# Genuine quantum many-body interference in fock space: coherent backscattering and many-body spin echo

Thomas Engl

Massey University

February 17, 2017

Theory

Numerics[1]

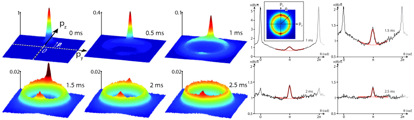
Regensburg

Liège

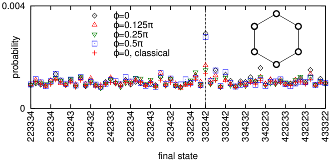
[1] "He's done it all" [S. Wimberger]

# Motivation

## Bosons



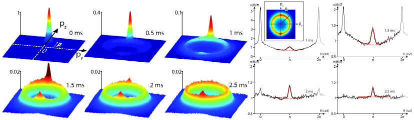

[cf. F. Jendrzejewski, K. Müller, J. Richard, A. Date, T. Plisson, P. Bouyer, A. Aspect and V. Josse, PRL **109** 195302 (2012)]




[TE, J. Dujardin, A. Argüelles, P. Schlagheck, K. Richter and J. D. Urbina, PRL **112** 140403 (2014)]

# Motivation

Bosons

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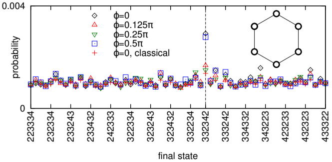
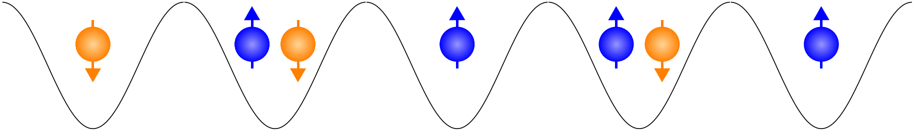



[TE, J. Dujardin, A. Argüelles, P. Schlagheck, K. Richter and J. D. Urbina, PRL **112** 140403 (2014)]

What about Fermions?

# Fermionic Hubbard model



$$|\mathbf{n}\rangle = |0,1,1,1,1,0,1,1,1,0\rangle$$

## Hamiltonian

$$\hat{H} = \sum_j \left\{ \sum_{\sigma=\uparrow,\downarrow} \left[ \epsilon_j \hat{c}^\dagger_{j\sigma}\hat{c}_{j\sigma} - J\left(\hat{c}^\dagger_{j\sigma}\hat{c}_{j+1\sigma} + \hat{c}^\dagger_{j+1\sigma}\hat{c}_{j\sigma}\right)\right] + U\hat{c}^\dagger_{j\uparrow}\hat{c}^\dagger_{j\downarrow}\hat{c}_{j\downarrow}\hat{c}_{j\uparrow} \right.$$
$$\left. + \kappa\left(\hat{c}^\dagger_{j,\downarrow}\hat{c}_{j+1,\uparrow} - \hat{c}^\dagger_{j+1,\downarrow}\hat{c}_{j,\uparrow}\right) + \kappa^*\left(\hat{c}^\dagger_{j+1,\uparrow}\hat{c}_{j,\downarrow} - \hat{c}^\dagger_{j,\uparrow}\hat{c}_{j+1,\downarrow}\right) \right\}$$

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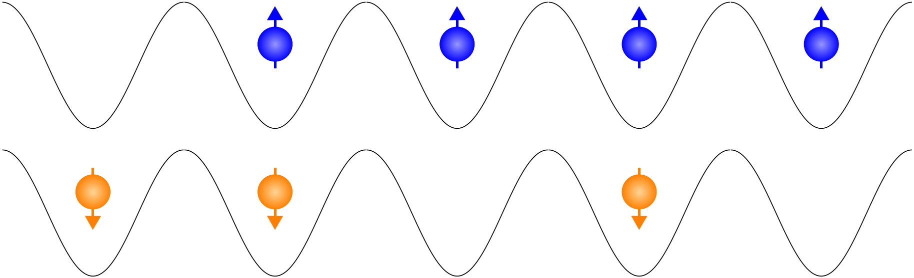

