%$ off)

J. Vidal, R. Thomale, K. P. Schmidt, and S. Dusuel, Phys. Rev. B **80**, 081104 (2009)

S. Dusuel and J. Vidal, Phys. Rev. B **92**, 125150 (2015)

Mean-field ansatz

String-net models with tension

$$H = -J\left(\sum_s A_s + \sum_p B_p\right) - h\sum_i V_i$$

- Degrees of freedom defined on the links of the honeycomb lattice
- Each link j can be in N different states $|1\rangle_j, |2\rangle_j, ..., |N\rangle_j$
- Restricted Hilbert space: $A_s|\psi\rangle = +|\psi\rangle$ (no charge excitation)
  $\rightarrow$ Constraints at each vertex (branching rules $\leftrightarrow$ fusion rules)
- Tension term: $[A_s, V_i] = 0$ (charge conserving)
- $V_j = \delta_{j,1}$: projector onto the state $|1\rangle_j$
- Simple ansatz: $\boxed{|\alpha\rangle = \mathcal{N}_\alpha \prod_p \left(\frac{\mathbb{1} + \alpha B_p}{2}\right)|1\rangle}$

M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)
S. Dusuel and J. Vidal, Phys. Rev. B **92**, 125150 (2015)

Mean-field ansatz

Mean-field transition point: $$h/J = \frac{D^2 - 1}{3D^2}$$ (D: total quantum dimension)

Results for $\mathbb{Z}_N$ anyon theories

- Mapping onto the transverse-field N-states Potts model (triangular lattice)

	$\mathbb{Z}_2$	$\mathbb{Z}_3$	...	$\mathbb{Z}_{N \gg 1}$
h/J (mean-field)	0.1667	0.2222	...	1/3
h/J (series)	0.2097	0.2466	...	1/3

H.-X. He, C. J. Hamer, and J. Oitmaa, J. Phys. A **23**, 1775 (1990)

C. J. Hamer, J. Oitmaa, and Z. Weihong, J. Phys. A **25**, 1821 (1992)

F. J. Burnell, S. H. Simon, and J. K. Slingerland, Phys. Rev. B 84, 125434 (2011)

Results for non-Abelian anyon theories

- No exact mapping known!

	Fibonacci	Ising
h/J (mean-field)	0.2412	0.25
h/J (series)	0.2618	0.267

M. D. Schulz, S. Dusuel, K. P. Schmidt, J. Vidal, Phys. Rev. Lett. 110, 147203 (2013)

M. D. Schulz, S. Dusuel, G. Misguich, K. P. Schmidt, J. Vidal, Phys. Rev. B 89, 201103 (2014)

Outline

Beyond the mean-field approximation

What is missing ?

- Correlations between fluxes
- Better approximation in the topological phase

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The filtering procedure

- Build an ansatz that is perturbatively exact (at a given order)
- Example: Toric code + X-field

Improved ansatz: $$\boxed{|\alpha,\beta\rangle = \mathcal{N}_{\alpha,\beta} \prod_i \mathrm{e}^{-\beta\sigma_i^x} \prod_p \left(\frac{\mathbb{1}+\alpha B_p}{2}\right)|+X\rangle}$$

→ Exact ground state for $h \ll J$ (order 1)

→ Nontrivial perimeter law (penalty to large loops)

J. Haegeman, K. Van Acoleyen, N. Schuch, J. I. Cirac, and F. Verstraete, Phys. Rev. X **5**, 011024 (2015)

J. Haegeman, V. Zauner, N. Schuch, and F. Verstraete, Nat. Commun. **6**, 8284 (2015)

L. Vanderstraeten, M. Mariën, J. Haegeman, N. Schuch, J. Vidal, F. Verstraete, Phys. Rev. Lett. **119**, 070401 (2017)

Beyond the mean-field approximation

Perturbative Projected Entangled Paired State

- Numerical implementation using tensor networks
- Ex: Toric code$+$ X-field $\rightarrow |\alpha, \beta\rangle$ is a PEPS with bond dimension $D = 2$
- Virtual symmetry breaking detects phase transition (G-injectivity)
  $\rightarrow \alpha$ **is still an order parameter**

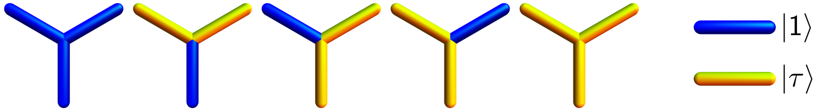
(Ask Norbert for details!)

