Dyonic Zero-Energy Modes

Michele Burrello

Niels Bohr International Academy

UNIVERSITY OF COPENHAGEN

Center for Quantum Devices

THE VELUX FOUNDATIONS
VILLUM FONDEN VELUX FONDEN

Niels Bohr Institute
University of Copenhagen

21st January 2019

Work in collaboration with:
M. I. K. Munk, A. Rasmussen, M. B., PRB 98, 245135 (2018)

Morten I. K. Munk
NBI, Copenhagen

Asbjørn Rasmussen
DTU, Copenhagen

What is this talk about?

M. I. K. Munk, A. Rasmussen, M. B., PRB 98, 245135 (2018)

Ising model **JW** ⟷ **Kitaev chain**

What is this talk about?

M. I. K. Munk, A. Rasmussen, M. B., PRB 98, 245135 (2018)

Symmetry	Model with Symmetry Breaking		
$\mathbb{Z}_2$	**Ising model**	**JW** ⟷	**Kitaev chain**

What is this talk about?

M. I. K. Munk, A. Rasmussen, M. B., PRB 98, 245135 (2018)

Symmetry	Model with Symmetry Breaking		Topological Model	Zero-energy Edge-modes
$\mathbb{Z}_2$	**Ising model**	JW ⟷	**Kitaev chain**	**Majorana**

What is this talk about?

M. I. K. Munk, A. Rasmussen, M. B., PRB 98, 245135 (2018)

Symmetry	Model with Symmetry Breaking		Topological Model	Zero-energy Edge-modes
$\mathbb{Z}_2$	Ising model	JW ⟺	Kitaev chain	Majorana
$\mathbb{Z}_N$	Chiral Potts		Fendley chain	Parafermions

What is this talk about?

M. I. K. Munk, A. Rasmussen, M. B., PRB 98, 245135 (2018)

Symmetry	Model with Symmetry Breaking		Topological Model	Zero-energy Edge-modes
$\mathbb{Z}_2$	Ising model	JW ⟺	Kitaev chain	Majorana
$\mathbb{Z}_N$	Chiral Potts		Fendley chain	Parafermions
Non-Abelian $\mathbb{G}$	Flux-Ladder model	$\mathbb{G}$JW ⟺	Dyonic model	Dyonic modes

What is this talk about?

M. I. K. Munk, A. Rasmussen, M. B., PRB 98, 245135 (2018)

Symmetry	Model with Symmetry Breaking		Topological Model	Zero-energy Edge-modes
$\mathbb{Z}_2$	Ising model	JW ⟷	Kitaev chain	Majorana
$\mathbb{Z}_N$	Chiral Potts		Fendley chain	Parafermions
Non-Abelian $\mathbb{G}$	Flux-Ladder model	$\mathbb{G}$JW ⟷	Dyonic model	Dyonic modes

Dyon:

Particle with both (non-Abelian!) *magnetic* flux and *electric* charge

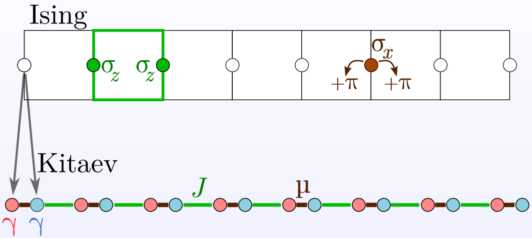
$$H = -J\sum_{r=1}^{L}\sigma_{z,r}\sigma_{z,r+1} - \mu\sum_{r=1}^{L}\sigma_{x,r}$$

- $\mathbb{Z}_2$ global symmetry:

  $$\mathcal{Q} = \prod_r \sigma_{x,r}$$

  such that $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$
- Ferromagnetic phase with broken symmetry
- Quasi-degenerate ground states, for $\mu = 0$:

$$||\psi_{\rm gs1}\rangle\rangle = |\uparrow\ldots\uparrow\rangle\,, \quad ||\psi_{\rm gs2}\rangle\rangle = |\downarrow\ldots\downarrow\rangle$$



$$H = -iJ\sum_r^L \gamma_{2r}\gamma_{2r+1} - i\mu\sum_r^L \gamma_{2r-1}\gamma_{2r}$$