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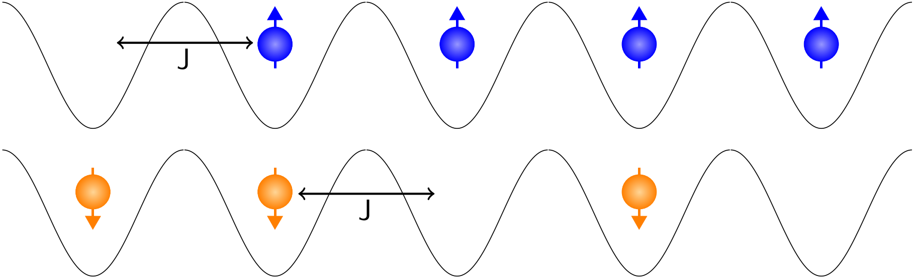


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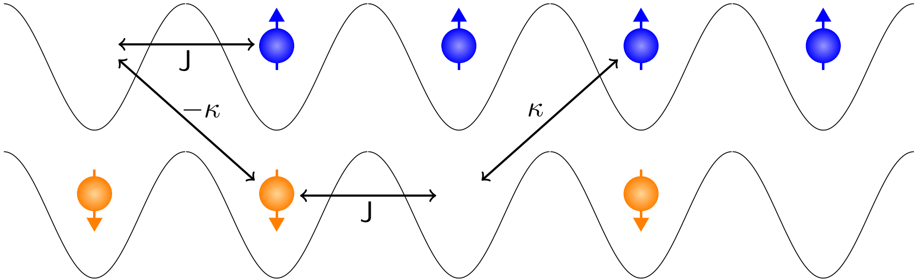


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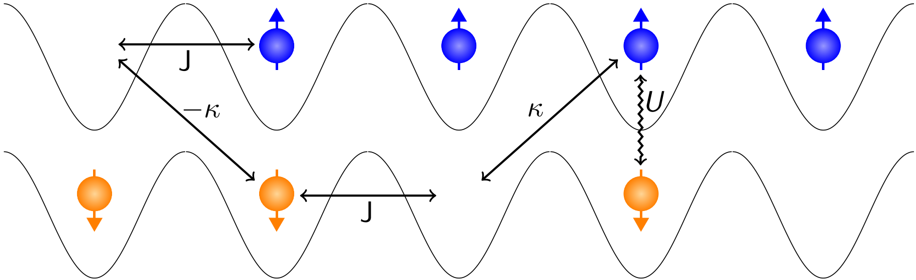


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# Semiclassical theory

## Semiclassical propagator ($\hbar_{eff} = 1/N$)

$$K(\mathbf{n},\mathbf{m};t) = \left\langle \mathbf{n} \middle| \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t} \middle| \mathbf{m} \right\rangle = \int \mathcal{D}\left[\phi(s)\right] \mathrm{e}^{\frac{\mathrm{i}}{\hbar}R[\phi(s);t]} \approx \sum_{\gamma:\mathbf{n}\to\mathbf{m}} \mathcal{A}_\gamma \mathrm{e}^{\frac{\mathrm{i}}{\hbar}R_\gamma}$$

$$\hat{c}^\dagger_{j\sigma}\hat{c}_{j\sigma} \to \left|\phi_{j\sigma}\right|^2$$

$$\hat{c}^\dagger_{j\sigma}\hat{c}_{j'\sigma'} \to \phi^*_{j\sigma}\phi_{j'\sigma'} \exp\left(-\left|\phi_{j\sigma}\right|^2 - \left|\phi_{j'\sigma'}\right|^2\right) \prod_{(k,\sigma'')=(j,\sigma)}^{(j',\sigma')} \left(1 - 2\left|\phi_{k\sigma''}\right|^2\right)$$

TE, J. D. Urbina and K. Richter, Theor. Chem. Acc. **133**, 1563; arXiv:1409.4196

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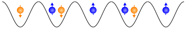
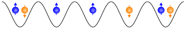
Classical trajectory $\gamma : \mathbf{n} \to \mathbf{m}$:



$|\psi_{j\sigma}(0)|^2 = n_{j\sigma}$



$|\psi_{j\sigma}(t)|^2 = m_{j\sigma}$

Classical action: $R_\gamma = \int\limits_0^t \mathrm{d}t' \left[\hbar\boldsymbol{\theta}(t')\,\dot{\mathbf{J}}(t') - H^{(\mathrm{cl})}(\boldsymbol{\psi}^*(t'), \boldsymbol{\psi}(t'))\right]$

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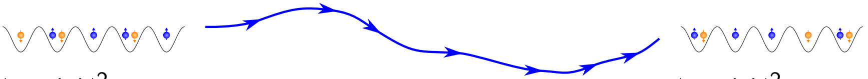