N. Schuch, J. I. Cirac, and D. Pérez-García, Ann. Phys. **325**, 2153 (2010).
L. Vanderstraeten, M. Mariën, J. Haegeman, N. Schuch, J. Vidal, F. Verstraete, Phys. Rev. Lett. **119**, 070401 (2017)

Beyond the mean-field approximation

The Fibonacci string-net model

- Two strings: $\{1, \tau\}$
- Fusion rules: $1 \times 1 = 1$, $1 \times \tau = \tau$, $\tau \times \tau = 1 + \tau$

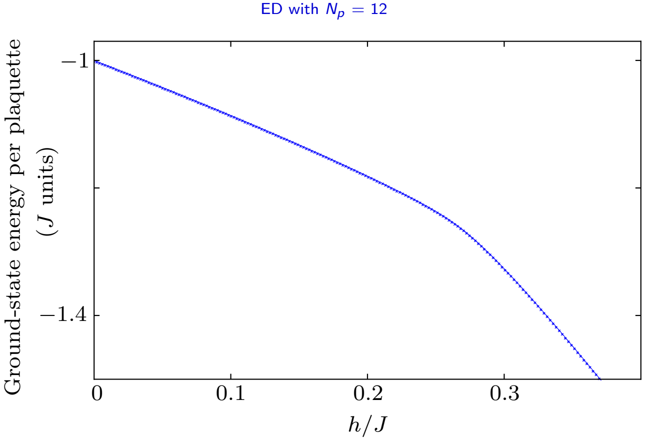


Hilbert space for any graph with N_v trivalent vertices

- Dim $\mathcal{H} = (1+\varphi^2)^{\frac{N_\mathrm{v}}{2}} + (1+\varphi^{-2})^{\frac{N_\mathrm{v}}{2}}$, $\quad \varphi = \frac{1+\sqrt{5}}{2}$ (golden ratio)

Beyond the mean-field approximation

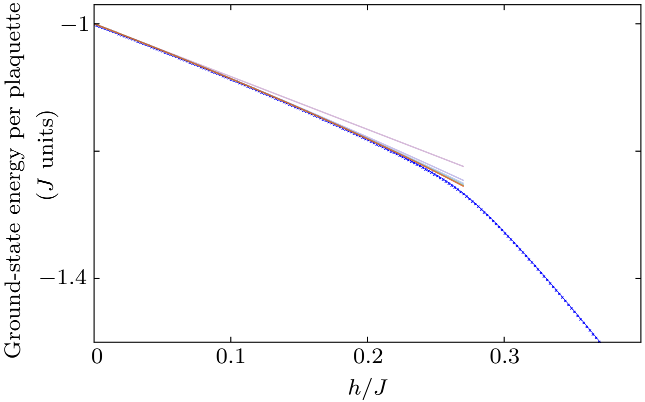
Fibonacci string-net model + tension



Beyond the mean-field approximation

Fibonacci string-net model + tension

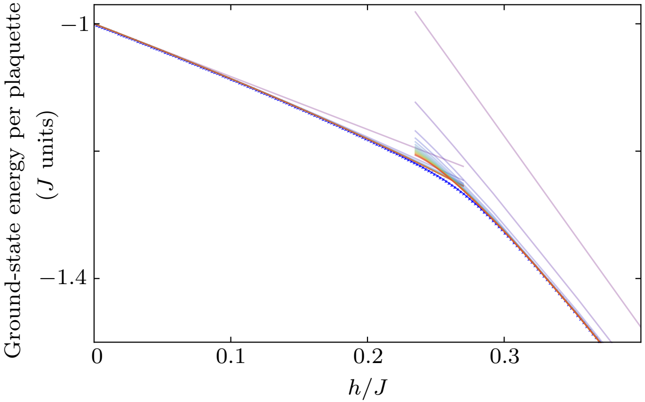
ED for $N_p = 12$ + Series $h \ll J$ (up to order 11)



Beyond the mean-field approximation

Fibonacci string-net model + tension

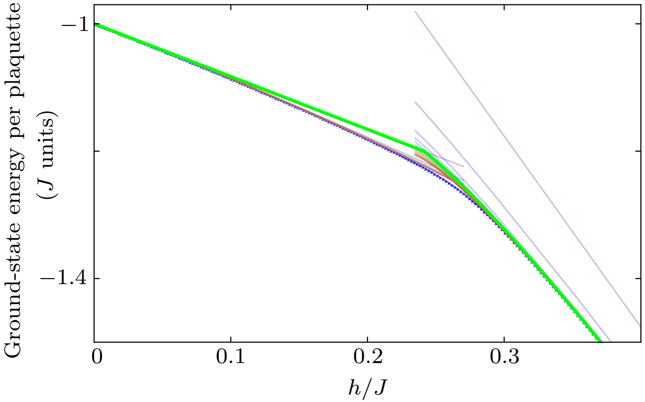
ED for $N_p = 12$ + Series $h \ll J$ (up to order 11) + Series $h \gg J$ (up to order 20)



