Quantum de Moivre-Laplace theorem for non-interacting indistinguishable particles in random networks

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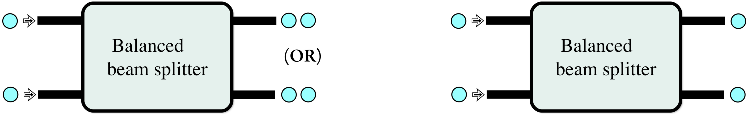
Reminder: Boson bunching and Fermion anti-bunching

Two indistinguishable particles & symmetric (Bell-type) unitary two-port, $M = 2$ and $N = 2$:

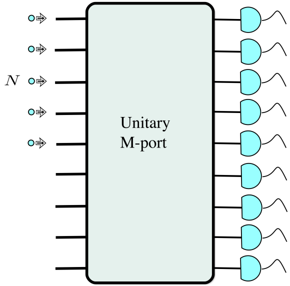
BOSONS FERMIONS



Experiments:

- with photons: Hong, C. K., Ou, Z. Y. & Mandel, L. *Phys. Rev. Lett.* **59**, 2044 (1987).
- with electrons: Liu, R. C., Odom, B., Yamamoto, Y. & Tarucha, S. *Nature* **391**, 263 (1998).
- with neutral bosonic atoms: Lopes, R., Imanaliev, A., Aspect, A., Cheneau, M., Boiron, D. & Westbrook, C.I. *Nature* **520**, 66 (2015).

Reminder: Identical particles in the linear multiports & importance



Given: N non-interacting indistinguishable particles at input of a M-port unitary network
Find: the distribution of particles at the network output.

- **Boson Sampling idea.** Aaronson, S. & Arkhipov, A. *Theory of Computing* **9**, 143 (2013).
  Proof of principle experiments:
  Broome, M. A. *et al. Science* **339**, 794 (2013)
  Spring, J. B. *et al. Science*, **339**, 798 (2013).
  Tillmann, M. *et al. Nat. Photon.* **7**, 540 (2013).
  Crespi, A. *et al. Nat. Photon.* **7**, 545 (2013).
- **Zero transmission laws for bosons & fermions.** Tichy, M. C., Tiersch, M., Mintert, F. & Buchleitner, A., *New J. Phys.* **14**, 093015 (2012); Crespi, A. *et al. Nat. Commun.* **7**, 10469 (2016).
- **Generalizations of boson bunching & fermion anti-bunching.** Spagnolo, N. *et al Phys. Rev. Lett.* **111**, 130503 (2013). Shchesnovich, V. S. *Phys. Rev. Lett.* **116**, 123601 (2016).

Particle counting in binned-together output ports of the random multiport

Quantum Statistical Mechanics:
Weakly-interacting identical particles & quantum-classical transition at a vanishing particle density.

In our case: the particle density

$$\alpha = N/M$$

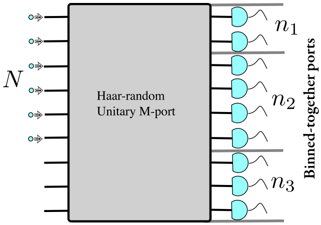
BOSONS: $0 < \alpha < \infty$
FERMIONS: $0 < \alpha < 1$

Why binning?
The average probability of an output configuration $\mathbf{m} = (m_1, \ldots, m_M)$ in the Haar-random M-port

$$\langle p^{(B,F)}(\mathbf{m})\rangle = \frac{1}{\#\text{Fock states}} \sim e^{-\gamma N},$$

$$\gamma = \ln(1/\alpha) + (1 \pm 1/\alpha)\ln(1 \pm \alpha) > 0,$$

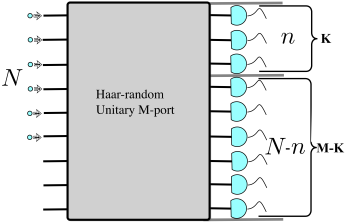
Why random multiport?
Use of a random multiport to average over the (multiport-specific) interference effects
& **reveal the effect of the particle statistics**.



Problem: The average probability distribution $\langle P(\mathbf{n})\rangle$, $\mathbf{n} = (n_1, \ldots, n_r)$ in r bins of the output ports as $N \gg 1$.