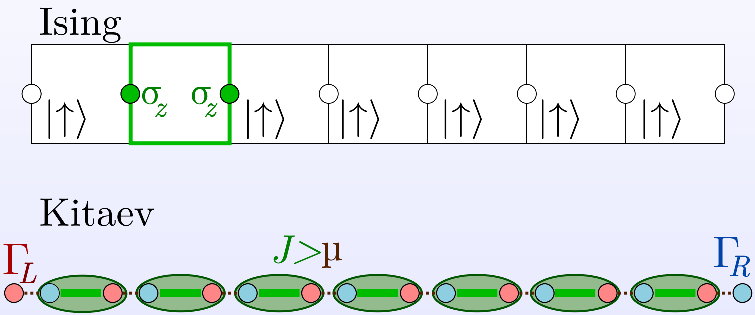
- Jordan-Wigner Transformation:

$$\gamma_{2r-1} = \sigma_{z,r}\prod_{j<r}\sigma_{x,j}\,, \qquad \gamma_{2r} = -i\sigma_{z,r}\sigma_{x,r}\prod_{j<r}\sigma_{x,j}\,.$$

- Majorana operators: $\{\gamma_i,\gamma_j\} = 2\delta_{ij}\,, \quad \gamma_i^2 = \mathbb{1}\,.$
- Symmetry $\mathcal{Q} = \prod_r \sigma_{x,r} \longrightarrow$ Parity $(-1)^F = \prod_r i\gamma_{2r-1}\gamma_{2r}$

The 1D Ising and Kitaev models

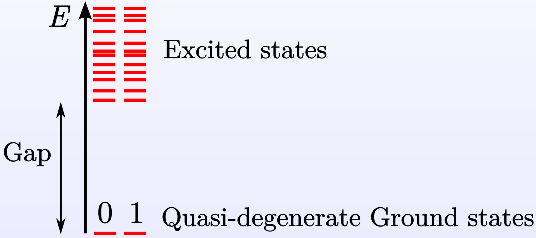
Ferromagnetic - topological phase:



1D Topological Order

The Kitaev model is topological:

1. Quasi-degenerate ground states $|\psi_{q_i}\rangle$:



1D Topological Order

The Kitaev model is topological:

1. Quasi-degenerate ground states $|\psi_{q_i}\rangle$
2. Robustness against bulk local operators $V(r)$:

$$\langle\psi_{q_1}|V(r)\psi_{q_2}\rangle = \bar{V}\delta_{q_1,q_2} + c(r, q_1, q_2)$$

where c decays exponentially with the distance from the edges.

1D Topological Order

The Kitaev model is topological:

1. Quasi-degenerate ground states $|\psi_{q_i}\rangle$
2. Robustness against bulk local operators $V(r)$:

$$\langle\psi_{q_1}|V(r)\psi_{q_2}\rangle = \bar{V}\delta_{q_1,q_2} + c(r, q_1, q_2)$$

   where c decays exponentially with the distance from the edges.
3. Local indistinguishability. For any local observable $O(r)$:

$$\langle\psi_{q_1}|O(r)\psi_{q_2}\rangle = \bar{O}\delta_{q_1,q_2} + o(L, q_1, q_2)$$

   where o decays exponentially with the system size.

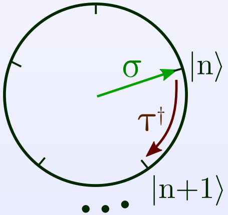
Chiral Potts model
Kadanoff and Fradkin, N. Phys. B 1980; Fendley, JSTAT 2012

- Topological model with $\mathbb{Z}_N$ symmetry:

$$H = -J \sum_{r=1}^{L} \left(e^{i\phi} \sigma^{\dagger}_{r+1} \sigma_r + \text{H.c.} \right) - \mu \sum_{k=1}^{N-1} \sum_{r=1}^{L} \tau_r^k \,,$$

- N states for each site $\{|1\rangle, \dots, |N\rangle\}$, such that:

$$\sigma|n\rangle = e^{i\frac{2\pi n}{N}} |n\rangle$$
$$\tau^{\dagger}|n\rangle = |n \oplus 1\rangle$$