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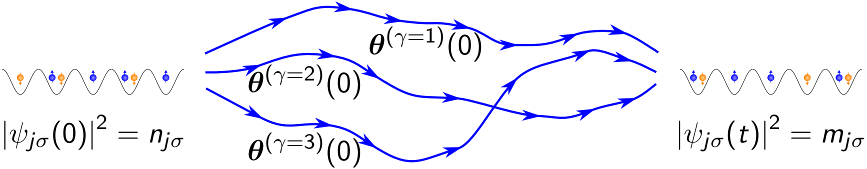


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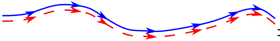
# Transition probability

$$P(\mathbf{n}, \mathbf{m}, t) = |K(\mathbf{n}, \mathbf{m}, t)|^2 = \sum_{\gamma, \gamma' : \mathbf{m} \to \mathbf{n}} \mathcal{A}_\gamma \mathcal{A}^*_{\gamma'} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(R_\gamma - R_{\gamma'}) + \mathrm{i}\frac{\pi}{2}(\mu_\gamma - \mu_{\gamma'})}$$

- diagonal approximation $\gamma' = \gamma$:  $=: P_{\mathrm{cl}}(\mathbf{n}, \mathbf{m}, t)$

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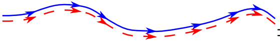
- diagonal approximation $\gamma' = \gamma$:  $=: P_{\mathrm{cl}}(\mathbf{n},\mathbf{m},t)$
- interference between time-reverse paths

$$\gamma' = \mathcal{T}\gamma$$

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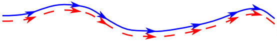
- diagonal approximation $\gamma' = \gamma$:  $=: P_{\mathrm{cl}}(\mathbf{n},\mathbf{m},t)$
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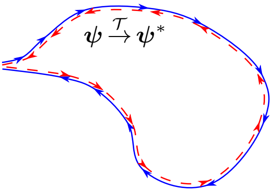
$$\gamma' = \mathcal{T}\gamma$$

$$R_\gamma - R_{\mathcal{T}\gamma} = 0$$

$\mathbf{n} = \mathbf{m}$



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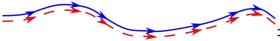
- diagonal approximation $\gamma' = \gamma$:  $=: P_{\mathrm{cl}}(\mathbf{n},\mathbf{m},t)$
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$\delta_{\mathbf{nm}} P_{\mathrm{cl}}(\mathbf{n},\mathbf{m},t)$

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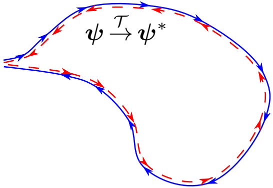

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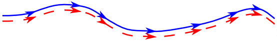
- diagonal approximation $\gamma' = \gamma$:  $=: P_{\mathrm{cl}}(\mathbf{n},\mathbf{m},t)$
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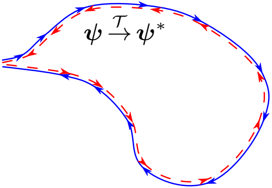


$$R_\gamma - R_{\mathcal{T}\gamma} = \hbar\pi \sum_j \left(n_{j\uparrow} - m_{j\uparrow}\right)$$

$$\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \xrightarrow{\mathcal{T}} \begin{pmatrix} -\psi_\downarrow \\ \psi_\uparrow \end{pmatrix}^*$$

$\mathbf{n}$  $\mathbf{m}$

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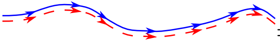
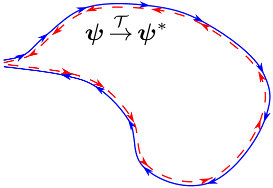
$$P\left(\mathbf{n},\mathbf{m},t\right) = \left|K\left(\mathbf{n},\mathbf{m},t\right)\right|^2 = \sum_{\gamma,\gamma':\mathbf{m}\to\mathbf{n}} \mathcal{A}_\gamma \mathcal{A}^*_{\gamma'} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(R_\gamma - R_{\gamma'}\right) + \mathrm{i}\frac{\pi}{2}\left(\mu_\gamma - \mu_{\gamma'}\right)}$$