Beyond the mean-field approximation

Fibonacci string-net model + tension

ED for $N_p = 12$ + Series $h \ll J$ (up to order 11) + Series $h \gg J$ (up to order 20) + $|\alpha\rangle$



Beyond the mean-field approximation

Fibonacci string-net model + tension

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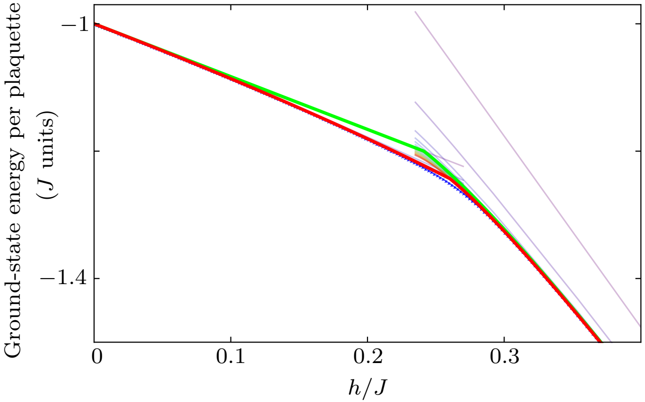

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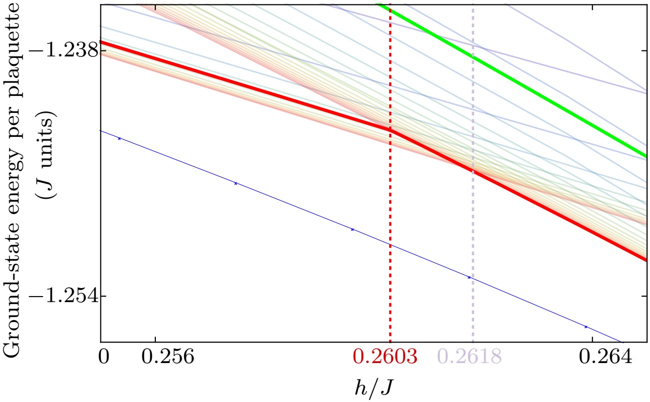
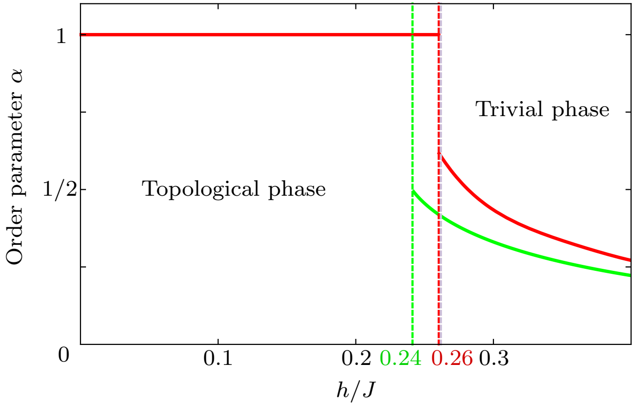


Beyond the mean-field approximation

Fibonacci string-net model + tension

Series $h \ll J$ (up to order 11) + Series $h \gg J$ (up to order 20) + $|\alpha\rangle$ + $|\alpha, \beta\rangle$



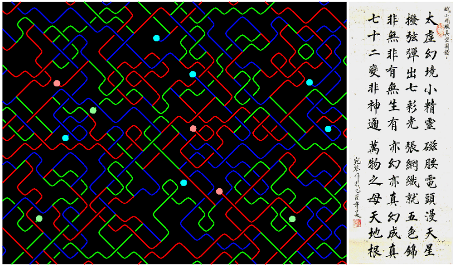
1^{st}-order transition!

J. Carrasco, et al. (in preparation)

Outlook

Take-home messages

- Simple mean-field ansatz $\rightarrow$ qualitatively good description
- Improved ansatz (perturbative PEPS) $\rightarrow$ quantitatively good description
- Perimeter law and area law captured
- To be tested in 3D models (Walker-Wang, fracton, X-cube, Haah's code,...)
- Not restricted to topologically-ordered systems



Artist view of string nets
(courtesy of X.-G. Wen)