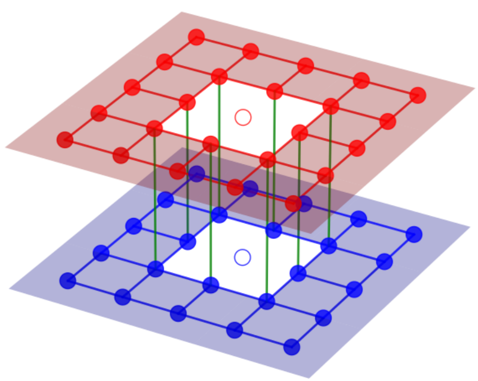
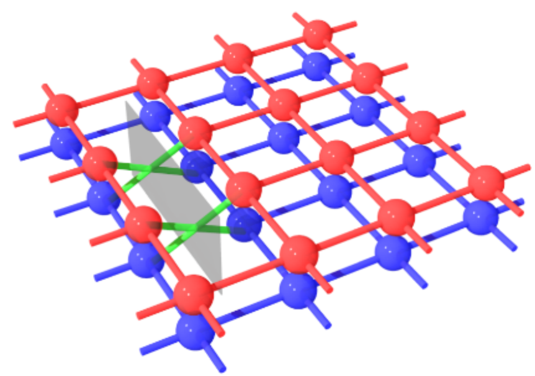
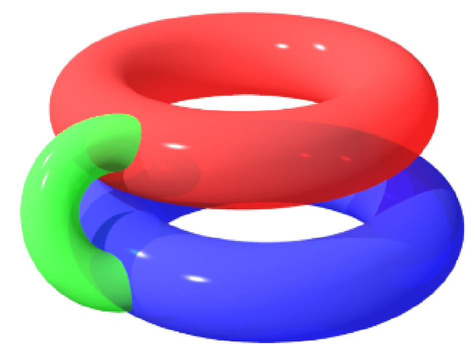
Lattice Fractional Quantum Hall States With Gapped Boundaries or Non-Abelian Defects – *a microscopic investigation*







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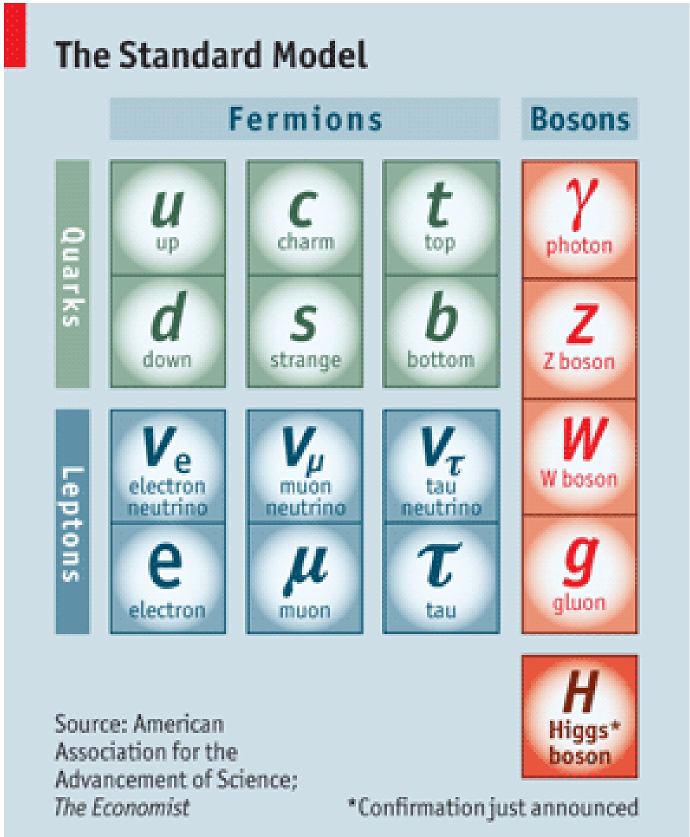
Gunnar Möller @ Kent

"Anyons in Quantum Many-Body Systems"
Dresden, Jan. 24, 2019

Anyons

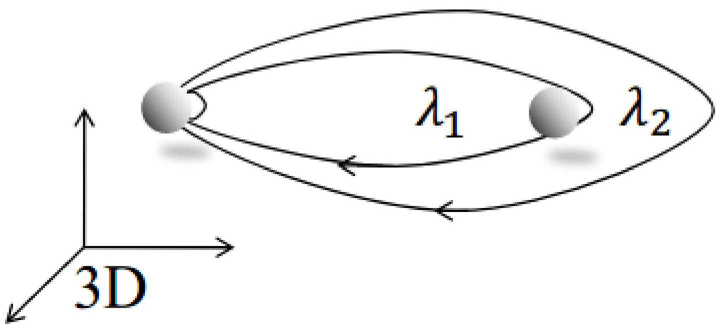
- All matters in nature are composed of bosons and fermions.

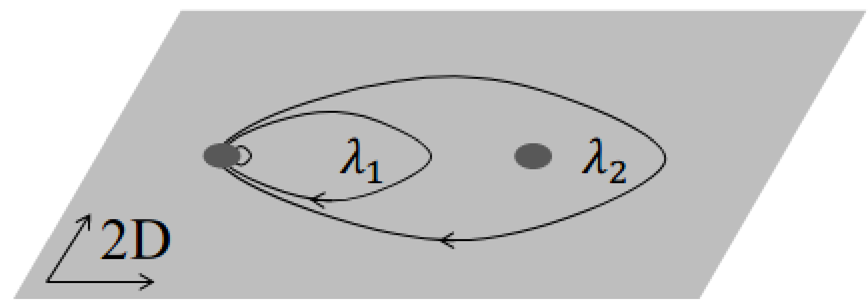


$$R|\Psi(\mathbf{r}_1,\mathbf{r}_2)\rangle = |\Psi(\mathbf{r}_2,\mathbf{r}_1)\rangle = \pm|\Psi(\mathbf{r}_1,\mathbf{r}_2)\rangle$$

Anyons

- Particles constrained in 2D can have statistics interpolating between those of bosons and fermions! *Leinaa and Myrheim (1977)*





$$|\Psi(\lambda_2)\rangle = |\Psi(\lambda_1)\rangle = |\Psi(0)\rangle$$

$$R^2|\Psi(0)\rangle = |\Psi(0)\rangle$$

$$R^2 = 1, R = \pm 1$$

either bosons or fermions

$$|\Psi(\lambda_2)\rangle \neq |\Psi(\lambda_1)\rangle = |\Psi(0)\rangle$$

$$R^2|\Psi(0)\rangle \neq |\Psi(0)\rangle, R^2 \neq 1$$

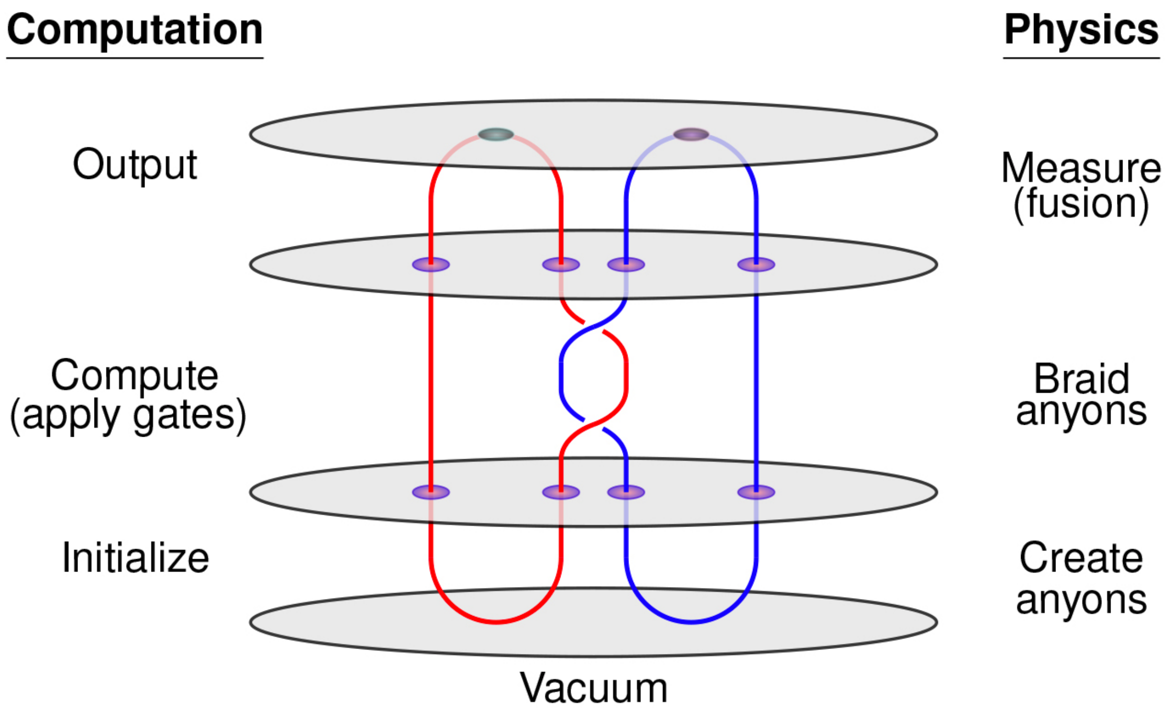
$R = e^{i\theta}$ *Abelian anyons*

$R = \mathrm{matrix}$ *non-Abelian anyons*

Wilczek (1982)

Anyons

- Braid non-Abelian anyons – the standard implementation of topological quantum computation.