- diagonal approximation $\gamma' = \gamma$:  $=: P_{\mathrm{cl}}\left(\mathbf{n},\mathbf{m},t\right)$
- interference between time-reverse paths

$$\gamma' = \mathcal{T}\gamma$$

$$\delta_{\mathbf{nm}} P_{\mathrm{cl}}\left(\mathbf{n},\mathbf{m},t\right) \qquad \left(-1\right)^N \delta_{\mathbf{n}_\uparrow \mathbf{m}_\downarrow} \delta_{\mathbf{n}_\downarrow \mathbf{m}_\uparrow} P_{\mathrm{cl}}\left(\mathbf{n},\mathbf{m},t\right)$$

$\mathbf{n} = \mathbf{m}$

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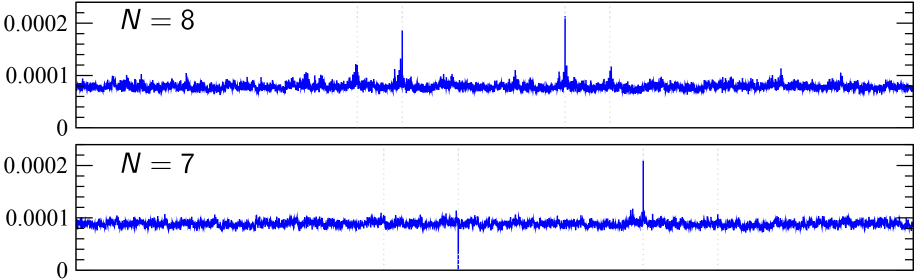

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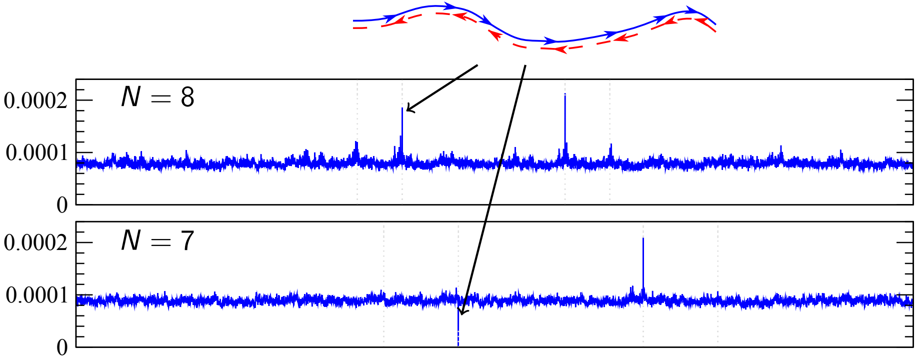

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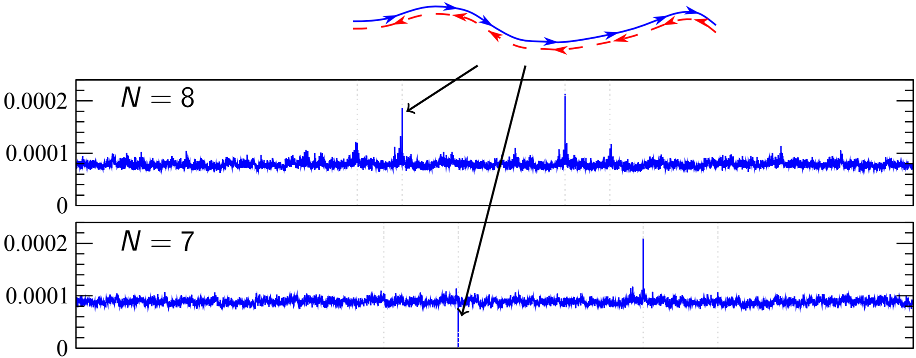


