Functional Integral Approach to Disordered Bosons in Traps

Robert Graham and Axel Pelster

UNIVERSITÄT DUISBURG ESSEN

1. Some Experimental Realizations
2. Theoretical Description
3. Huang-Meng Theory (T=0)
4. Collective Oscillations
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SFB/TR 12: Symmetries and Universality in Mesoscopic Systems

1.1 Overview of Set-Ups

- **Superfluid Helium in Porous Media**:

  Reppy *et al.*, PRL **51**, 666 (1983)

- **Laser Speckles**:

  Inguscio *et al.*, PRL **95**, 070401 (2005)

  Aspect *et al.*, PRL **95**, 170409 (2005)

- **Wire Traps:**

  Schmiedmayer *et al.*, eprint: `cond-mat/0504686`

- **Localized Atomic Species:**

  Gavish and Castin, PRL **95**, 020401 (2005)

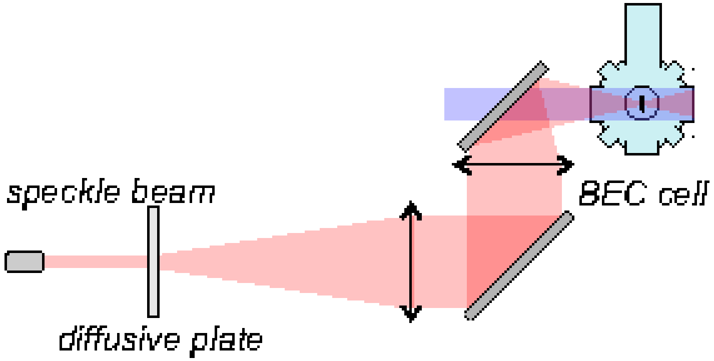
- **Incommensurate Lattices:**

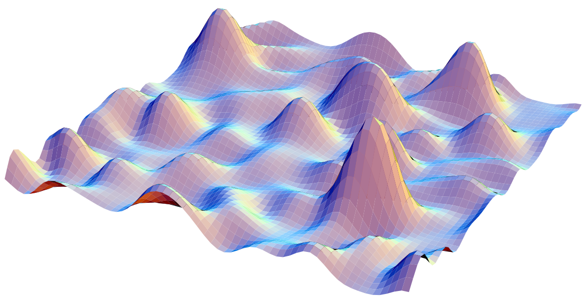
  Lewenstein *et al.*, PRL **91**, 080403 (2003)

  Ertmer *et al.* PRL **95**, 170411 (2005)

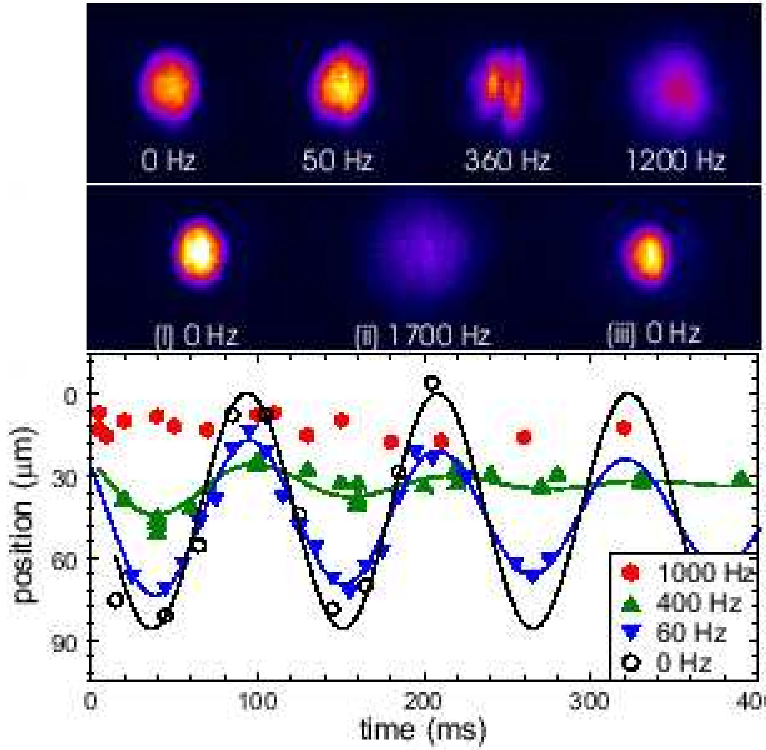
1.2 Magneto-Optical Trap

Laser Speckles:



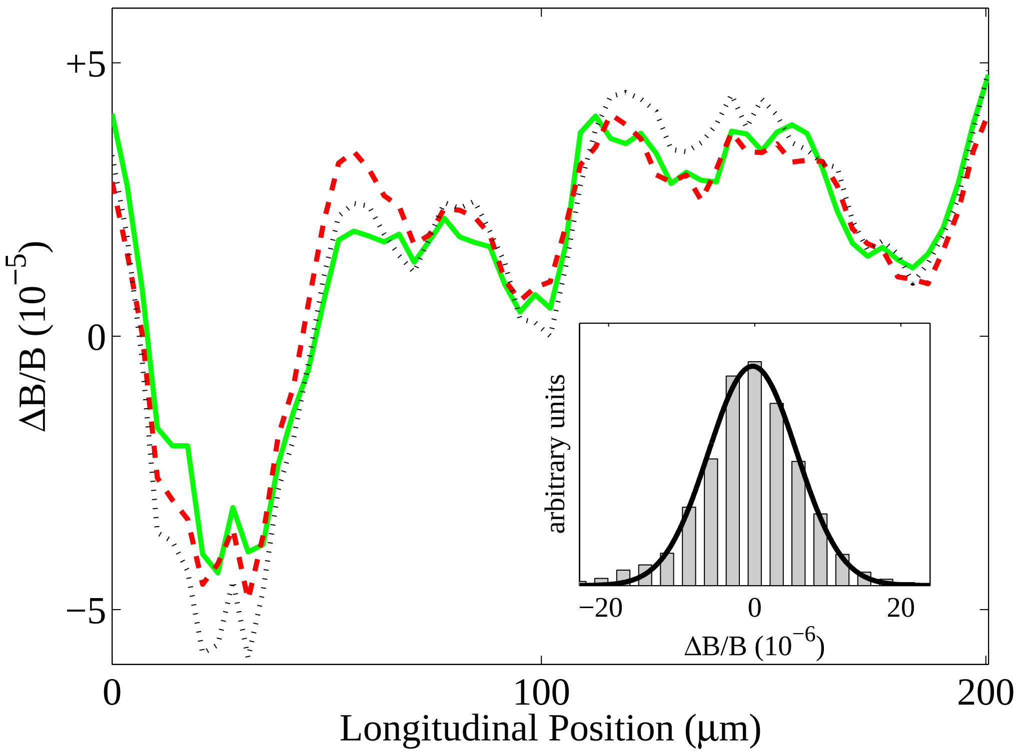


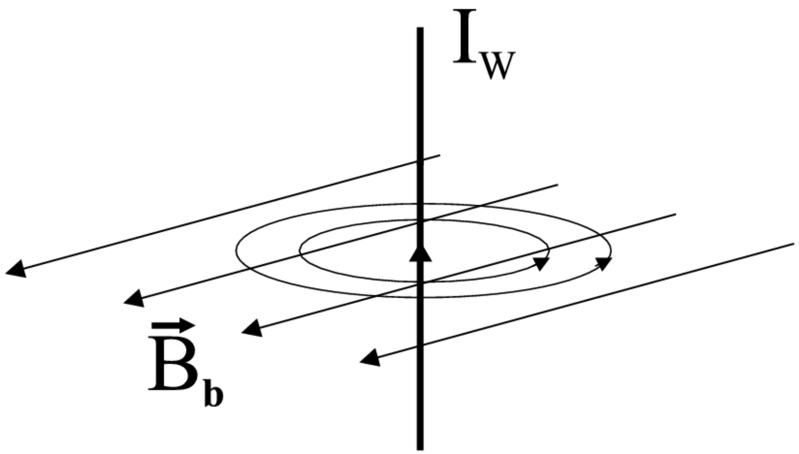
Inguscio *et al.*, PRL **95**, 070401 (2005)



global condensate vanishes

1.3 Wire Trap





Distance: $d = 10\ \mu$m

Wire Width: $100\ \mu$m

Magnetic Field: 10 G, 20 G, 30 G

Deviation: $\Delta B/B \approx 10^{-4}$

Krüger, Schmiedmayer *et al.*, eprint: `cond-mat/0504686`

2.1 Model System

Action of a Bose gas:

$$\mathcal{A} = \int_0^{\hbar\beta} d\tau \int d^3x \left\{ \psi^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M}\mathbf{\Delta} + U(\mathbf{x}) + V(\mathbf{x}) - \mu\right]\psi(\mathbf{x},\tau) + \frac{g}{2}\psi^*(\mathbf{x},\tau)^2\psi(\mathbf{x},\tau)^2 \right\}$$