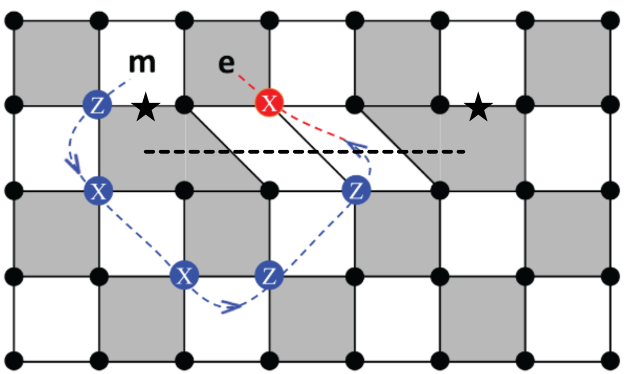


It is challenging to realize non-Abelian anyons.
Other routes to achieve and even facilitate the realization of TQC?

Non-Abelian defects (genons)

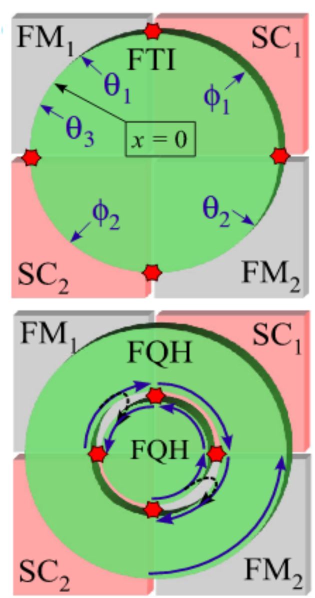
- Defects created in topologically ordered states can carry non-Abelian features, even when hosted by Abelian states.
  – enhance the GS degeneracy; *Barkeshli, Jian, and Qi (2013)*
  associated to a degenerate ground-state manifold;
  – obey projective non-Abelian braiding statistics.

lattice dislocations
in toric code model



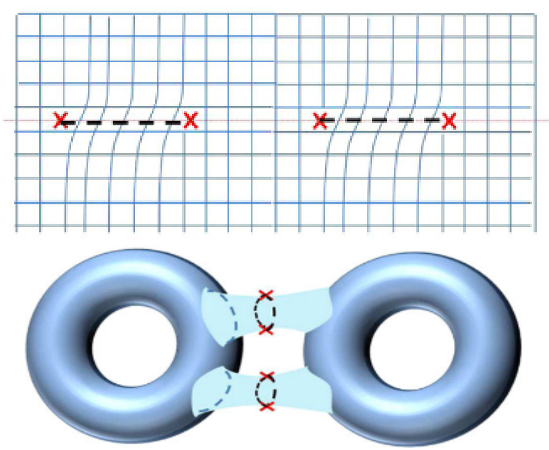
Bombin (2010)

SC-FM domain walls
in FTI/FQH



Lindner, Berg, Refael, and Stern (2012)

lattice dislocations
in C=2 bands



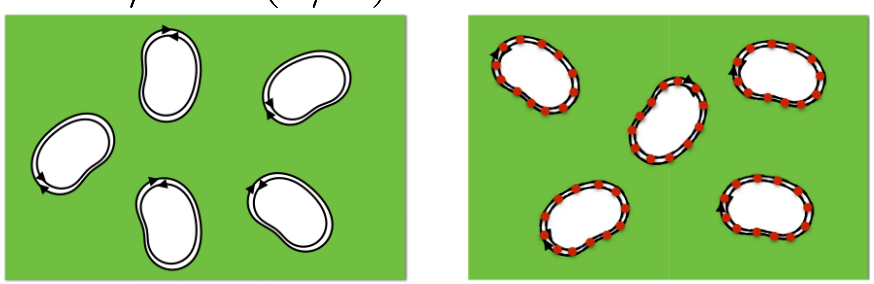
Barkeshli and Qi (2012)

Gapped boundaries

- Gapless states on boundaries are often mentioned as a defining feature of topologically ordered states. However, we can gap out the counter propagating edge modes of nonchiral topological states.
- The GS degeneracy may be enhanced by gapped boundaries, depending on the boundary gapping condition on each boundary.

Wang and Wen (2015); Hung and Wan (2015)

$\nu = 1/k + (1/k)^*$ $\quad$ $\mathrm{GSD} = k^{N-1}$



Apply the same boundary gapping condition (tunneling or pairing) for all boundaries of an FQSH state
→ an FQH state on a higher-genus surface

Barkeshli (2016); Ganeshan et al. (2017); Repellin et al. (2018)

- Braiding gapped boundaries + topological charge measurements + modular transformations in the mapping class group of the *g*>0 surface → universal TQC even though the intrinsic anyons of the underlying phase do not support it.

Barkeshli and Freedman (2016); Cong, Cheng, and Wang (2017)

Our motivation

- The first step of implementing these beautiful ideas is to investigate the realizations of topological phases compatible with non-Abelian defects or gapped boundaries.

Effective field theory and exactly solvable models:
Bombin (2010); You and Wen (2012); Lindner, Berg, Refael, and Stern (2012); Barkeshli and Qi (2012); Barkeshli, Jian, and Qi (2013); Barrett et al. (2013); Vaezi (2013, 2014); Kapustin (2014); Wang and Wen (2015); Hung and Wan (2015); Barkeshli (2016); Barkeshli and Freedman (2016); Ganeshan, Gorshkov, Gurarie, and Galitski (2017); Cong, Cheng, and Wang (2017); and more ...

Numerical simulations in microscopic models:
Liu, Bergholtz, and Möller (2017); M.-S. Vaezi and A. Vaezi (2017); Repellin, Cook, Neupert, and Regnault (2018); Liu and Bergholtz (2019)

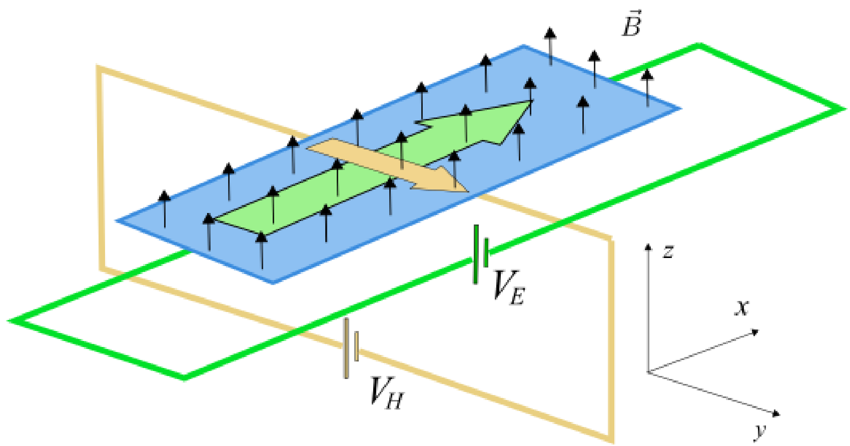
- Microscopic investigations are still relatively rare, but they are indispensable for guiding experiments and identifying problems obscured by the effective field theory.

This talk

- The building block: Kapit-Mueller model
  - an elegant lattice model mimicking the lowest Landau level
- Two layers of KM models with opposite chiralities + holes created by removing lattice sites + interlayer tunneling around holes + interactions
  - bosonic fractional quantum Hall states observed by extensive exact diagonalization
- Two layers of KM models with the same chirality + wormhole like branch cuts
- Outlook