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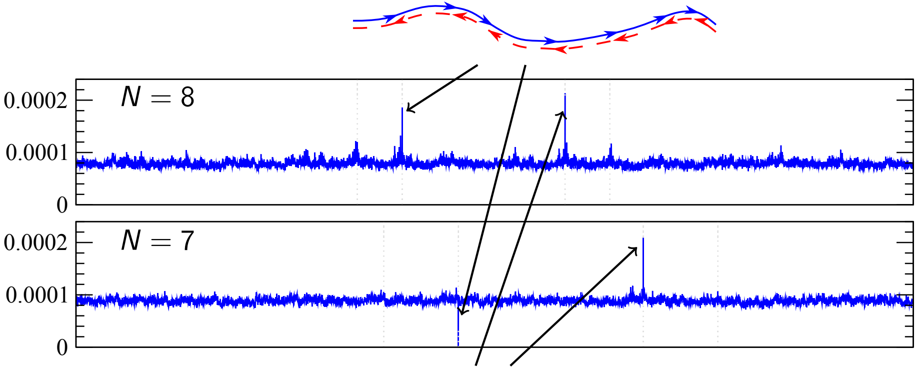
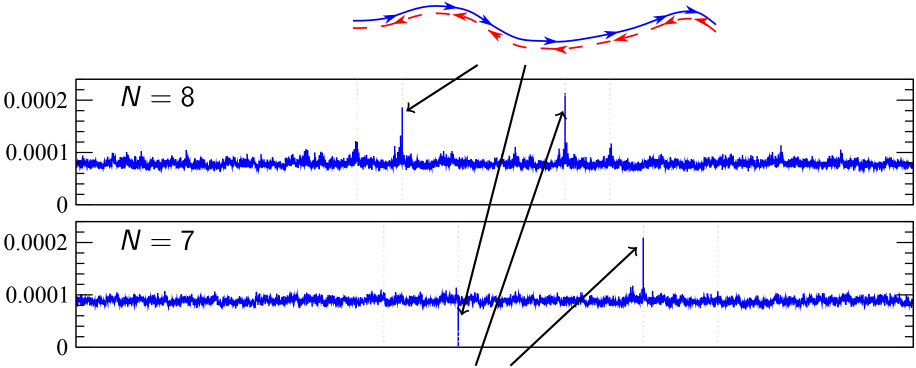



symmetry under $\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \rightarrow \begin{pmatrix} \psi_\downarrow e^{i\arg(\kappa)} \\ \psi_\uparrow e^{-i\arg(\kappa)} \end{pmatrix}$

# Transition probability

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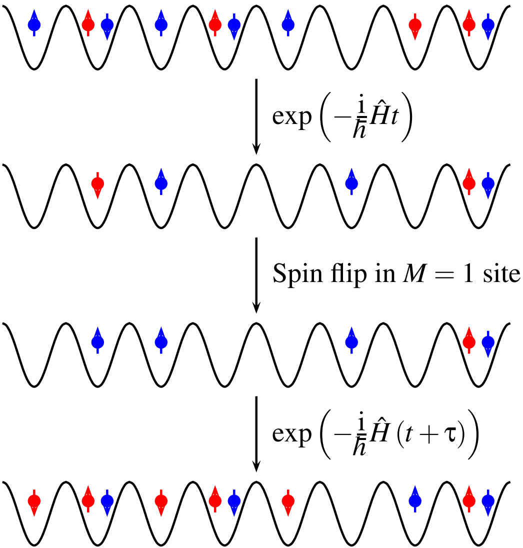
**Every discrete (antiunitary) symmetry gives another peak/dip!**

# Many-Body Spin Echo



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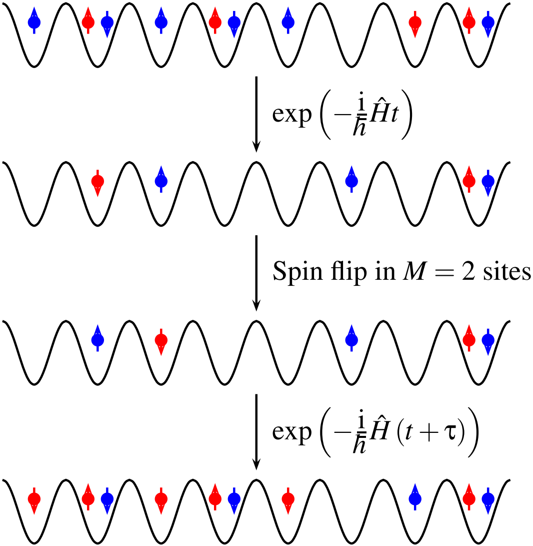