Distinguishable particles & two bins: the de Moivre-Laplace theorem

Distinguishable (classical) particles:



The asymptotic distribution $\langle P^{(D)}(n)\rangle$ as $N \to \infty$?

One particle at a time: the probability $k \to l$

$$\langle p(l|k)\rangle = \langle |U_{kl}|^2\rangle = \frac{1}{M}.$$

For N particles in two output bins:

$$\langle P^{(D)}(n)\rangle = \frac{N!}{n!(N-n)!} q^n (1-q)^{N-n}, \quad q \equiv \frac{K}{M}$$

The de Moivre-Laplace theorem: as $N \to \infty$

$$\langle P^{(D)}(n)\rangle \sim \frac{\exp\left\{-\frac{(n-Nq)^2}{2Nq(1-q)}\right\}}{\sqrt{2\pi Nq(1-q)}}$$

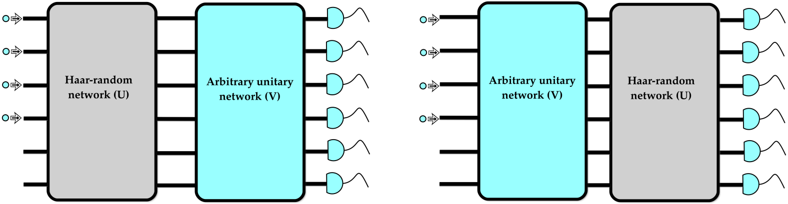
That is

$$n \approx qN + \sqrt{Nq(1-q)}\,x$$

$x \in \mathcal{N}(0,1)$.

Indistinguishable particles: the average transition probability $\mathbf{k} \to \mathbf{l}$ is uniform in $\mathbf{k}, \mathbf{l}$

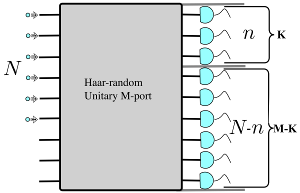
The unitary invariance of the Haar measure $\mu(U) = \mu(UV) = \mu(VU)$:



The average transition probability of N particles from input ports $\mathbf{k}$ to output ports $\mathbf{l}$:

$$\langle p^{(B,F)}(\mathbf{l}|\mathbf{k})\rangle = \frac{1}{\#\text{Fock states}} = \frac{N!}{(M \pm N \mp 1)\dots M}.$$

Indistinguishable particles & two-bins: quantum de Moivre-Laplace theorem



$$\langle P^{(B,F)}\rangle = \frac{\#\mathrm{Fock\,states\,(n,N-n)}}{\#\mathrm{all\,Fock\,states}} = \underbrace{\langle P^{(D)}\rangle}_{\substack{\textit{Classical}\\ \textit{binomial}}} Q^{(B,F)},$$

with the "quantum factor"

$$Q^{(B,F)} \equiv \frac{\prod_{s=0}^{n-1}\left(1\pm\frac{s}{K}\right)\prod_{s=0}^{N-n-1}\left(1\pm\frac{s}{M-K}\right)}{\prod_{s=0}^{N-1}\left(1\pm\frac{s}{M}\right)}$$

As $N\to\infty$ ($\alpha = N/M$ particle density)

$$Q^{(B,F)}(n) \sim \frac{\exp\left\{\pm\frac{\alpha}{1\pm\alpha}\frac{(n-Nq)^2}{2Nq(1-q)}\right\}}{\sqrt{1\pm\alpha}}$$

The quantum version of the de Moivre-Laplace theorem: as $N\to\infty$

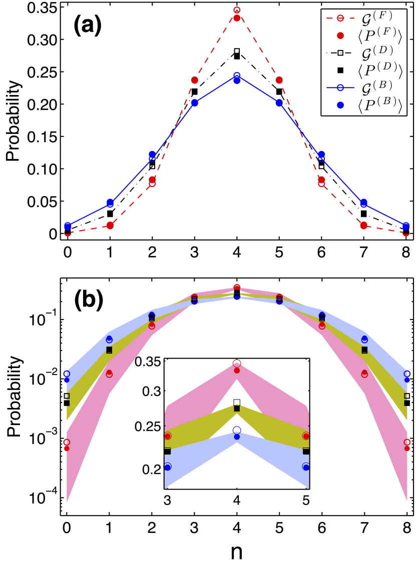
$$\langle P^{(B,F)}(n)\rangle \sim \frac{\exp\left\{-\frac{(n-Nq)^2}{2(1\pm\alpha)Nq(1-q)}\right\}}{\sqrt{2\pi(1\pm\alpha)Nq(1-q)}}$$

That is

$$n \approx qN + \sqrt{(1\pm\alpha)Nq(1-q)}\,x$$

$x \in \mathcal{N}(0,1)$.