The quantum Hall effect

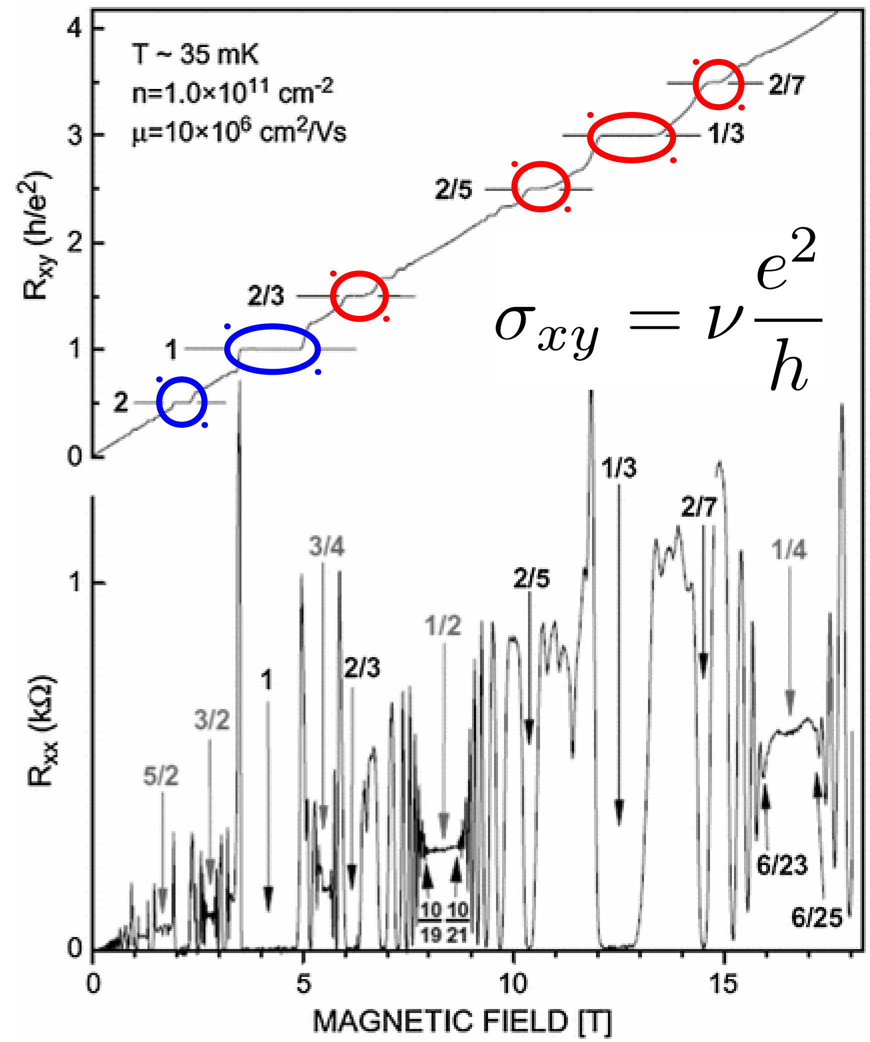
- Cold 2DES in a strong magnetic field



Landau levels formed:
exactly flat carrying Chern number C=1.

Integer QHE:
non-interacting problem;
a result of Landau level quantization
Klitzing, Dorda, Pepper (1980)

Fractional QHE:
produced by strong interactions;
a notoriously difficult problem
Tsui, Stormer, Gossard (1982)



$$\sigma_{xy} = \nu \frac{e^2}{h}$$

The fractional quantum Hall effect

- Single-particle states in the lowest LL: $\mathbf{A} = \frac{B}{2}(-y, x)$

$$\psi_m = z^m e^{-\frac{|z|^2}{4\ell^2}} \quad z = x + iy, \ell = \sqrt{(\hbar c)/(eB)} \quad m = 0, 1, 2, \ldots$$

- Ansatz wave functions in the lowest LL:

$$\nu = 1/3 : \Psi_{\mathrm{Lau}} = \prod_{i<j}^{N_e} (z_i - z_j)^3 e^{-\sum_{i=1}^{N_e} \frac{|z_i|^2}{4\ell^2}}$$

Laughlin (1983)

$$\nu = 5/2 : \Psi_{\mathrm{MR}} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{N_e} (z_i - z_j)^2 e^{-\sum_{i=1}^{N_e} \frac{|z_i|^2}{4\ell^2}}$$

Moore and Read (1991)

- Read-Rezayi series for $\nu = k/(kM + 2), k \geq 1, M \geq 0$

Read and Rezayi (1998)

– odd *M* for fermions and even *M* for bosons
– Laughlin and MR states as special examples
– short-range parent Hamiltonians
(*k*+1)-body contact interaction for the $\nu = k/2$ bosonic RR state

$$\sum_{i_1 < i_2 < \cdots < i_{k+1}}^{N_b} \delta^2(z_{i_1} - z_{i_2}) \delta^2(z_{i_2} - z_{i_3}) \cdots \delta^2(z_{i_k} - z_{i_{k+1}})$$

E. Kapit and E. Mueller, PRL **105**, 215303 (2010)

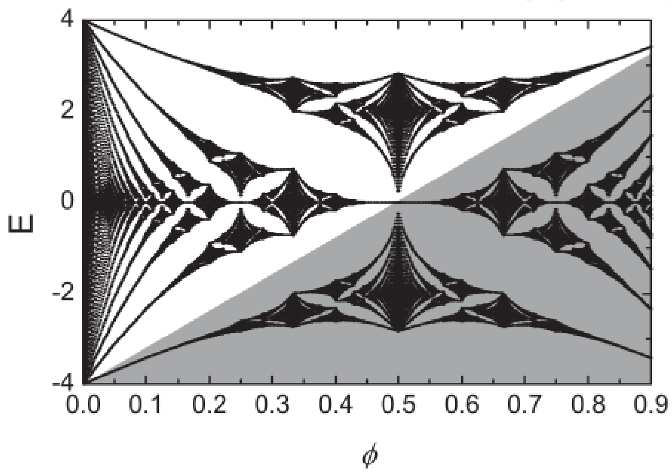
The Kapit-Mueller model

- Modified Hofstadter model:
  – square lattice, flux quanta ϕ piercing each plaquette
  – infinite-range, but exponentially decayed hopping

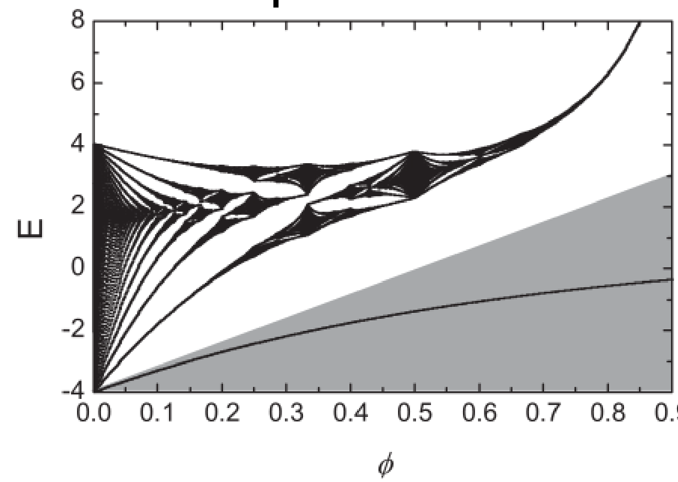
$$t_{jk} = t_0(-1)^{x+y+xy} e^{-\frac{\pi}{2}(1-|\phi|)(|x|^2+|y|^2)} e^{-i\pi\phi(x_j+x_k)y}$$

$$x = x_j - x_k,\ y = y_j - y_k$$

Hofstadter (NN hopping)



Kapit-Mueller



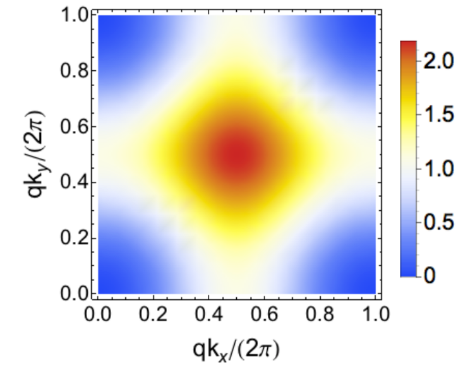
- The lowest band is exactly flat at any ϕ.

E. Kapit and E. Mueller, PRL **105**, 215303 (2010)

The Kapit-Mueller model

- The lowest band somehow mimics the LLL:
  - $\mathcal{C}=1$, energetically exactly flat

    Berry curvature still varying! $\phi=1/2$



  - spanned by discretized lowest LL wave functions

$$\psi_n(z_j)=z_j^n\exp\Big(-\frac{\pi|\phi|}{2}|z_j|^2\Big)$$ **orthogonality lost on the lattice!**

  - (k+1)-body onsite repulsion $\sum n_i(n_i-1)\cdots(n_i-k)$

    gives discretized ν=$k/2$ bosonic RR states as the ground states.

excitations changed!

$$k=1:\Psi_{\mathrm{Lau}}=\prod_{i<j}(z_i-z_j)^2e^{-\frac{\pi|\phi|}{2}\sum_i|z_i|^2}\qquad k=2:\Psi_{\mathrm{MR}}=\mathrm{Pf}\Big(\frac{1}{z_i-z_j}\Big)\prod_{i<j}(z_i-z_j)e^{-\frac{\pi|\phi|}{2}\sum_i|z_i|^2}$$

Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

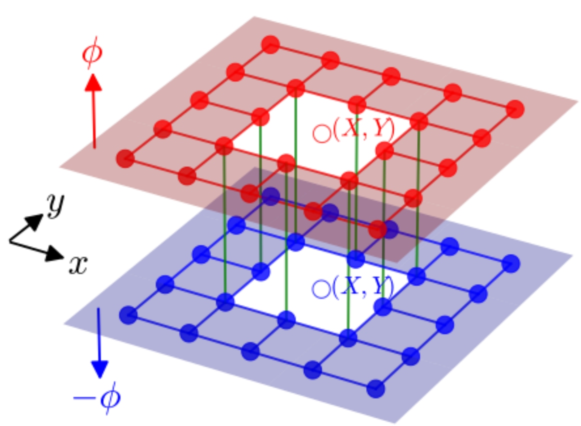
KM model + gapped boundaries

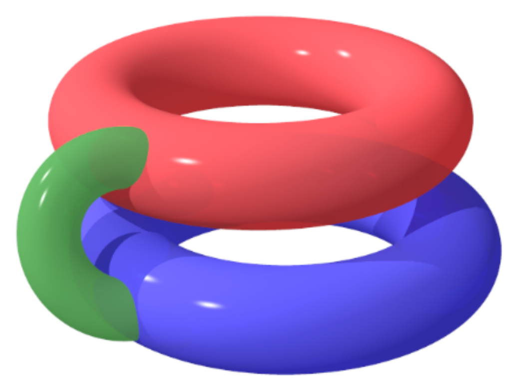
- Punch *M* pairs of holes through two layers of KM model with opposite chiralities.
  - An extreme limit: each hole only contains a single removed lattice site, such that its edge contains eight sites.
  - Couple the two edges of each pair of holes with vertical interlayer tunneling, which gives an effective genus *g*=*M*+1 surface.