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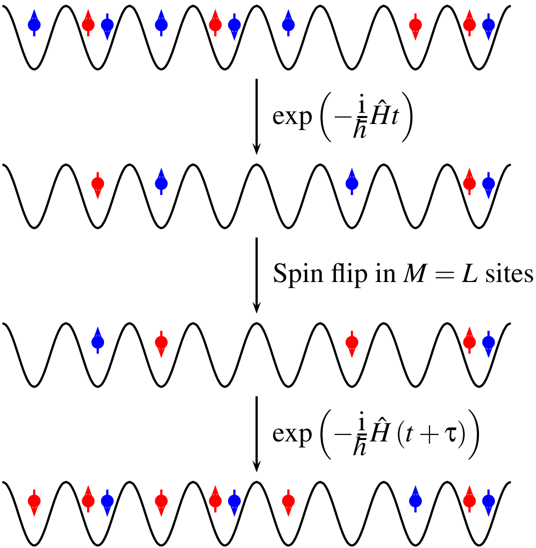
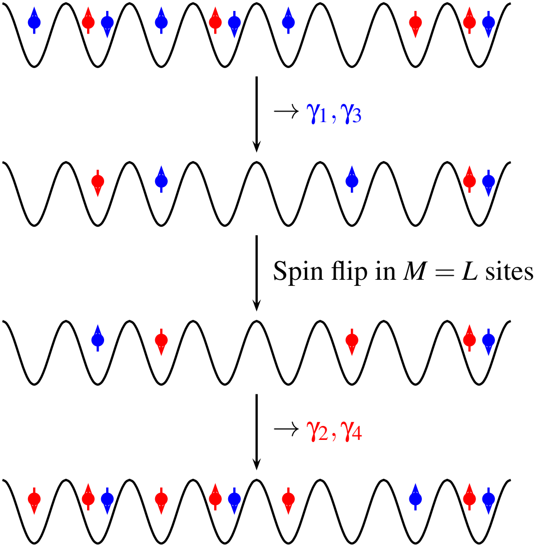
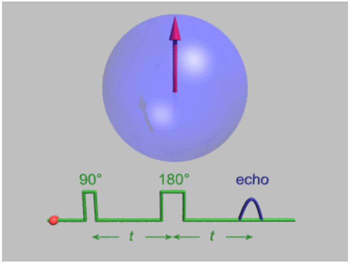


# Many-Body Spin Echo



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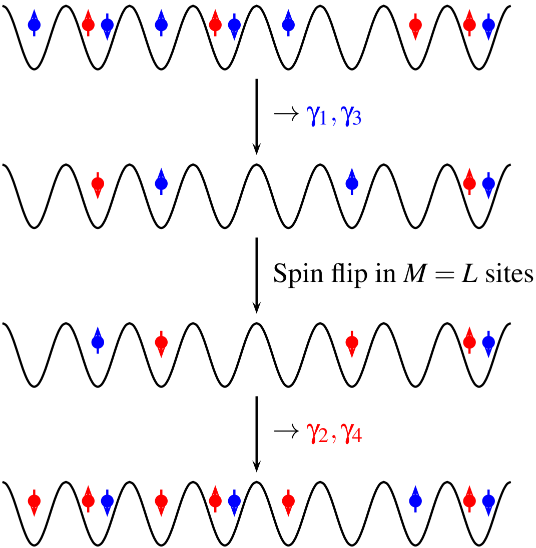
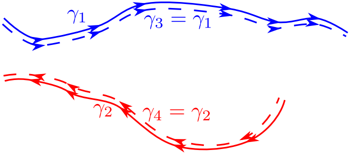
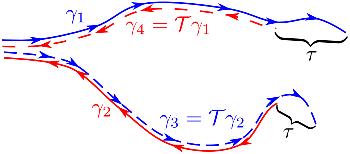