Properties:

- harmonic trap potential: $U(\mathbf{x}) = \frac{M}{2}\sum_{i=1}^{3}\omega_i^2 x_i^2$
- disorder potential: $V(\mathbf{x})$; bounded from below, i.e. $V(\mathbf{x}) > V_0$
- chemical potential: μ
- repulsive interaction: $g = \frac{4\pi\hbar^2 a}{M}$
- periodic Bose fields: $\psi(\mathbf{x},\tau+\hbar\beta) = \psi(\mathbf{x},\tau)$

2.2 Random Potential

Disorder Ensemble Average:

$$\overline{\bullet} = \int \mathcal{D}V \bullet P[V]\,, \qquad \int \mathcal{D}V \, P[V] = 1\,, \qquad P\,[V < V_0] = 0$$

Assumption:

$$\overline{V(\mathbf{x}_1)} = 0\,, \qquad \overline{V(\mathbf{x}_1)V(\mathbf{x}_2)} = R^{(2)}(\mathbf{x}_1 - \mathbf{x}_2) = \frac{R}{(2\pi\xi^2)^{3/2}}\, e^{-\frac{(\mathbf{x}_1-\mathbf{x}_2)^2}{2\xi^2}}$$

Characteristic Functional:

$$\overline{\exp\left\{ i \int d^D x \, j(\mathbf{x}) V(\mathbf{x}) \right\}}$$

$$= \exp\left\{ \sum_{n=2}^{\infty} \frac{i^n}{n!} \int d^D x_1 \cdots \int d^D x_n \, R^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\, j(\mathbf{x}_1) \cdots j(\mathbf{x}_n) \right\}$$

2.3 Grand-Canonical Potential

Aim:

$$\begin{aligned} \Omega &= -\frac{1}{\beta}\,\overline{\ln \mathcal{Z}} \\ \mathcal{Z} &= \oint D^2\psi \, e^{-\mathcal{A}[\psi^*,\psi]/\hbar} \end{aligned}$$

Problem:

$$\overline{\ln \mathcal{Z}} \neq \ln \overline{\mathcal{Z}}$$

Solution: Replica Trick

$$\Omega = -\frac{1}{\beta} \lim_{N\to 0} \frac{\overline{\mathcal{Z}^N} - 1}{N}$$

2.4 Replica Trick

Disorder Averaged Partition Function:

$$\overline{\mathcal{Z}^N} = \overline{\oint \left\{ \prod_{\alpha'=1}^{N} D^2\psi_{\alpha'} \right\} e^{-\sum_{\alpha=1}^{N} \mathcal{A}([\psi_\alpha^*,\psi_\alpha])/\hbar}} = \oint \left\{ \prod_{\alpha=1}^{N} D^2\psi_\alpha \right\} e^{-\mathcal{A}^{(N)}/\hbar}$$

Replicated Action:

$$\mathcal{A}^{(N)} = \int_0^{\hbar\beta} d\tau \int d^D x \sum_{\alpha=1}^{N} \left\{ \psi_\alpha^*(\mathbf{x},\tau) \left[\hbar \frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M}\mathbf{\Delta} + U(\mathbf{x}) - \mu \right] \psi_\alpha(\mathbf{x},\tau) \right.$$

$$\left. + \frac{g}{2} \left|\psi_\alpha(\mathbf{x},\tau)\right|^4 \right\} + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\frac{-1}{\hbar} \right)^{n-1} \int_0^{\hbar\beta} d\tau_1 \cdots \int_0^{\hbar\beta} d\tau_n \int d^D x_1 \cdots \int d^D x_n$$

$$\times \sum_{\alpha_1=1}^{N} \cdots \sum_{\alpha_n=1}^{N} R^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \left|\psi_{\alpha_1}(\mathbf{x}_1,\tau_1)\right|^2 \cdots \left|\psi_{\alpha_n}(\mathbf{x}_n,\tau_n)\right|^2$$

