Cold atoms in quasi-1D traps: beyond the zero-range approximation

Krzysztof Jachymski, Hagar Veksler, Paul S. Julienne and Shmuel Fishman

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Why ultracold atoms?

- large number of bosons/fermions close to zero temperature
- easy to manipulate with optical fields
- optical lattices, reduced dimensional systems
- control of the type and strength of interactions
- precise measurements
- great toolkit for quantum simulations

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Realization of Lieb-Liniger model

- low density, quasi-1D gas with short-range interactions
- \( V(x) \approx g_{1D} \delta(x) \)
- control over \( g \) to probe different regimes achieved using Feshbach resonances
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Interactions & Feshbach resonances

- neutral, closed-shell atoms - van der Waals interactions
  \[ V(r) \xrightarrow{r \to \infty} -\frac{C_6}{r^6} \]

- length \( R_6 = (2\mu C_6/\hbar^2)^{1/4} \) or \( a \approx 0.477R_6 \); \( E_6 = \hbar^2/2\mu R_6^2 \)

- collision energy \( E \sim nK \), very dilute gas, s-wave scattering is enough (bosons)

- scattering length \( a_{3D} = \lim_{k \to 0} \left( -\frac{\tan \delta(k)}{k} \right) \) can be tuned using Feshbach resonances (see Chin et al, RMP 2010)
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- scattering length \( a_{3D} = a_{bg} \left( 1 - \frac{\Delta}{B-B_0} \right) \)
- universal weakly bound state \( E_b \propto 1/a^2 \)

Figure: Scattering length and bound state energy near a Feshbach resonance (Chin et al, RMP 2010).
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**Figure:** Scattering length and bound state energy near a Feshbach resonance (Chin et al, RMP 2010).
Introducing the pseudopotential

- replace the interaction by pseudopotential which reproduces scattering properties

\[
V_{\text{eff}} = \frac{2\pi \hbar^2 a_{3D}}{\mu} \delta(r) \frac{\partial}{\partial r} r
\]

- energy dependence can be included using effective range

\[
k \cot \delta_{3D}(k) = -\frac{1}{a_{3D}(k)} = -\frac{1}{a_{3D}} + \frac{1}{2} r_{3D} k^2 + \ldots
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- energy-dependent pseudopotential for trapped particles

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Figure: Energy dependence of the phase shift (Chin et al, RMP 2010)
Energy and length scales

- transverse harmonic confinement $U_{tr} = \frac{1}{2} \mu \omega^2 \rho^2$
- new length scale $d = \sqrt{\frac{\hbar}{\mu \omega}}$, energy scale $\hbar \omega$
- typically $d \gg R_6$
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Solving energy-dependent problem

- start with Schrödinger equation

\[
\left( -\frac{\hbar^2}{2\mu} \nabla^2 + V(\mathbf{r}) + \frac{1}{2}\mu \omega^2 \rho^2 \right) \psi = E \psi
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- asymptotic boundary conditions

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\psi \xrightarrow{r \to \infty} \psi_{nm}(\rho) e^{ipz} + \sum_{n' m'} f^{(+)}_{nm,n'm'}(p) \psi_{n'm'} e^{ip'|z|}
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- odd part vanishes (bosons)
- restrict to lowest transverse mode
- extract the scattering amplitude \( f^{(+)} \)

Olshanii, PRL 1999; Naidon et al, NJP 2007; Idziaszek et al, NJP 2015
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1D physics

- one-dimensional phase shift $\delta_{1D}$

$$f^{(+)}(p) = -\frac{1}{1 + i \cot \delta_{1D}(p)}$$

- 1D even scattering length

$$a^{(+)}_{1D}(p) = \frac{1}{p \tan \delta_{1D}(p)}$$

- effective interaction strength in 1D $V_{\text{eff}}(x) = g_{1D}(x)$

$$g_{1D}(p) = -\frac{\hbar^2}{\mu a_{1D}(p)}$$

(note that $g_{3D} = \frac{2\pi \hbar^2 a_{3D}}{\mu}$)
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Energy-dependent interaction strength