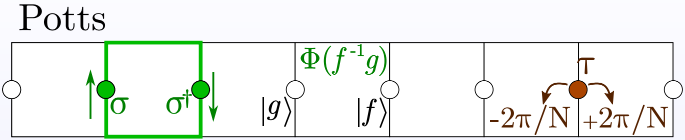
Clock operators:

$$\sigma\tau = e^{i\frac{2\pi}{N}} \tau\sigma \,, \quad \sigma^N = \tau^N = \mathbb{1}$$

- Global Symmetry: $\mathcal{Q}_n = \prod_r \tau_r^n$
- For $J \gg \mu$ and $|\phi| < \pi/N$ we get a ferromagnetic phase.
- N ground states with aligned clock degrees of freedom:

$$||\psi_n\rangle\rangle = |nn\dots n\rangle$$

Chiral Potts model and flux ladder



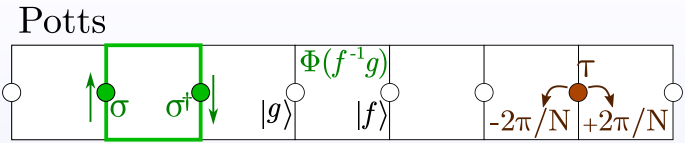
$$H = -J\sum_{r=1}^{L}\left(e^{i\phi}\sigma_{r+1}^{\dagger}\sigma_r + \mathrm{H.c.}\right) - \mu\sum_{k=1}^{N-1}\sum_{r=1}^{L}\tau_r^k\,,$$

- Each state in the chain is associated to an element $g \in \mathbb{Z}_N$:

$$\{|n_g\rangle,\ \text{s.t. } n_g = 1,\dots,N\} \longrightarrow \{|g\rangle,\ \text{s.t. } g \in \mathbb{Z}_N\}$$

- Each plaquette is associated with a flux $\Phi = 2\pi\left(n_g - n_f\right)/N$

Chiral Potts model and flux ladder



$$H = -J \sum_{r=1}^{L} \left(e^{i\phi} \sigma^{\dagger}_{r+1} \sigma_r + \text{H.c.} \right) - \mu \sum_{k=1}^{N-1} \sum_{r=1}^{L} \tau_r^k \,,$$

- Each state in the chain is associated to an element $g \in \mathbb{Z}_N$:

$$\{ |n_g\rangle, \text{ s.t. } n_g = 1, \ldots, N \} \longrightarrow \{ |g\rangle, \text{ s.t. } g \in \mathbb{Z}_N \}$$

- Each plaquette is associated with a flux $\Phi = 2\pi \left(n_g - n_f \right) / N$
- For $\phi = 0$, the plaquette $J-$term returns an energy:

$$m_\Phi = -2J \cos \left[2\pi \left(n_g - n_f \right) / N \right]$$

- The ferromagnetic ground states have only trivial fluxes
- The $\mu-$term varies the fluxes in the plaquettes

Can we generalize it to a non-Abelian gauge group G?

The building blocks from lattice gauge theory (E. Zohar and M.B., PRD 2015)

- We will build a model symmetric under any transformation $h \in G$.
- For each site we consider $|G|$ states: $\{|g\rangle$ with $g \in G\}$.

- We will build a model symmetric under any transformation $h \in G$.
- For each site we consider $|G|$ states: $\{|g\rangle$ with $g \in G\}$.

- **Gauge** operators: $\tau^{n_h} \rightarrow \theta_h$:

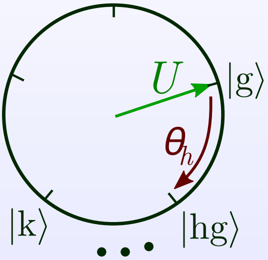
$$\theta_h |g\rangle = |hg\rangle \,,$$
$$\theta_h^\dagger |g\rangle = |h^{-1}g\rangle$$

θ constitutes a left group multiplication.