The two-bin asymptotic Gaussian law: Numerical test



The asymptotic Gaussian law as $N \to \infty$:

$$\langle P^{(\sigma)}(n)\rangle \sim \mathcal{G}^{(\sigma)}(n) \equiv \frac{\exp\left\{-\frac{(n-Nq)^2}{2(1+\sigma\alpha)Nq(1-q)}\right\}}{\sqrt{2\pi(1+\sigma\alpha)Nq(1-q)}}$$

$\sigma = \{-, 0, +\}$ for {Fermions,Classical,Bosons}

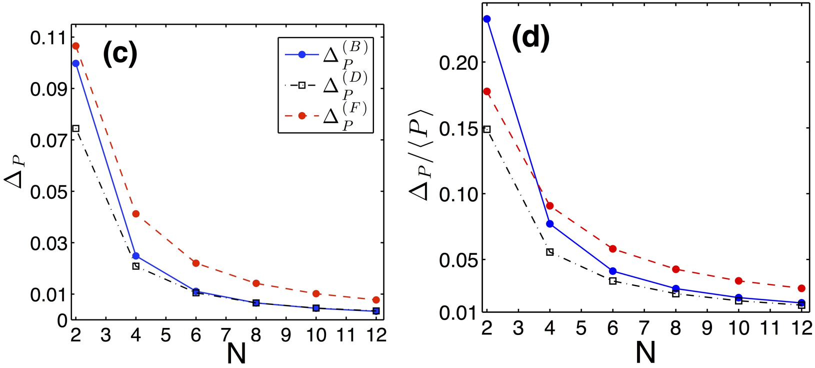
Fig. (a): $\mathcal{G}^{(\sigma)}(n)$ vs. $\langle P^{(\sigma)}(n)\rangle$.

The standard deviation from the average probability

$$\Delta_P^2 = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \sum_{j=1}^{\mathcal{T}} \left[P(n; U^{(j)}) - \langle P(n)\rangle\right]^2.$$

Fig. (b): Δ_P = half the hight of a filled region.

The standard deviation Δ_P from the average probability at $n = N/2$

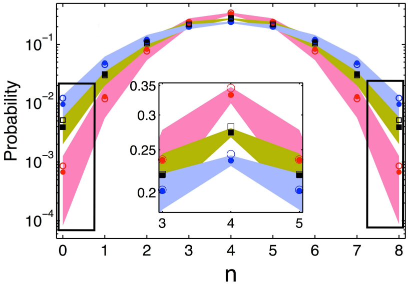


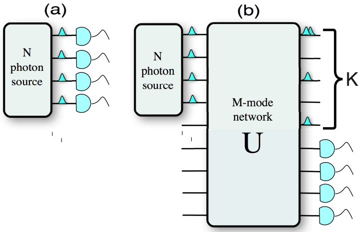
The standard deviation from the average probability

$$\Delta_P^2 = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \sum_{j=1}^{\mathcal{T}} \left[P(n; U^{(j)}) - \langle P(n)\rangle \right]^2 .$$

The generalized boson bunching & fermion anti-bunching

Bosons: blue   Fermions: red





Probabilities to count all N particles in $K < M$ output ports in a given multiport:

$$P^{(F)} = \det(H) \leq P^{(D)} = \prod_{i=1}^{N} H_{i,i} \leq P^{(B)} = \mathrm{per}(H), \quad H_{i,j} = \sum_{l=1}^{K} U_{k_i,l} U^*_{k_j,l}.$$

"Universality of Generalized Bunching and Efficient Assessment of Boson Sampling"

Shchesnovich, V. S. *Phys. Rev. Lett.* **116**, 123601 (2016).