$$H_0 = \sum_{j,k\notin\mathcal{R}} \sum_{\sigma=\uparrow,\downarrow} t^{\sigma}_{jk} a^{\dagger}_{j\sigma} a_{k\sigma} + \sum_{m=1}^{M} \sum_{e\in\mathcal{E}_m} \left(t^{\perp}_{e} a^{\dagger}_{e\uparrow} a_{e\downarrow} + h.c. \right)$$

$$t^{\sigma}_{jk} = t_0 (-1)^{x+y+xy} e^{-\frac{\pi}{2}(1-|\phi_\sigma|)(x^2+y^2)} e^{i\pi\phi_\sigma (x_j+x_k) y}, \;\; \phi_\uparrow = -\phi_\downarrow = \phi > 0$$

$M = 1$



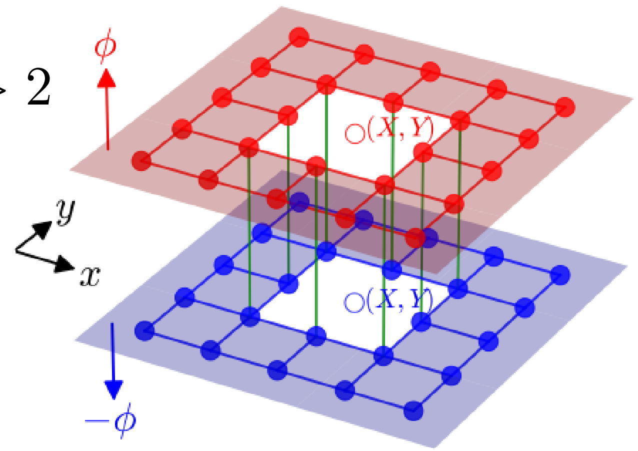


$g = M + 1 = 2$

Single-particle physics

- In the absence of holes: two decoupled KM models, the lowest band of each contains $\phi\, L_x L_y$ eigenstates
  $\rightarrow$ the lowest $2\phi\, L_x L_y$ eigenstates of H_0 are exactly degenerate.

$\phi = 1/q, q > 2$



$L_x \times L_y$

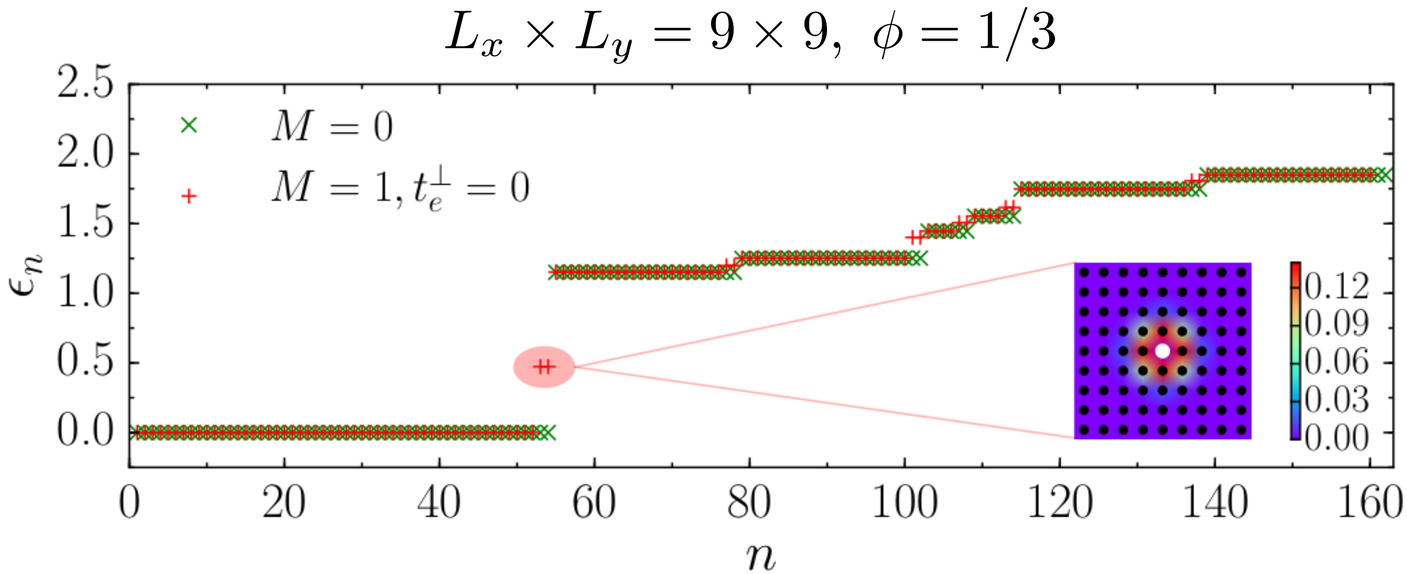
$L_x \times L_y$

- How do the holes change the band structure?

Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

Single-particle physics

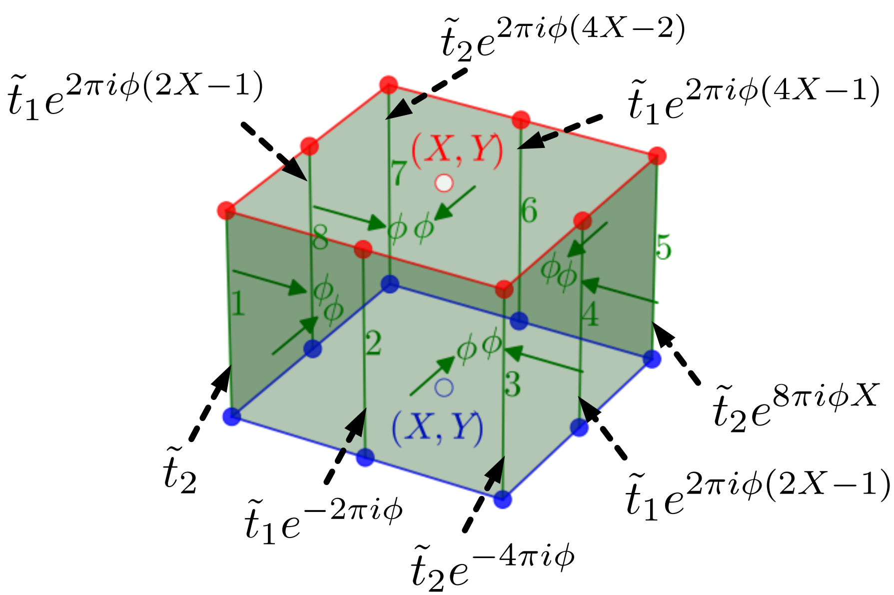
- *M* pairs of holes distort the exactly flat lowest band: without interlayer tunneling, 2*M* states move into the band gap, but other $2\phi\, L_x L_y$-2*M* states stay at original energies.
- The 2*M* ingap states are edge states – remnants of *M* pairs of counter propagating continuum edge modes (not visible in the minimal hole limit).



Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

Single-particle physics

- Interlayer tunneling splits the ingap edge states, such that a band gap is reopened (boundaries are gapped out). But the band structure can be further distorted.
- To make interactions dominant, can we design suitable interlayer tunneling to restore a nearly flat lowest band?