- **Connection** operators: $\sigma \rightarrow U^K_{mn}$.
U^K_{mn} is a matrix of operators such that:

$$U^K_{mn}|g\rangle = D^K_{mn}(g)|g\rangle \,,$$
$$U^{K\dagger}_{mn}|g\rangle = D^{K\dagger}_{mn}(g)|g\rangle$$

where $D^K(g)$ is the unitary matrix representation of g with respect to the irreducible representation K.



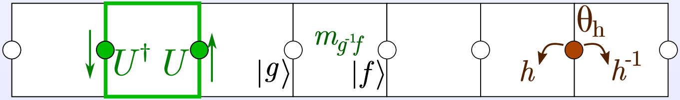
Commutation rule:

$$\theta_h^\dagger U_{mn} = D_{mm'}(h) U_{m'n} \theta_h^\dagger$$

From the clock model to the flux ladder

$$H_{\text{CLOCK}} = -J\sum_r \left(e^{i\phi}\sigma^\dagger_{r+1}\sigma_r + \text{H.c.}\right) - \mu\sum_r\sum_{k=1}^{N-1}\tau_r^k$$

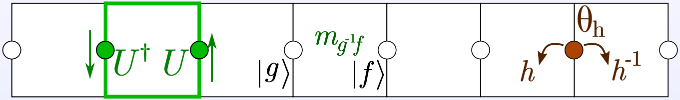
$$H_{\text{FLUX}} = \underbrace{-J\left(\sum_r \text{Tr}[U^F(r+1)CU^{F\dagger}(r)] + \text{H.c.}\right)}_{H_J}\underbrace{-\mu\sum_r\sum_{g\neq e\in G}\chi^A(g^{-1})\theta_g(r)}_{H_\mu}$$



- H_J assigns a mass m_h to all the $h \in G$ fluxes
- C is a unitary matrix (parameter). It breaks time reversal
- H_μ nucleates pairs of fluxes and gives them kinetic energy
- $\chi^A(g) = \text{Tr}\left[D^A(g)\right]$ depends on (the conjugacy class of) g

Flux ladder

$$H_{\mathsf{FLUX}} = \underbrace{-J\left(\sum_r \mathrm{Tr}[U^F(r+1)\,C\,U^{F\dagger}(r)] + \mathrm{H.c.}\right)}_{H_J} \underbrace{-\mu \sum_r \sum_{g\neq e\in G} \chi^A(g^{-1})\theta_g(r)}_{H_\mu}$$



- For $C = \mathbb{1}$:

$|g_r\rangle_r = |g_{r+1}\rangle_{r+1} \quad \Rightarrow \quad$ No flux: H_J is minimized

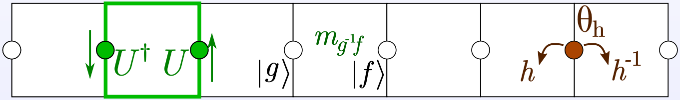
$|g_r\rangle_r \neq |g_{r+1}\rangle_{r+1} \quad \Rightarrow \quad$ Non-trivial flux Φ!

H_J defines the masses m_Φ

- We want C such that $m_h \neq m_k$ for $h \neq k$.

Flux ladder

$$H_{\text{FLUX}} = \underbrace{-J\left(\sum_r \mathrm{Tr}[U^F(r+1)\,C\,U^{F\dagger}(r)] + \mathrm{H.c.}\right)}_{H_J} \underbrace{-\mu \sum_r \sum_{g\neq e\in G} \chi^A(g^{-1})\theta_g(r)}_{H_\mu}$$



- For $\mu = 0$ and $C \approx \mathbb{1}$ we are in a ferromagnetic phase.
- $|G|$ degenerate ground states with parallel degrees of freedom:

$$||\Psi_g\rangle\rangle = |gg\ldots g\rangle \quad \forall g \in G$$

- Global (broken) left G symmetry
- For a weak $\mu \ll J$, the ground states split with $\Delta E \propto \frac{\mu^L}{J^{L-1}}$