Singular binning & the probability as $N \to \infty$ for a fixed $\alpha = N/M$

Quantum Statistical Mechanics:

System of weakly-interacting identical particles $\to$ classical behavior at the vanishing particle density.

Random multiports: non-interacting identical particles, density $\alpha = N/M$.

(i) General two-bin partition, the Asymptotic Gaussian law:

$$\frac{n}{N} \sim q + \frac{\sqrt{(1 \pm \alpha)q(1-q)}}{\sqrt{N}} x, \quad x \in \mathcal{N}(0,1).$$

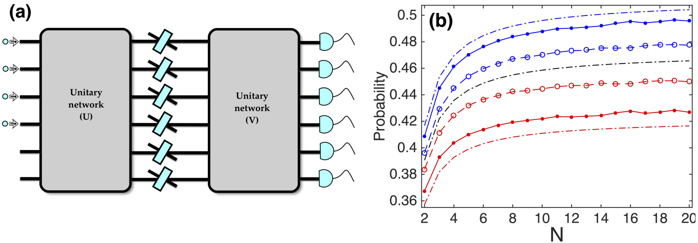
$$\langle P^{(B,F,D)}(n) \rangle \xrightarrow[N\to\infty]{} \delta_{n,qN}.$$

(ii) Singular binning: count N particles in $K = M - m$ modes, m being fixed.

The Gaussian asymptotic law does not apply:

$$\langle P^{(B,F)} \rangle \xrightarrow[N\to\infty]{} (1 \pm \alpha)^{\mp m}, \quad \langle P^{(D)} \rangle \xrightarrow[N\to\infty]{} e^{-m\alpha}.$$

Application: the survival probability of N identical particles in a random lossy multiport



Panel (a): a lossy M-port $A = UDV$ as a unitary $2M$-port, where the beamsplitters have the transmission coefficients $\eta_1, \ldots, \eta_M$ and $D = \mathrm{diag}(\sqrt{\eta_1}, \ldots, \sqrt{\eta_M})$. Panel (b): the average survival probability of N indistinguishable bosons (above the middle dash-dot line) and fermions (below the middle dash-dot line) for $M = 4N$ and $m = 3$ lossy channels with $\eta_{1,2,3} = 0.1$ (dots) and $\eta_{1,2,3} = 0.3$ (open circles).

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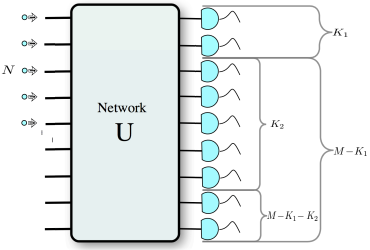
The asymptotic Gaussian law for the r bin partition



Layered division of the r-bin partition into $r-1$ two-bin partitions.

Valid for both classical and quantum cases.

The de Moivre-Laplace theorem can be applied sequentially, via the factorization identity:

$$P(n_1,\ldots,n_r|K_1,\ldots,K_r) = P(n_1,N_1-n_1|K_1,M_1-K_1)P(n_2,N_2-n_2|K_2,M_2-K_2) \times\ldots\times P(n_{r-1},N_{r-1}-n_{r-1}|K_{r-1},M_{r-1}-K_{r-1}),$$

where $N_1=N$, $M_1=M$ and for $s=2,\ldots,r-1$

$$N_s = N-\sum_{i=1}^{s-1} n_i, \quad M_s = M-\sum_{i=1}^{s-1} K_i.$$

The asymptotic Gaussian law for r-bin partition: rigorous formulation

Theorem (1)

Consider the Haar-random unitary M-port with the binned together output ports into sets of $K_1, \dots, K_r$ ports. Then, for fixed $q_i = K_i/M > 0$, as $N, M \to \infty$ the average probability to count $\mathbf{n} = (n_1, \dots, n_r)$ identical particles into the r bins such that

$$|n_i - Nq_i| \le AN^{\frac{2}{3}-\epsilon}, \quad A > 0, \quad 0 < \epsilon < \frac{1}{6} \tag{1}$$

has the following asymptotic form

$$\langle P^{(\sigma)}(\mathbf{n})\rangle = \frac{\exp\left\{-\sum_{i=1}^r \frac{(n_i - q_i N)^2}{2N(1+\sigma\alpha)q_i}\right\}}{(2\pi[1+\sigma\alpha]N)^{\frac{r-1}{2}}\prod_{i=1}^r \sqrt{q_i}}\left\{1 + \mathcal{O}\left(\frac{(1-\alpha\delta_{\sigma,-})^{-3}}{N^{3\epsilon}} + \frac{\alpha\delta_{\sigma,+}}{N}\right)\right\} \tag{2}$$

(bosons $\sigma =$ "$+$", fermions $\sigma =$ "$-$", classical $\sigma =$ "0").