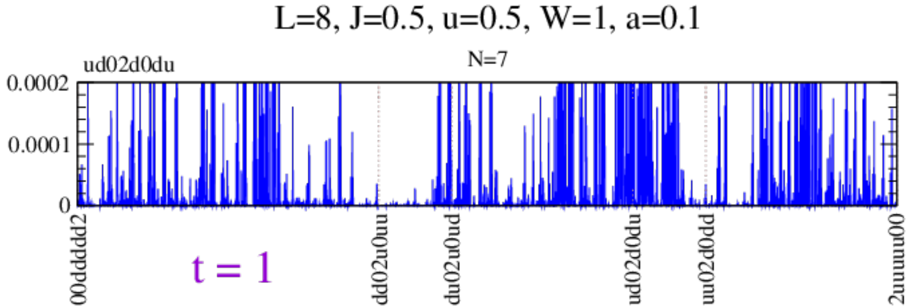




$$P^{(\mathrm{cl})}(\mathbf{n}, \mathbf{m}; t, \tau)$$

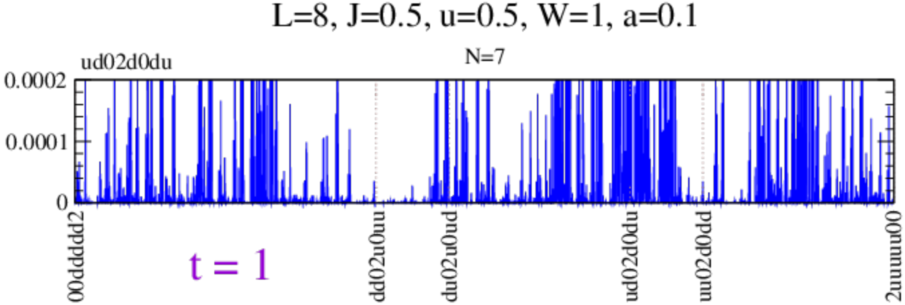



$$P^{(\mathrm{cl})}(\mathbf{n}, \mathbf{m}; t, \tau) \times \begin{cases} \delta_{\mathbf{n}, \mathcal{T}\mathbf{m}} & \tau = 0 \\ 0 & \tau \not\approx 0 \end{cases}$$

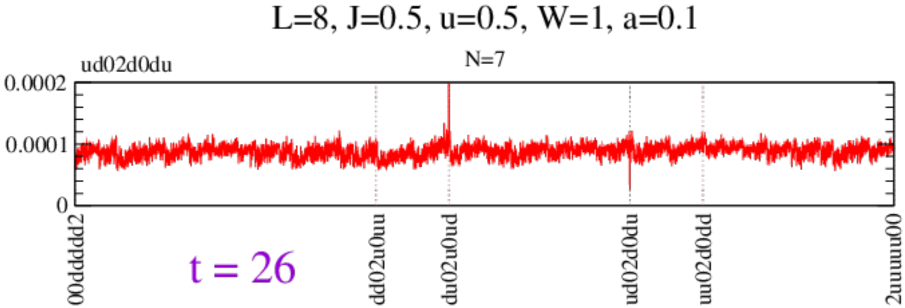
# Many-Body Spin Echo



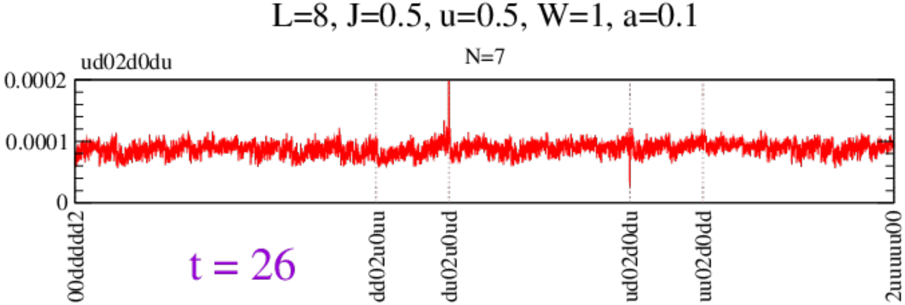
# Many-Body Spin Echo



# Many-Body Spin Echo



# Many-Body Spin Echo



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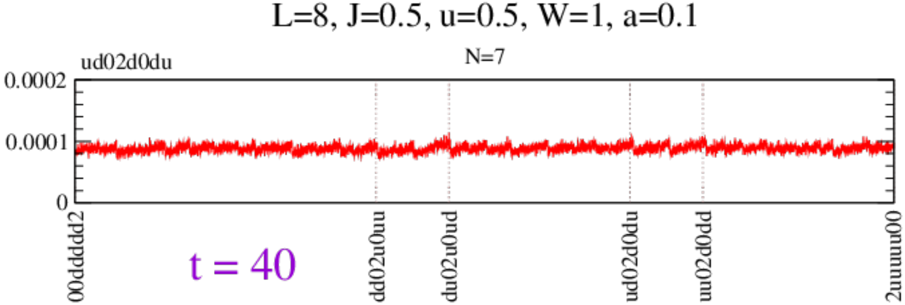

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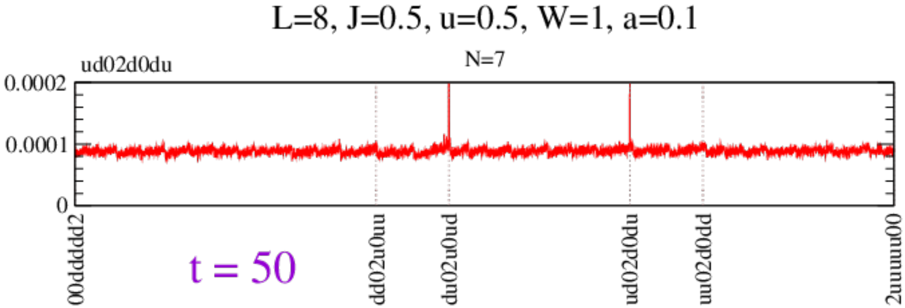