Example: $G = S_3$

- S_3 is the group of the triangle with 6 elements:

$$R(n) = \begin{pmatrix} \cos\frac{2\pi n}{3} & -\sin\frac{2\pi n}{3} \\ \sin\frac{2\pi n}{3} & \cos\frac{2\pi n}{3} \end{pmatrix}, \; I(n) = \begin{pmatrix} \cos\frac{2\pi n}{3} & \sin\frac{2\pi n}{3} \\ \sin\frac{2\pi n}{3} & -\cos\frac{2\pi n}{3} \end{pmatrix}, \; n = 0, 1, 2$$

- Generators: $c = R(1), b = I(0)$, such that:

$$g(p,q) = b^p c^q \, , \; p = 0, 1 \, , \; q = 0, 1, 2 \; \Rightarrow |g(p,q)\rangle \equiv |p\rangle \otimes |q\rangle$$

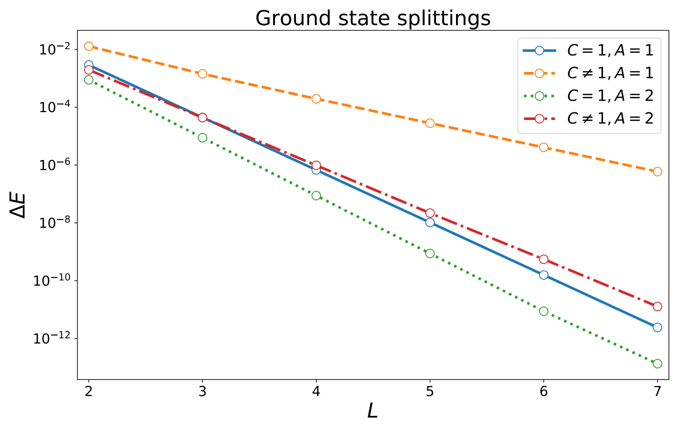
- Gauge operators:

$$\theta_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1},$$

$$\theta_c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

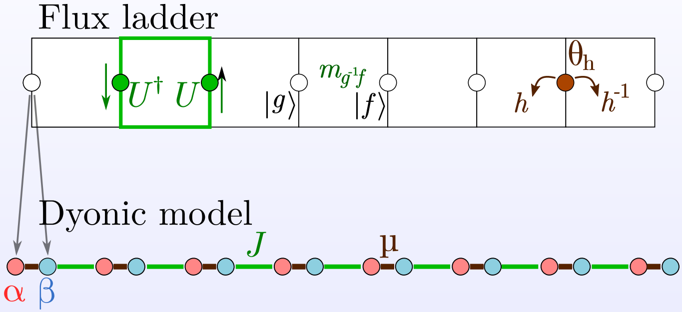
Example: $G = S_3$

$\mu/J = 0.03$



How to build a topological model?

So far we have a ferromagnetic model with G symmetry



We need a Jordan-Wigner transformation to define operators α and β:

$$\alpha^{F,A}_{g,mn,\mathsf{ab}}(2r-1) = \mathcal{L}^{A}_{g,\mathsf{ab}}(r-1)U^{F\dagger}_{mn}(r)\,,$$
$$\beta^{F,A}_{g,mn,\mathsf{ab}}(2r) = \mathcal{L}^{A}_{g,\mathsf{ab}}(r)U^{F\dagger}_{mn}(r)\,,$$

The G-Jordan-Wigner transformation

Irreducible representations

$$\alpha^{F,A}_{g,mn,\mathsf{ab}}(2r-1) = \mathcal{L}^{A}_{g,\mathsf{ab}}(r-1)\, U^{F\dagger}_{mn}(r)$$

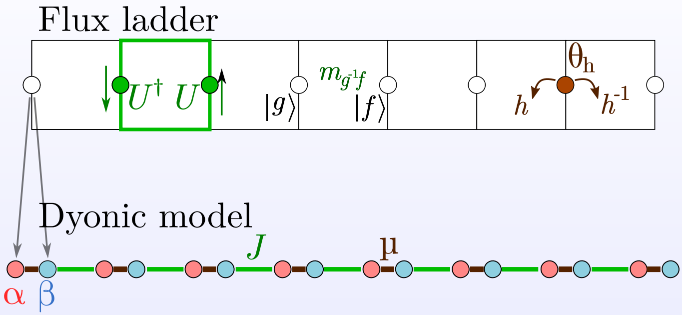
Group element

Jordan-Wigner string

- $\mathcal{L}^A_g(r)$ is a Jordan-Wigner string which adds a flux g in the r^{th} plaquette.
- The string is built from "dressed" gauge transformations Θ^A_g:

$$\mathcal{L}^A_g(r) = \prod_{j\leq r} \Theta^{A\dagger}_g(j)\,, \qquad \Theta^A_g(j) = U^{A\dagger}(j)\theta_g U^A(j)$$