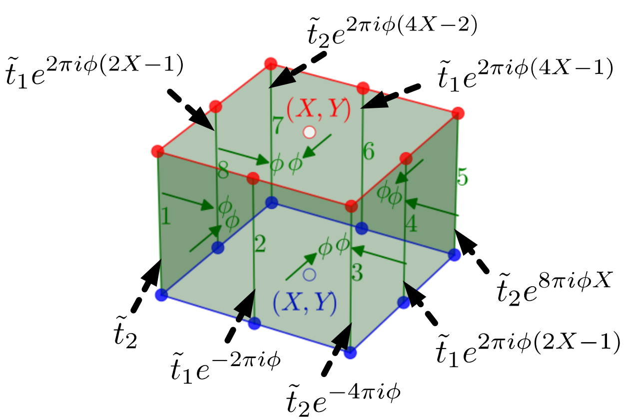


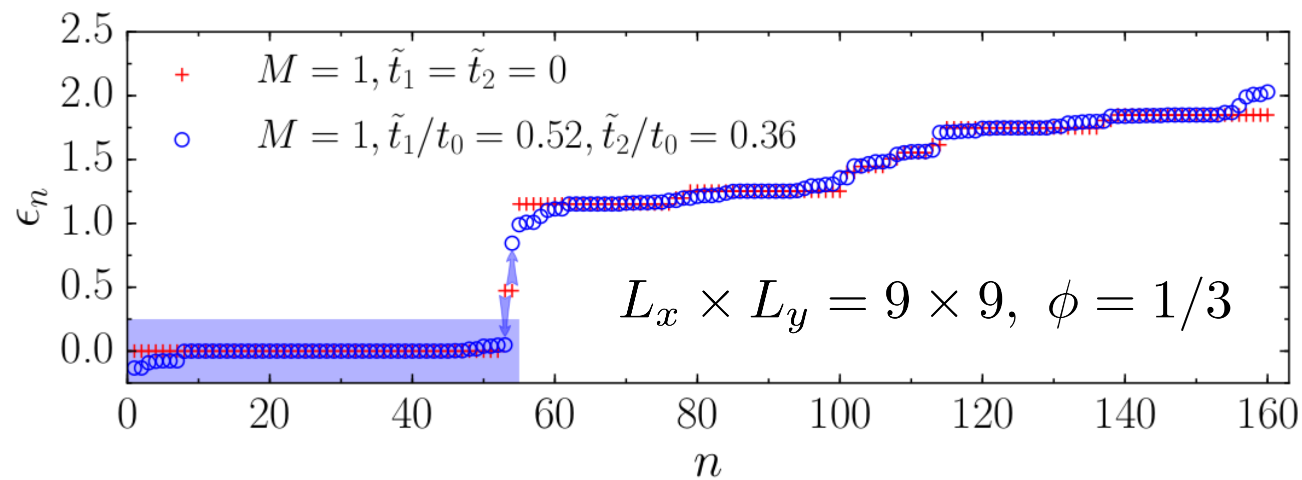
- Two tunneling strengths for the eight edge sites of each hole.
- Suitable tunneling phases to mimic a magnetic field consistent with that in each layer: each vertical plaquette between a pair of holes pierced inwardly by effective flux ϕ.

Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

Single-particle physics

- With suitable tunneling strength, our scheme of interlayer tunneling can indeed restore a flat lowest band containing $2\phi L_x L_y - M$ eigenstates of H_0: a higher-genus flat band.





- With the decreasing of flux density ϕ, we can get a flatter lowest band with weaker tunneling strength.

ϕ	$\tilde{t}_1/t_0$	$\tilde{t}_2/t_0$	f
1/3	0.52	0.36	4.40
1/4	0.42	0.24	5.81
1/5	0.36	0.19	6.94
1/6	0.33	0.15	7.96

Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

The potential FQH states

- What topological states can we stabilize in this new flat band? Due to the relevance with the cold-atom implementation, we focus on the possibility of the $\nu=k/2$ bosonic RR state on a single $g=M+1$ surface.

- In the continuum, the $\nu=k/2$ RR state of N_b bosons on a genus-g surface resides in $N_s=2N_b/k-(1-g)$ exactly degenerate single-particle states in the lowest Landau level.

  Wen and Zee (1992)

- The correct system size in our lattice model:

$$N_s = 2\phi L_x L_y - M, g = M + 1 \to N_b = k(\phi L_x L_y - M)$$

- Switch on (k+1)-body onsite repulsion between bosons:

$$H_{\text{int}} = U \sum_{\sigma=\uparrow,\downarrow} \sum_{i\notin\mathcal{R}} : n_{i,\sigma} n_{i,\sigma} \cdots n_{i,\sigma} :$$

Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

Topological degeneracy

- We use exact diagonalization to identify the nature of the ground state.
- Can we observe ground-state topological degeneracies consistent with the $\nu = k/2$ RR state?

Ardonne, Bergholtz, Kailasvuori, and Wikberg (2008)

State	GS degeneracy	g=2 (M=1)	g=3 (M=2)
k=1 Laughlin	2^g	4	8
k=2 MR	$2^{g-1}(2^g+1)$	10	36
k=3 Z_3 RR	$2[(5+\sqrt{5})^{g-1}+(5-\sqrt{5})^{g-1}]$	20	120

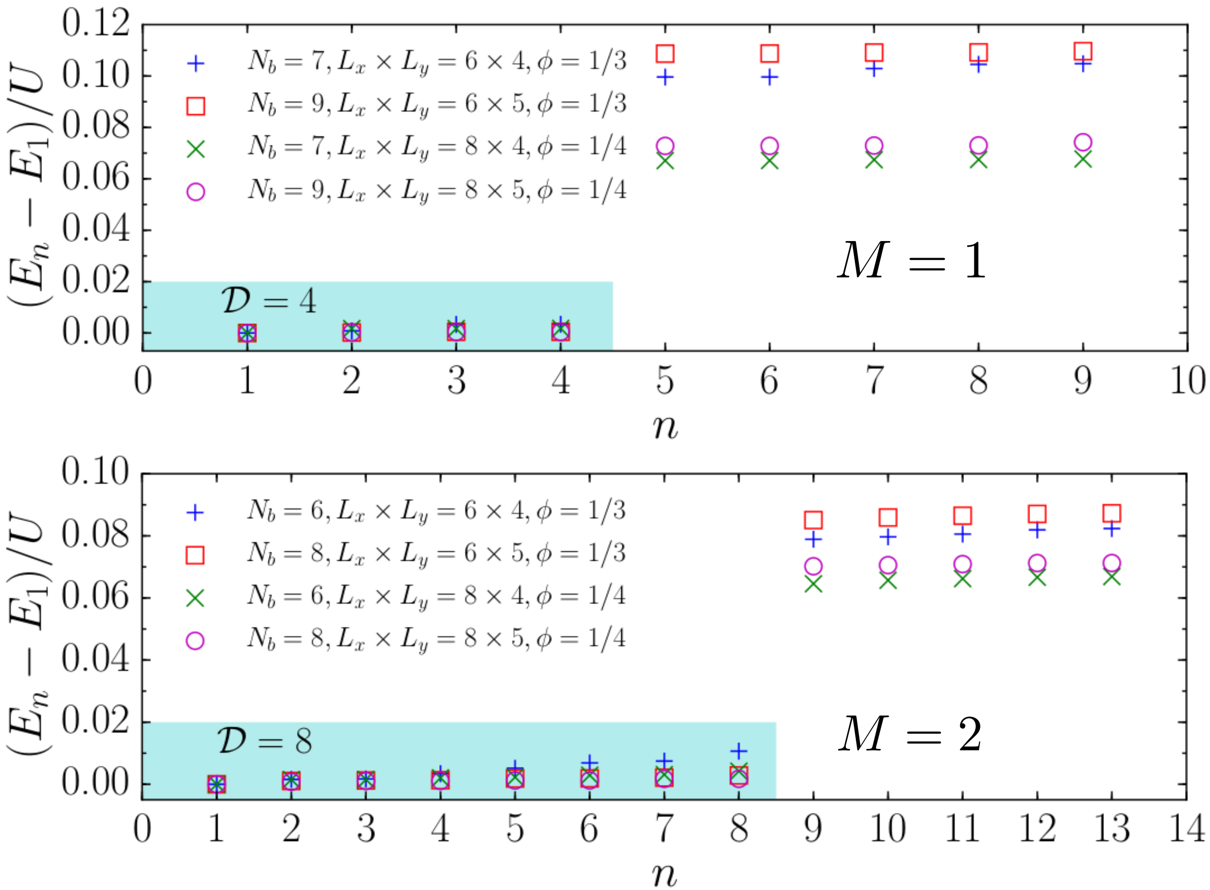
- For large numerical efficiency, we project the interaction to the restored flat band, and neglect its dispersion.

Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

k=1: Laughlin state ?