$\Longrightarrow$ Disorder amounts to attractive interaction for $n = 2$

$\Longrightarrow$ Higher-order disorder cumulants negligible in replica limit $N \to 0$

3.1 Condensate Density

Assumption: homogeneous Bose gas $U(\mathbf{x}) = 0$, $T = 0$

Bogoliubov Theory:

background method: $\psi_\alpha(\mathbf{x}, \tau) = \Psi_\alpha + \delta\psi_\alpha(\mathbf{x}, \tau)$

replica symmetry: $\Psi_\alpha = \sqrt{n_0}$

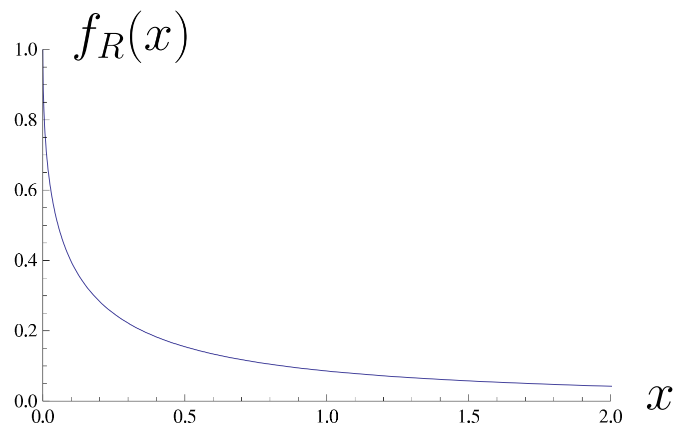
$\Longrightarrow$ In the replica limit $N \to 0$ only $R^{(2)}(\mathbf{x})$ contributes

Result: $n_0 = n - n_g - n_R$

interaction: $n_g = \frac{8}{3\sqrt{\pi}} \sqrt{a\, n_0}^3$

disorder: $n_R = \frac{M^2 R}{8\pi^{3/2}\hbar^4} \sqrt{\frac{n_0}{a}}\, f_R\left(4\pi n_0 \xi^2 a\right)$

$$f_R(x) = e^{2x}(1+4x)\,\mathrm{Erfc}\left(\sqrt{2x}\right) - 2\sqrt{\frac{2x}{\pi}}$$



Kobayashi, Tsubota, PRB **66**, 174516 (2002)

3.2 Superfluid Density

Galilei Boost:

$$\Delta\mathcal{A} = \int_0^{\hbar\beta} d\tau \int d^3x\, \psi^*(\mathbf{x},\tau)\, \mathbf{u}\, \frac{\hbar}{i} \boldsymbol{\nabla}\, \psi(\mathbf{x},\tau)$$

$$d\Omega = -S\, dT - p\, dV - N\, d\mu - \mathbf{p}\, d\mathbf{u}$$

$$\mathbf{p} = -\left.\frac{\partial\Omega(T,V,\mu,\mathbf{u})}{\partial\mathbf{u}}\right|_{T,V,\mu} = MV\, n_n\, \mathbf{u} + \ldots$$

Result for $T = 0$: $$n_s = n - n_n = n - \frac{4}{3} n_R$$

Huang and Meng, PRL **69**, 644 (1992)

4.1 Hydrodynamic Equations

Phenomenological Two-Fluid Model:

$$\begin{aligned} \frac{\partial}{\partial t} n + \boldsymbol{\nabla} \left(\mathbf{v}_s n_s + \mathbf{v}_n n_n \right) &= 0 \\ M \frac{\partial}{\partial t} \mathbf{v}_s + \boldsymbol{\nabla} \left(\mu + \frac{1}{2} M \mathbf{v}_s^2 \right) &= 0 \end{aligned}$$

Pinning of normal component (= fourth sound): $\mathbf{v}_n = 0$

Decomposition: $n(\mathbf{x}, t) = n(\mathbf{x}) + \delta n(\mathbf{x}, t)$

Wave Equation:

$$M \frac{\partial^2}{\partial t^2} \delta n \left(\mathbf{x}, t \right) - \boldsymbol{\nabla} \left[n_s \left(\mathbf{x} \right) \boldsymbol{\nabla} \left(\frac{\partial \mu \left[n \left(\mathbf{x} \right) \right]}{\partial n \left(\mathbf{x} \right)} \delta n \left(\mathbf{x}, t \right) \right) \right] = 0$$