- general result for $a_{1D}(p)$ ($C = -\zeta(1/2)$) (Olshanii, Naidon)

$$a_{1D}(p) = -\frac{d^2}{2a_{3D}(k)} \left(1 - C \frac{a_{3D}(k)}{d}\right)$$

- incorporate energy dependence

$$a_{3D}(k) \approx \frac{a_{3D}}{1 - k^2 r_{3D} a_{3D}/2}$$

- include zero-point energy of the transverse oscillator!

$$k^2 = p^2 + 2/d^2$$

- low energy expansion $g_{1D}(p) = g_{1D}(1 + g' p^2) + \ldots$

$$g_{1D} = \frac{2\hbar^2}{\mu d} \left(\frac{d}{a_{3D}} - C - \frac{r_{3D}}{d}\right)^{-1}$$

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Numerical verification

Figure: Naidon, NJP 2007; Bergeman, PRL 2003, Lennard-Jones potential

theory remains valid for $d \gtrsim \bar{a}$, independently of $a_{3D}$
Transmission coefficient

convenient analysis in terms of transmission coefficient

\[ T(p) = |1 + f|^2 = \frac{1}{1 + \tan^2 \delta_{1D}(p)} \]
Role of effective range

Figure: Wide trap $d = 20\bar{a}$, wide resonance - typical conditions in Innsbruck experiment
Wide vs. narrow resonances

- role of closed-channel contribution close to the Feshbach resonance
- “pole strength“ $s_{\text{res}} = \frac{a_{\text{bg}}}{a} \frac{\delta \mu \Delta}{\bar{E}}$
- large $s_{\text{res}}$ - open channel-dominated (“broad“)
- $s_{\text{res}} \ll 1$ “narrow“ resonance
- effective range at the broad resonance - single-channel formula

$$r_{3D} = \frac{\Gamma(1/4)^2 \bar{a}}{6\pi^2} \left(1 - \frac{2\bar{a}}{a_{3D}} + \frac{2\bar{a}^2}{a_{3D}^2}\right)$$

- narrow resonances - nonuniversal behavior

$$r_{3D} \approx \frac{\nu + r_0(a_{3D} - a_{\text{ex}})^2}{a_{3D}^2}$$

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- Effective range at the broad resonance - single-channel formula

$$r_{3D} = \frac{\Gamma(1/4)^2 \bar{a}}{6\pi^2} \left(1 - \frac{2\bar{a}}{a_{3D}} + \frac{2\bar{a}^2}{a_{3D}^2}\right)$$

- Narrow resonances - nonuniversal behavior

$$r_{3D} \approx \nu + r_0 \frac{(a_{3D} - a_{\text{ex}})^2}{a_{3D}^2}$$

Bo Gao, PRA 1998; Blackley et al, PRA 2014
Role of effective range

Cs, \( \sim 47G \) resonance with very small \( s_{\text{res}} \)

Figure: left: \( d = 20\bar{a} \), right: \( d = 5\bar{a} \)

Theory without effective range corrections fails!
Role of effective range II

Figure: $g_{1D}$ with (blue) and without corrections for two different narrow resonances at $a_{3D} = 10\bar{a}$; note that red curve remains the same!
Introducing the GLL

- length scale $\ell = \sqrt{2|g'|}$ associated with the correction
- $V(x)\psi(x) = g_{1D}\delta(x) \left(1 - g'\partial^2_x\right)\psi(x)$
- discretize the derivative
- resulting effective model
  
  $V(x) = c_0\delta(x) + c_\ell(\delta(x - \ell) + \delta(x + \ell))$

  $c_0 = 2g_{1D}$, $c_\ell = -g_{1D}/2$ or $c_0 = 0$, $c_\ell = g_{1D}/2$ depending on the sign of $g'$