State	GS degeneracy	*g*=2 (*M*=1)	*g*=3 (*M*=2)
k=1 Laughlin	2^g	4	8

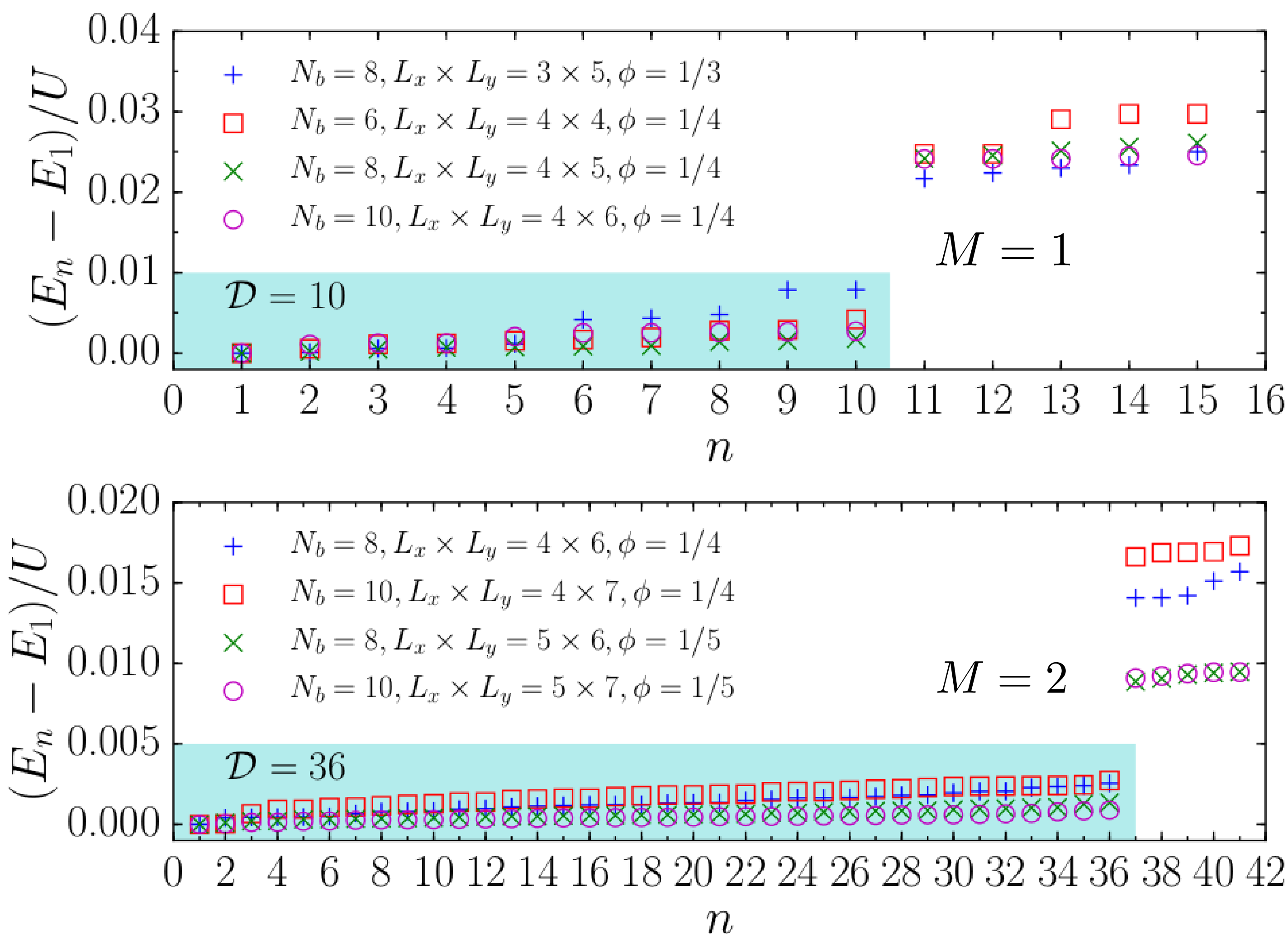
- Nice (approximate) ground-state degeneracies exist!
- For a fixed ϕ, the ground-state splitting is reduced relative to the gap as the system size is increased.



Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

k=2: Moore-Read state ? ✓

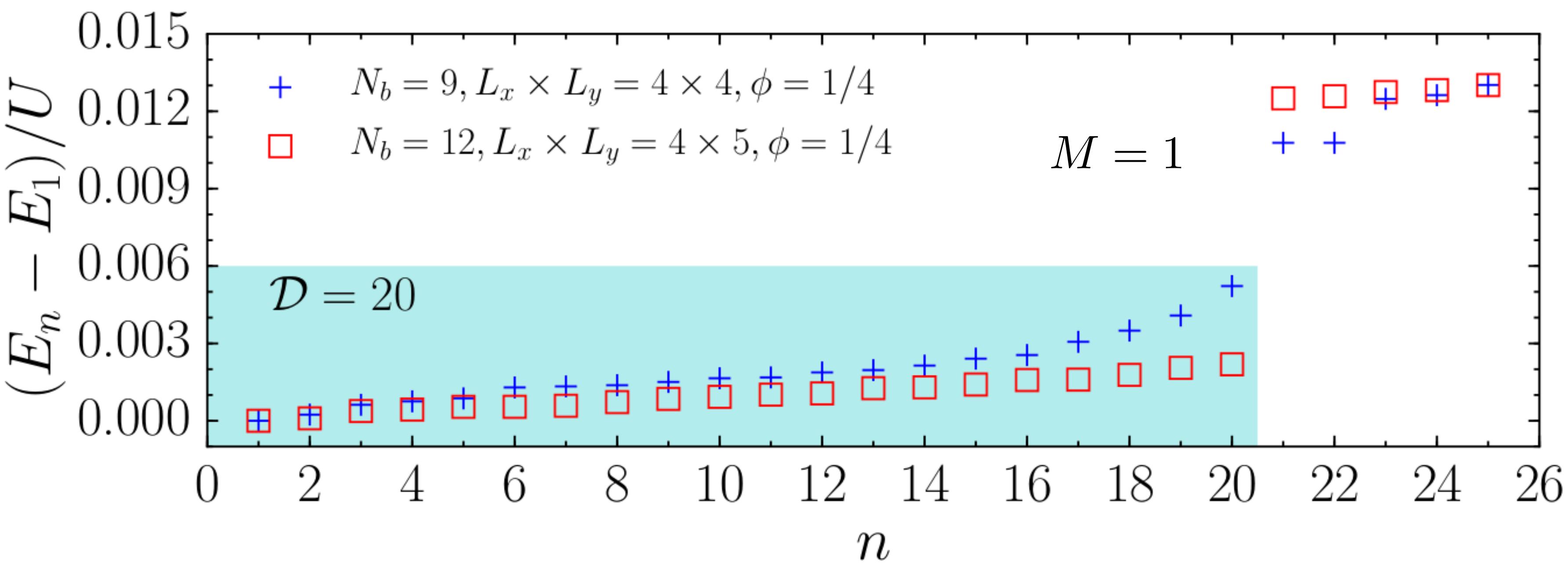
State	GS degeneracy	g=2 (M=1)	g=3 (M=2)
k=2 MR	$2^{g-1}(2^g+1)$	10	36



Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

k=3: Z_3 Read-Rezayi state ?

State	GS degeneracy	g=2 (M=1)	g=3 (M=2)
k=3 Z_3 RR	$2[(5+\sqrt{5})^{g-1}+(5-\sqrt{5})^{g-1}]$	20	120



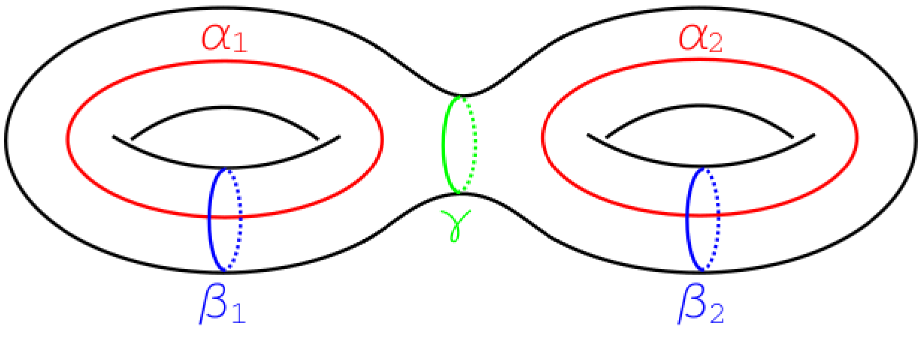
- Compared with the Laughlin case, non-Abelian states require multibody interactions, and lower flux densities.

Evidence beyond degeneracy?

- The modular *S* matrix contains the information of anyonic statistics of underlying quasiparticles.
- For the simplest case of Abelian states on a *g*=2 surface, the direct product of two *S* matrices gives the transformation between two special bases of the ground-state manifold.

$$|a'cb'\rangle_{\beta_1\gamma\beta_2} = \sum_{a,b} \mathcal{S}_{aa'}\mathcal{S}_{bb'}|acb\rangle_{\alpha_1\gamma\alpha_2}$$

$|acb\rangle_{\alpha_1\gamma\alpha_2}$ – quasiparticles *a*, *c* and *b* threading the nonintersecting, noncontractible circles α_1, γ, and α_2, respectively.
– *c* must be identity for Abelian states.

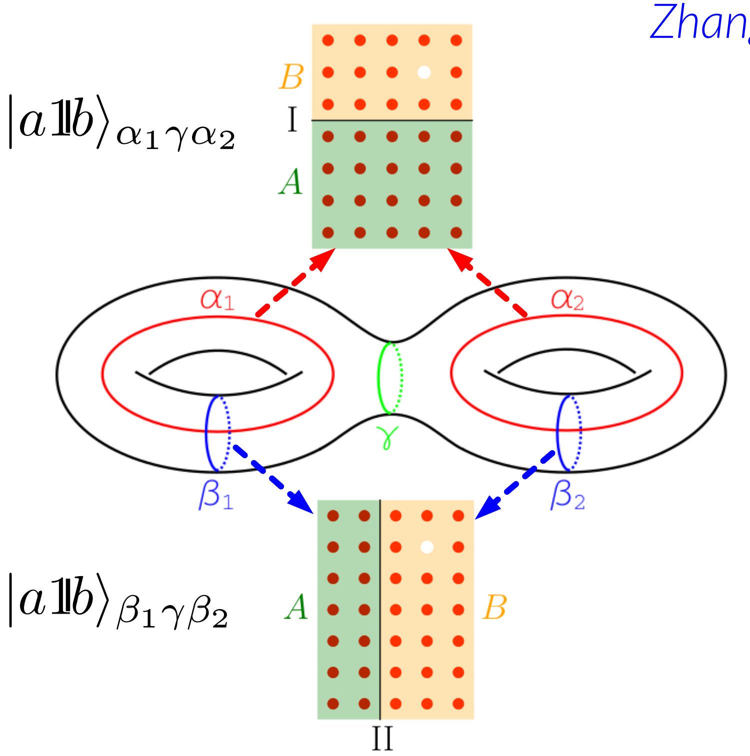


Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

Quasiparticle statistics

- The basis states are minimally entangled states with respect to a specific bipartition of the whole system.