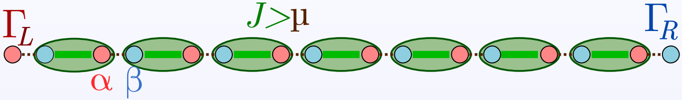
Θ^A_g is defined based on non-Abelian dualities
(Cobanera, Ortiz, Knill 2013)



$$H_{\mathsf{Dyon}} = -\frac{J}{\dim(A)} \left(\sum_r \mathrm{Tr}_F\, \mathrm{Tr}_A \left[\alpha_h^\dagger(2r+1) C \beta_h(2r) \right] + \mathrm{H.c.} \right) - \frac{\mu}{\dim(F)} \sum_r \sum_{g \neq e \in G} \mathrm{Tr}_F\, \mathrm{Tr}_A \left[\beta_g^\dagger(2r) \alpha_g(2r-1) D^{F\dagger}(g) \right]$$

Dyonic model: Topological phase

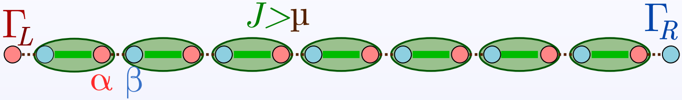
$\mu = 0$



- For $\mu = 0$, $\alpha(1)$ and $\beta(2L)$ do not enter the Hamiltonian: **Dyonic zero-energy edge modes!**
- In the bulk $\mathrm{Tr}[\alpha_h^\dagger(2r+1)\beta_h(r)] = \dim(A)$: **All the ground states share the same bulk**

Dyonic model: Topological phase

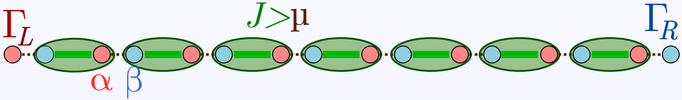
$\mu = 0$



- For $\mu = 0$, $\alpha(1)$ and $\beta(2L)$ do not enter the Hamiltonian: **Dyonic zero-energy edge modes!**
- In the bulk $\mathrm{Tr}[\alpha_h^\dagger(2r+1)\beta_h(r)] = \dim(A)$: **All the ground states share the same bulk**
- Bulk operators are either trivial or they create fluxes: **Bulk operators do not cause ground-state transitions**
- The only observables distinguishing the ground states are built with $\alpha(1)$ <u>and</u> $\beta(2L)$: **Local indistinguishability!**

Dyonic model: Topological phase

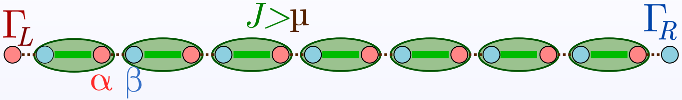
$\mu = 0$



- For $\mu = 0$, $\alpha(1)$ and $\beta(2L)$ do not enter the Hamiltonian: **Dyonic zero-energy edge modes!**
- In the bulk $\mathrm{Tr}[\alpha_h^\dagger(2r+1)\beta_h(r)] = \dim(A)$: **All the ground states share the same bulk**
- Bulk operators are either trivial or they create fluxes: **Bulk operators do not cause ground-state transitions**
- The only observables distinguishing the ground states are built with $\alpha(1)$ and $\beta(2L)$: **Local indistinguishability!**
- For $\mu = 0$ the system is **topological**!