Theorem (2)

The average probability of the particle counts $\mathbf{n}$ violating Eq. (1) for $N, M \to \infty$ and a fixed α satisfies

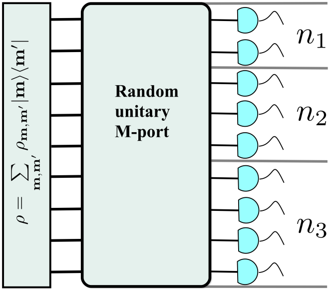
$$\langle P^{(\sigma)}(\mathbf{n})\rangle = \mathcal{O}\left(\exp\left\{-\frac{A^2}{1+\sigma\alpha}N^{\frac{1}{3}-2\epsilon}\right\}\right), \tag{3}$$

where A is from Eq. (1).

General input state & random multiport: the asymptotic Gaussian law

The asymptotic Gaussian law is valid for **arbitrary** state ρ of N identical particles at the input of a Haar-random unitary M-port:

$$\rho = \sum_{\mathbf{m},\mathbf{m}'} \rho_{\mathbf{m},\mathbf{m}'} |\mathbf{m}\rangle\langle\mathbf{m}'|, \quad |\mathbf{m}\rangle = |m_1,\ldots,m_M\rangle,\; m_1+\ldots+m_M = N.$$



The asymptotic probability of particles counts in r output bins as $N\to\infty$:

$$\langle P^{(\sigma)}(\mathbf{n}|\rho)\rangle_U \sim \frac{\exp\left\{-\sum_{i=1}^r \frac{(n_i-q_iN)^2}{2N(1+\sigma\alpha)q_i}\right\}}{(2\pi[1+\sigma\alpha]N)^{\frac{r-1}{2}}\prod_{i=1}^r\sqrt{q_i}}.$$

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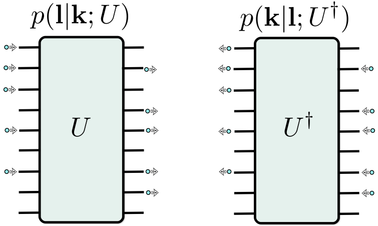
The time-inversion symmetry

$$\boxed{p(\mathbf{l}|\mathbf{k};U) = p(\mathbf{k}|\mathbf{l};U^\dagger)}$$



Distinguishable particles (the classical case):

$$p(l|k;U) = |U_{kl}|^2 = |(U^\dagger)_{l,k}|^2 = p(k|l;U^\dagger)$$

Indistinguishable particles: the time-inversion symmetry for a unitary M-port U:

$$\det(U[\mathbf{k},\mathbf{l}]) = \left[\det(U^\dagger[\mathbf{l},\mathbf{k}])\right]^*, \quad \mathrm{per}(U[\mathbf{k},\mathbf{l}]) = \left[\mathrm{per}(U^\dagger[\mathbf{l},\mathbf{k}])\right]^*.$$

Averaging over the input configurations: the asymptotic Gaussian law

The averaging over the Haar-random U and the **uniform** averaging over the Fock states (output $\mathbf{l}$ or input $\mathbf{k}$) for a fixed U give the same results:

$$\langle p(\mathbf{l}|\mathbf{k};U)\rangle_U = \langle p(\mathbf{l}|\mathbf{k};U)\rangle_{\mathbf{k}} = \frac{1}{\#\text{Fock states}}.$$

The asymptotic Gaussian law applies for a fixed multiport U with binned-together output ports & uniform averaging over the input Fock states $\mathbf{k}$:

$$\langle P^{(\sigma)}(\mathbf{n}|\mathbf{k})\rangle_{\mathbf{k}} \sim \frac{\exp\left\{-\sum_{i=1}^{r}\frac{(n_i-q_iN)^2}{2N(1+\sigma\alpha)q_i}\right\}}{(2\pi[1+\sigma\alpha]N)^{\frac{r-1}{2}}\prod_{i=1}^{r}\sqrt{q_i}}$$