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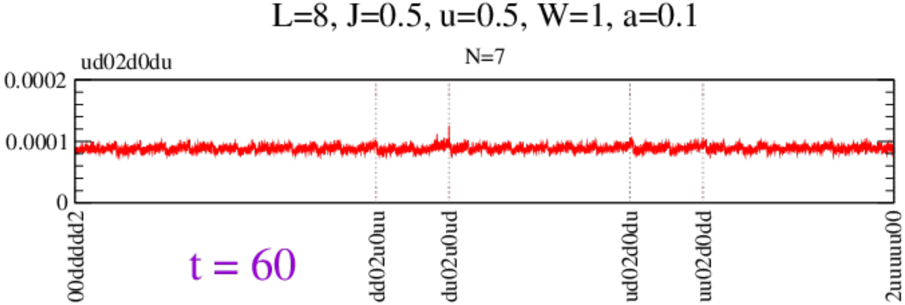

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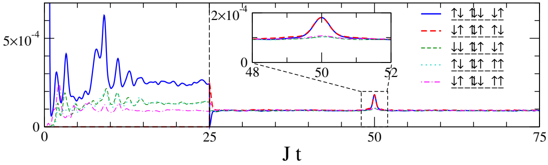
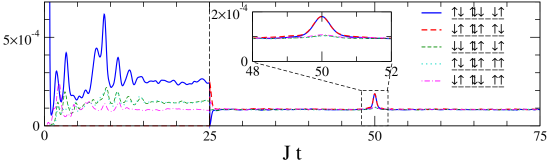


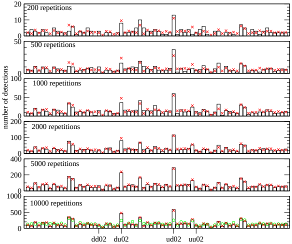

$$\frac{P^{\mathrm{MBSE}}(\mathbf{n},\mathbf{m};t,\tau)}{P^{(\mathrm{cl})}(\mathbf{n},\mathbf{m};t,\tau)} \approx 1 + \begin{cases} c_T(L,N,M)\, f(\tau) & \tau \sim 0,\ \mathbf{n} = T\mathbf{m} \\ c_{\mathrm{id}}(L,N,M;\varphi)\, f(\tau) & \tau \sim 0,\ \mathbf{n} = \mathbf{m} \\ 0 & \textit{else} \end{cases}$$

$$\varphi = \arg(\kappa)$$

TE, J.D. Urbina, P.Schlagheck, K. Richter, arXiv:1409.5684

# Many-Body Spin Echo

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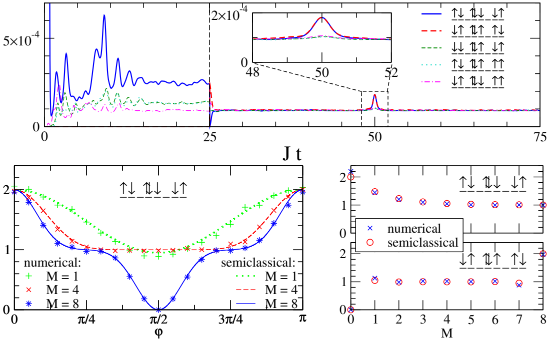


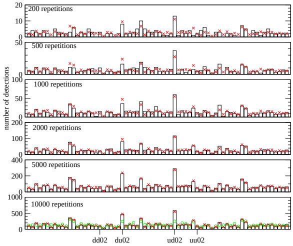


$$\frac{P^{\mathrm{MBSE}}(\mathbf{n},\mathbf{m};t,\tau)}{P^{(\mathrm{cl})}(\mathbf{n},\mathbf{m};t,\tau)} \approx 1 + \begin{cases} c_T(L,N,M)\,f(\tau) & \tau\sim 0,\ \mathbf{n}=T\mathbf{m} \\ c_{\mathrm{id}}(L,N,M;\varphi)\,f(\tau) & \tau\sim 0,\ \mathbf{n}=\mathbf{m} \\ 0 & \mathit{else} \end{cases}$$

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