Topological phase and weak zero-energy modes

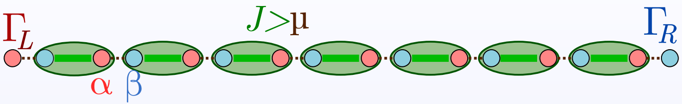
$\mu \ll J$



- $\mu \neq 0$: quasi-adiabatic continuation! (Hastings & Wen 2005)
- The system remains topological!

Topological phase and weak zero-energy modes

$\mu \ll J$

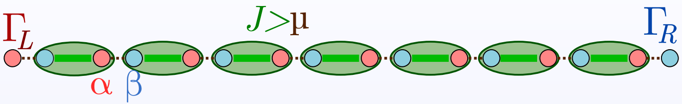


- $\mu \neq 0$: quasi-adiabatic continuation! (Hastings & Wen 2005)
- The system remains topological!
- ***Weak*** topological zero-energy modes (*ground states only!*):

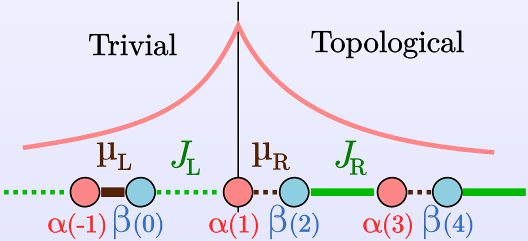
$$\mathcal{V}(\mu)\alpha(1)\mathcal{V}^{\dagger}(\mu) = \alpha(1)+\mu \sum_{h\neq e} \frac{\mathrm{Tr}\left[\beta_h^{\dagger}(2)\alpha_h(1)D^{\dagger}(h)\right]}{m_h - m_e}\alpha(1)\left(\mathbb{1} - D^{\dagger}(h)\right)+O\left(\frac{\mu^2}{J^2}\right)$$

Topological phase and weak zero-energy modes

$\mu \ll J$

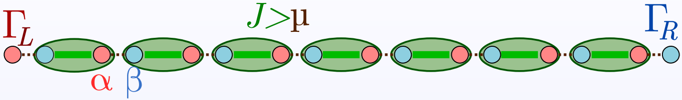


- $\mu \neq 0$: quasi-adiabatic continuation! (Hastings & Wen 2005)
- The system remains topological!
- ***Weak*** topological zero-energy modes (*ground states only!*):

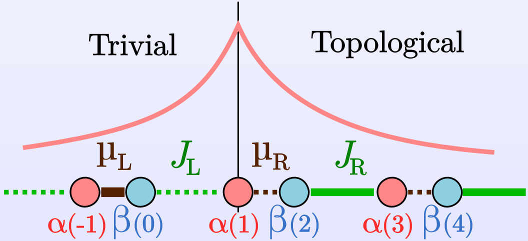


Topological phase and weak zero-energy modes

$\mu \ll J$



- $\mu \neq 0$: quasi-adiabatic continuation! (Hastings & Wen 2005)
- The system remains topological!
- ***Weak*** topological zero-energy modes (*ground states only!*):



- ***Strong*** topological zero-energy modes (*all the spectrum*!) require a breaking of translational symmetry

Conclusions

M. I. K. Munk, A. Rasmussen, M. B., PRB 98, 245135 (2018)

- We built a **gauge-flux model** with a discrete **non-Abelian** symmetry group G and a **symmetry-broken phase**
- A G-Jordan-Wigner transformation defines dyonic operators
- We obtained a **dyonic model with topological order**
- We obtained **weak zero-energy dyonic modes**
- Strong dyonic zero-energy modes require position-dependent terms

Conclusions

M. I. K. Munk, A. Rasmussen, M. B., PRB 98, 245135 (2018)

- We built a **gauge-flux model** with a discrete **non-Abelian** symmetry group G and a **symmetry-broken phase**
- A G-Jordan-Wigner transformation defines dyonic operators
- We obtained a **dyonic model with topological order**
- We obtained **weak zero-energy dyonic modes**
- Strong dyonic zero-energy modes require position-dependent terms

Postdoc and PhD positions available!

Contact me if you are interested in working in Copenhagen University

THE VELUX FOUNDATIONS
VILLUM FONDEN VELUX FONDEN