Zhang, Grover, Turner, Oshikawa, and Vishwanath (2012)

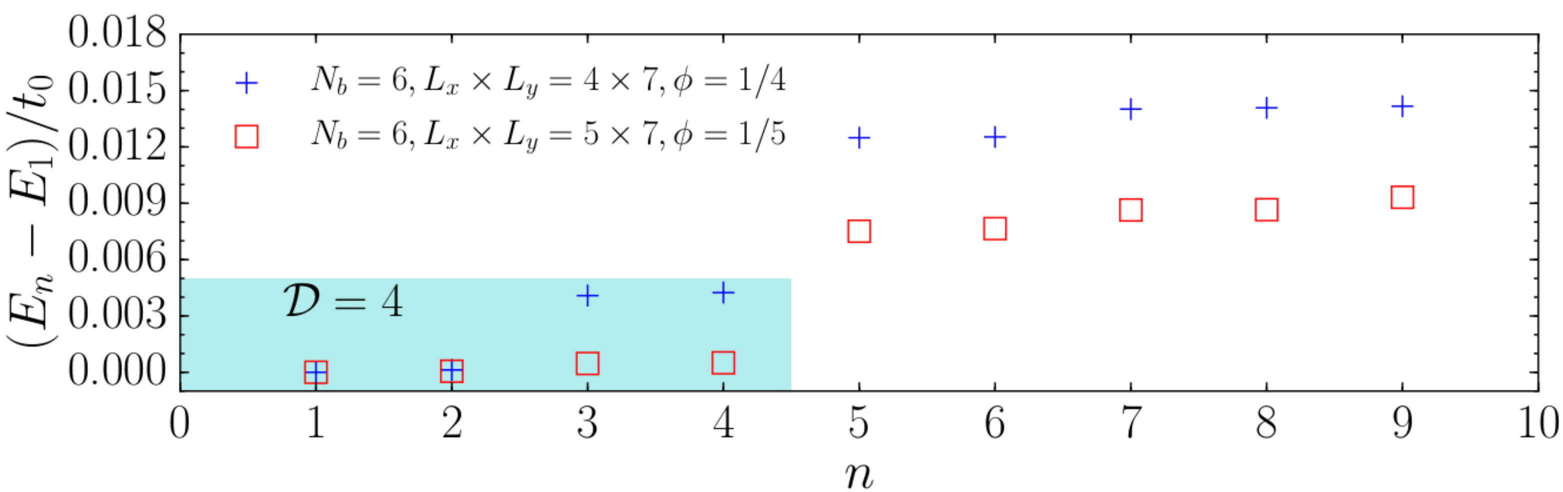


Cuts go through both layers.
A and *B* are bilayer subsystems.

- We must diagonalize the full Hamiltonian in real space.
  – limited to smaller systems compared with band projection
  – hardcore condition imposed to increase numerical efficiency

Quasiparticle statistics

- We focus on the ½ filling with M=1 pair of holes (i.e., g=2). We do observe four-fold degeneracies by real-space ED!



- We minimize the Renyi-2 entropy $S_2 = -\ln \mathrm{Tr}\rho_A^2$ in this ground-state subspace for two cuts.
For each cut, we indeed find four (almost orthogonal) minimally entangled states with similar S_2.

	$N_b = 6, L_x \times L_y = 4 \times 7, \phi = 1/4$	$N_b = 6, L_x \times L_y = 5 \times 7, \phi = 1/5$
cut I	$S_2 = 1.37908, 1.36319, 1.36319, 1.37908$	$S_2 = 1.76580, 1.71694, 1.71694, 1.76580$
cut II	$S_2 = 2.86280, 2.82103, 2.91412, 2.86280$	$S_2 = 3.12519, 3.27780, 3.27780, 3.42259$
cut between two layers	$S_2 = 0.357869, 0.357887, 0.530498, 0.536709$	$S_2 = 0.327539, 0.327342, 0.350278, 0.356425$

Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

Quasiparticle statistics

- What is the overlap matrix between MESs? $\mathcal{O}_{mn} = \langle \Sigma^{\mathrm{I}}_m | \Sigma^{\mathrm{II}}_n \rangle$

$L_x \times L_y = 4 \times 7$

$$\mathcal{O} \approx \begin{pmatrix} 0.523 & 0.525 & 0.517 & 0.523 \\ 0.477 & -0.472 & 0.483 & -0.477 \\ 0.477 & 0.472 & -0.483 & -0.477 \\ 0.523 & -0.525 & -0.517 & 0.523 \end{pmatrix}$$

$L_x \times L_y = 5 \times 7$

$$\mathcal{O} \approx \begin{pmatrix} 0.493 & 0.494 & 0.494 & 0.496 \\ 0.507 & -0.505 & 0.505 & -0.503 \\ 0.507 & 0.505 & -0.505 & -0.503 \\ 0.493 & -0.494 & -0.494 & 0.496 \end{pmatrix}$$

$$\boxed{|a'cb'\rangle_{\beta_1\gamma\beta_2} = \sum_{a,b} \mathcal{S}_{aa'}\mathcal{S}_{bb'} |acb\rangle_{\alpha_1\gamma\alpha_2}}$$

- *O* is very close to the direct product of two modular *S* matrices of the Laughlin state!

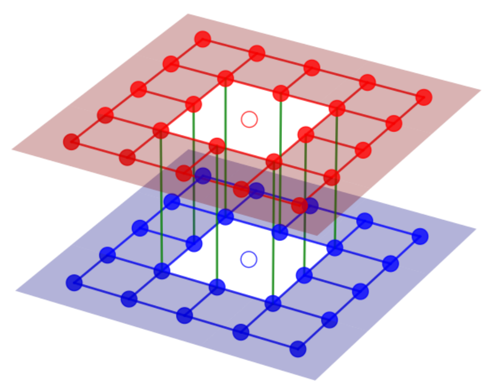
$$\mathcal{S} \otimes \mathcal{S} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

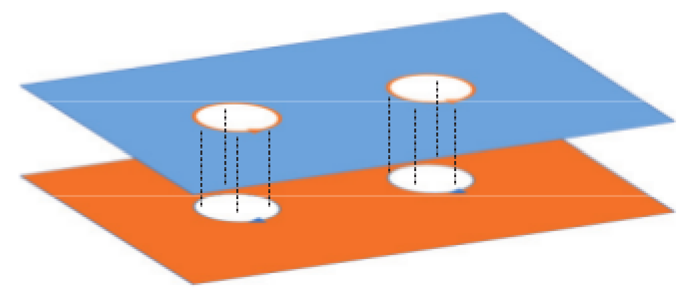
- Together with the nonzero interlayer entropy, we confirm that the ground state is the Laughlin state on a single *g*=2 surface.

Z. Liu and E. J. Bergholtz, arXiv:1901.05119 (2019)

KM model + gapped boundaries

- The key message from our results:
  the idea of gapped boundaries works even in the most extreme lattice limit with minimal holes + relatively high flux densities, even though it would appear doubtful that the insights from low-energy field theory would apply in this limit even qualitatively.