$\sigma=\{-,0,+\}$ for {Fermions,Classical,Bosons}

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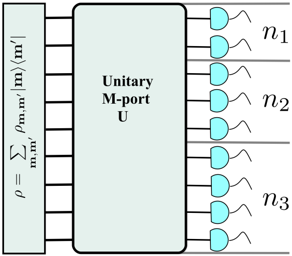
Given input ρ & a unitary M-port, find the asymptotic form of $P(\mathbf{n}|\rho; U)$



The asymptotic probability of particles counts in r output bins as $N \to \infty$:

$$P^{(\sigma)}(\mathbf{n}|\rho; U) \sim ?$$

For the uniform averaging over the inputs (e.g., the random-phase bosons)

$$\rho^{(u)} \propto \sum_{|\mathbf{m}|=N} |\mathbf{m}\rangle\langle\mathbf{m}|$$

$$P^{(\sigma)}(\mathbf{n}|\rho^{(u)}) \sim \frac{\exp\left\{-\sum_{i=1}^{r} \frac{(n_i - q_i N)^2}{2N(1+\alpha)q_i}\right\}}{(2\pi[1+\alpha]N)^{\frac{r-1}{2}} \prod_{i=1}^{r} \sqrt{q_i}}.$$

Problem: General input ρ of N identical particles, arbitrary M-port, find $P(\mathbf{n})$?

The inverse problem $P(\mathbf{n}) \to \rho$, application for certification of the Boson Sampling.

Related: "Statistical benchmark for BosonSampling", Walschaers, M. *et al*, *New J. Phys.* **18** 032001 (2016).

The partial indistinguishability and the quantum-to-classical transition

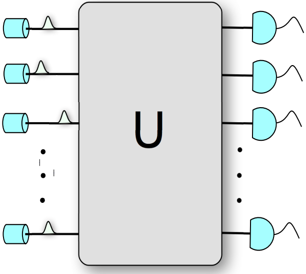
Probability of the transition
$\mathbf{n} = (n_1, \ldots, n_M) \to \mathbf{m} = (m_1, \ldots, m_M)$:

$$p(\mathbf{m}|\mathbf{n}) = \frac{1}{\mathbf{m}!\mathbf{n}!} \sum_{\sigma,\tau \in S_N} J_{\sigma,\tau} \left[\prod_{\alpha=1}^{N} U^*_{k_{\sigma(\alpha)}, l_\alpha} U_{k_{\tau(\alpha)}, l_\alpha} \right]$$

$$J(\sigma) = \varepsilon(\sigma) \boxed{\mathrm{Tr}\left(\rho^{(int)} P_\sigma\right)},$$

$$\varepsilon(\sigma) = \begin{cases} 1, & \text{Bosons}, \\ \mathrm{sgn}(\sigma), & \text{Fermions} \end{cases}$$

and $P_\sigma \prod_{a=1}^{N}{}^{\otimes} |j_a\rangle = \prod_{a=1}^{N}{}^{\otimes} |j_{\sigma^{-1}(a)}\rangle$.



More details: Shchesnovich, V. S., *Phys. Rev. A* **89**, 022333 (2014); *ibid* **91**, 013844 (2015).

Problem: **Partial distinguishability effect on the transition from the quantum to classical asymptotic Gaussian Law?**

Related: "Multiparticle Correlations in Mesoscopic Scattering: Boson Sampling, Birthday Paradox, and Hong-Ou-Mandel Profiles", Urbina, J.-D., *et al*, *Phys. Rev. Lett* **116**, 100401 (2016)

References

(i) Heuristic derivation, numerical data & discussion:

Shchesnovich, V.S. "Asymptotic Gaussian law for noninteracting indistinguishable particles in random networks", *Sci. Reports.* **7**:31 (2007).

(ii) Rigorous formulation and proof:

Shchesnovich, V.S., "Quantum De Moivre-Laplace theorem for noninteracting indistinguishable particles" *arXiv:1609.05007v3 [quant-ph].*