- mapping on the Lieb-Liniger model for dilute system

  $c_{\text{eff}} = c_0 + 2c_\ell + \frac{mc_\ell\ell}{\hbar^2} \left(2c_0 + 2c_\ell + \frac{mc_0c_\ell\ell}{\hbar^2} + \frac{mc_0^2\ell}{2\hbar^2}\right)$

  $\frac{1}{1 - \frac{m^2c_0c_\ell\ell^2}{2\hbar^4} - \frac{m_c\ell\ell}{\hbar^2}}$

- $c_{\text{eff}} \neq c_0 + 2c_\ell$, unit transmission possible for finite interactions

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$$c_{\text{eff}} = c_0 + 2c_\ell + \frac{mc_\ell\ell}{\hbar^2} \left(2c_0 + 2c_\ell + mc_0c_\ell\ell + mc_0^2\ell^2 \right) \left(1 - \frac{m^2c_0c_\ell\ell^2}{2\hbar^4} - \frac{mc_\ell\ell}{\hbar^2}\right)$$

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- mapping on the Lieb-Liniger model for dilute system
  $$c_{\text{eff}} = c_0 + 2c_\ell + \frac{mc_\ell\ell}{\hbar^2}\left(2c_0 + 2c_\ell + \frac{mc_0c_\ell\ell}{\hbar^2} + \frac{mc_0^2\ell^2}{2\hbar^2}\right)$$
  $$\frac{1}{1 - \frac{m^2c_0^2c_\ell^2\ell^2}{2\hbar^4} - \frac{mc_\ell\ell^2}{\hbar^2}}$$
- $c_{\text{eff}} \neq c_0 + 2c_\ell$, unit transmission possible for finite interactions

Introducing the GLL

- length scale \( \ell = \sqrt{2|g'|} \) associated with the correction
- \( V(x)\psi(x) = g_{1D}\delta(x) \left(1 - g' \partial_x^2\right) \psi(x) \)
- discretize the derivative
- resulting effective model
  \[
  V(x) = c_0\delta(x) + c_\ell(\delta(x - \ell) + \delta(x + \ell))
  \]
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  \[
  c_{\text{eff}} = c_0 + 2c_\ell + \frac{mc_\ell \ell}{\hbar^2} \left(2c_0 + 2c_\ell + \frac{mc_0 c_\ell \ell}{\hbar^2} + \frac{mc_0^2 \ell}{2\hbar^2}\right) \left(1 - \frac{m^2 c_0 c_\ell \ell^2}{2\hbar^4} - \frac{m c_\ell \ell}{\hbar^2}\right)^{-1}
  \]
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Introducing the GLL

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  $c_{\text{eff}} \neq c_0 + 2c_\ell$, unit transmission possible for finite interactions
Cold atoms in quasi-1D traps: beyond the zero-range approximation

Generalized Lieb-Liniger model

Introducing the GLL

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- discretize the derivative
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  $V(x) = c_0\delta(x) + c_\ell(\delta(x - \ell) + \delta(x + \ell))$

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  $c_{\text{eff}} = c_0 + 2c_\ell + \frac{mc_\ell \ell}{\hbar^2} \left( 2c_0 + 2c_\ell + \frac{mc_0 c_\ell \ell}{\hbar^2} + \frac{mc_0^2 \ell}{2\hbar^2} \right)$

  $1 - \frac{m^2 c_0 c_\ell \ell^2}{2\hbar^4} - \frac{m c_\ell \ell}{\hbar^2}$

- $c_{\text{eff}} \neq c_0 + 2c_\ell$, unit transmission possible for finite interactions

validity of GLL

**Figure:** Transmission for two narrow resonances; GLL denoted by black dashed line
Conclusions & outlook

- Atomic interactions in traps need to include finite energy corrections.
- Can be described in terms of universal quantities.
- Effective range affects the stability of the trapped gas.
- Relevant especially for narrow resonances.
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