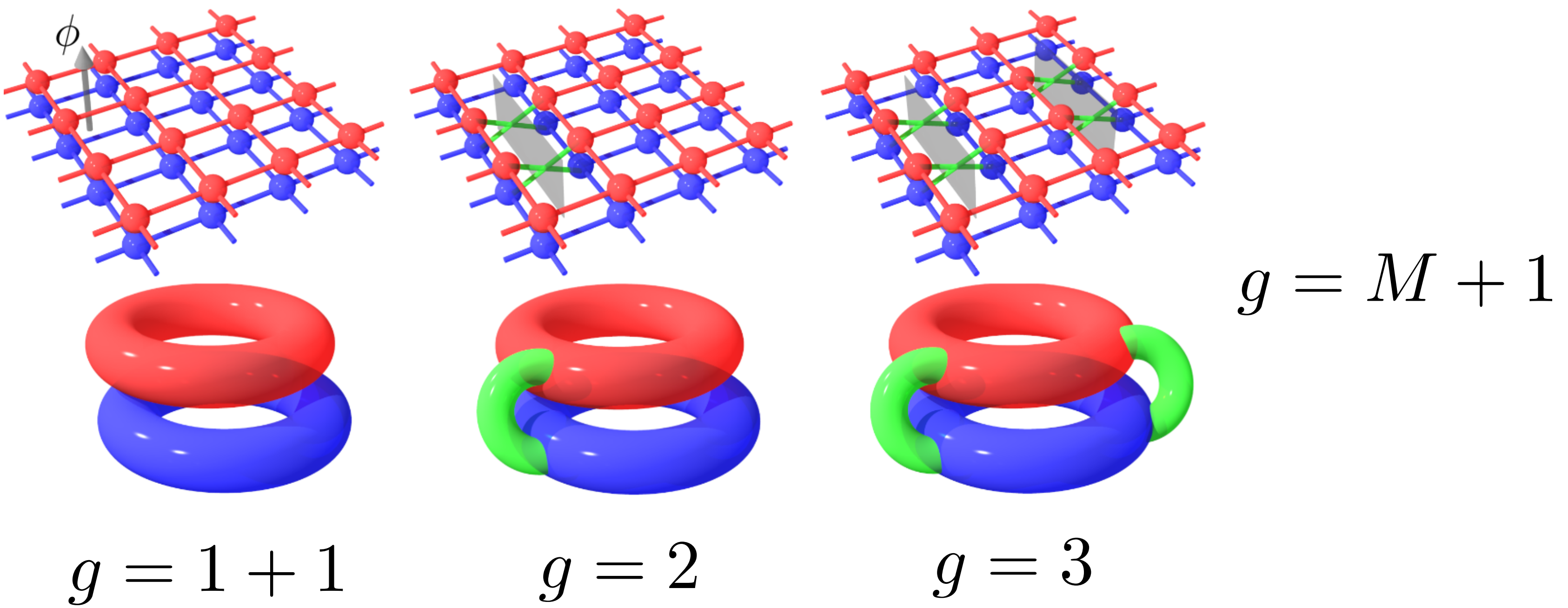


- This limit is the most attractive regime from a practical point of view since the involved energy scales are much larger than in the dilute limit. Our results are thus encouraging in the context of experimental realizations.

Z. Liu, E. J. Bergholtz, G. Möller, PRL ***119****, 106801 (2017)*

KM model + defects

- Introduce defects in two layers of KM model with the same chirality.
  - A pair of defects is connected by a straight branch cut. Hopping across a branch cut is switched from intralayer to interlayer.
  - With wormhole-like branch cuts, we again have an effective high-genus surface.



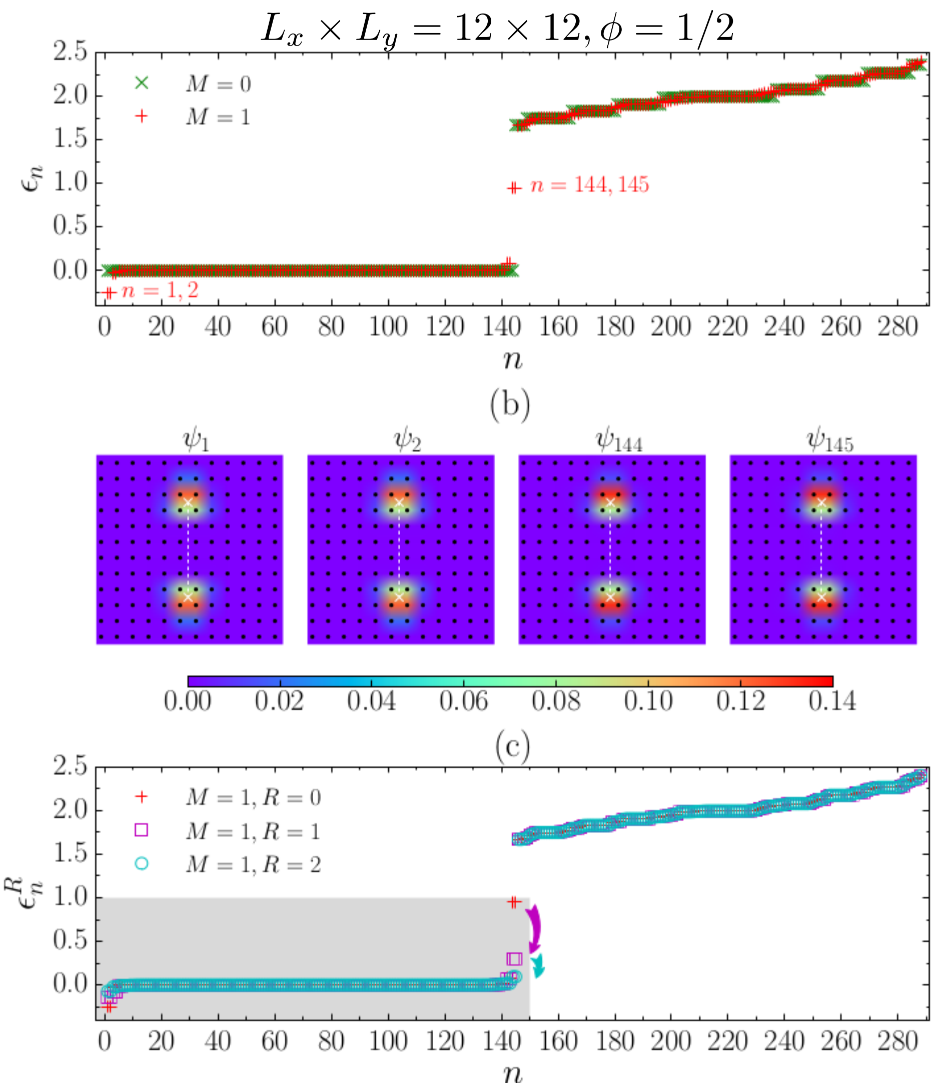
What we find for defects

- Single-particle states localized at defects exist in the band gap

- The lowest flat band can be restored by a local potential around defects.

$$V = -\sum_{n=1}^{2\phi L_x L_y + M} \epsilon_n \mathcal{T}_R(|\psi_n\rangle\langle\psi_n|)$$

- Switching on interactions in this new flat band also gives bosonic RR states.



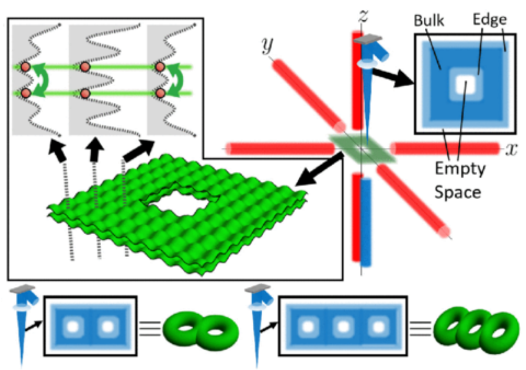
*See Z. Liu, E. J. Bergholtz, G. Möller, PRL **119**, 106801 (2017) for details.*

Relevance to experiments

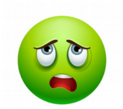
Good news 

– Long-range hopping is NOT necessary! GS degeneracies still exist for the Hofstadter model (but with larger finite-size effects).

– Key ingredients available in experiments:
the Hofstadter model;
lattice shaking / pairs of beams → bilayer;
beam shaping → holes, branch cuts
→ high-genus surface



Kim, Zhu, Porto, and Hafezi (2018)

Challenges 

– More realistic schemes to restore the lowest flat band, which may be very important for making interactions dominant.

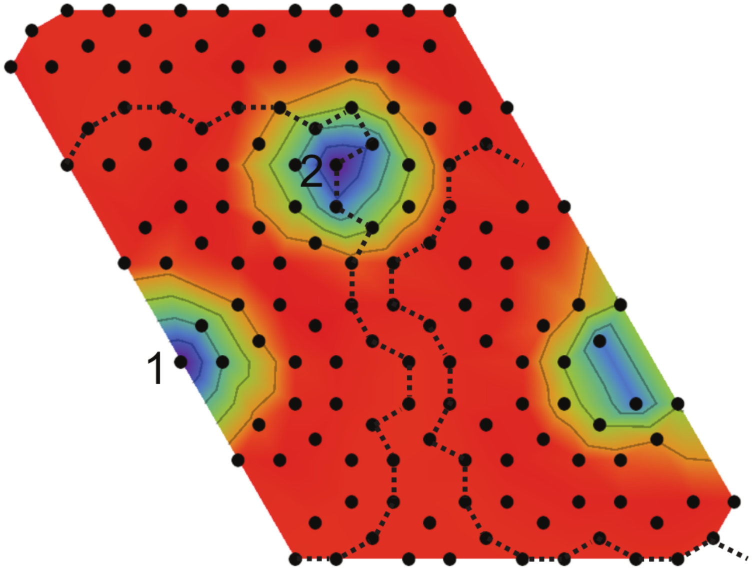
– Multibody interactions needed for non-Abelian states.

– Realistic planar geometry works?

Outlook

- More complicated states if we use a higher Chern number model as the building block?
- Microscopic lattice models of dislocations, pairing?
- Microscopic investigation of anyons in lattice FQH systems: the quasiparticle tunneling, the interplay between intrinsic anyons and defects/gapped boundaries, ...

Anyons in Quantum Many-Body Systems



Microscopic characterization of Abelian quasiholes on lattices:
density profile, quasihole size, braiding, effective lattice magnetic length:

$$\ell_B^{\mathrm{lat}} = \sqrt{A/(2\pi)}$$

Zhao Liu, R. N. Bhatt, and Nicolas Regnault (2015)
Błażej Jaworowski, Nicolas Regnault, and Zhao Liu (2019)

Thank you!
Welcome to visit Hangzhou in the future!

a sister city of Dresden