Huang-Meng theory, trap in Thomas-Fermi approximation

4.2 Collective Excitations

Hydrodynamic Equation in Trap With Disorder:

$$m\,\frac{\partial^2}{\partial t^2}\,\delta n(\mathbf{x},t) - \boldsymbol{\nabla}\left[g n_{\mathrm{TF}}(\mathbf{x})\boldsymbol{\nabla}\delta n(\mathbf{x},t)\right]$$
$$= -\boldsymbol{\nabla}^2\left[3 g n_R(\mathbf{x})\delta n(\mathbf{x},t)\right] - \boldsymbol{\nabla}\left[\frac{4g}{3} n_R(\mathbf{x})\boldsymbol{\nabla}\delta n(\mathbf{x},t)\right]$$

$n_R(\mathbf{x})$: Huang-Meng depletion in trap

$n_{\mathrm{TF}}(\mathbf{x}) = \left[\mu - V(\mathbf{x})\right]/g$: Thomas-Fermi density

Violation of Kohn Theorem:

Surface Dipole Mode
$(n = 0\,, l = 1)$:

$$\frac{\delta\omega_{\mathrm{dip}}(\xi = 0)}{\omega_{\mathrm{trap}}} = -\frac{5\pi}{16}\,\frac{M^2 R}{8\pi^{3/2}\hbar^4\sqrt{n_{\mathrm{TF}}(\mathbf{0})a}}$$

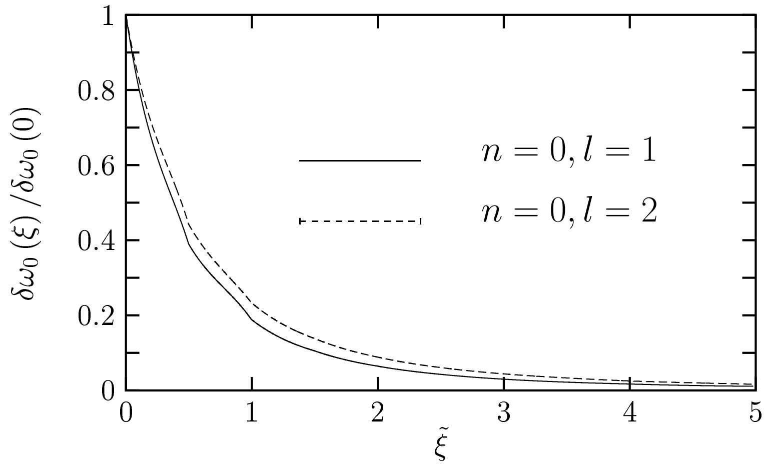
Falco, Pelster, and Graham, Phys. Rev. A **76**, 013624 (2007)

4.3 Comparison With Experiment

Typical Values:

Inguscio *et al.*, PRL **95**, 070401 (2005)

$$\left.\begin{array}{rcl} \xi & = & 10\ \mu\text{m} \\ R_{\rm TF} & = & 100\ \mu\text{m} \\ a_{\rm HO} & = & 10\ \mu\text{m} \end{array}\right\} \quad \tilde{\xi} = \frac{\xi R_{\rm TF}}{a_{\rm HO}^2\sqrt{2}} \approx 7$$



$\Longrightarrow$ Disorder effect vanishes in laser speckle experiment

Improvement:

laser speckle setup with correlation length $\xi = 1\ \mu$m

Aspect *et al.*, New J. Phys. **8**, 165 (2006)

$\Longrightarrow$ Disorder effect should be measurable

5. Summary and Outlook

- **Frozen Disorder Potential**:

  arises both artificially (laser speckles) or naturally (wire trap)

- **Bosons:**

  local condensates in minima + global condensate + thermally excited

- **Localization Versus Transport:**

  disorder reduces superfluidity

- **Phase Diagram:**

  Timmer, Pelster, and Graham, Europhys. Lett. **76**, 760 (2006)

  Navez, Pelster, and Graham, Appl. Phys. B **86**, 395 (2007)

  Yukalov and Graham, Phys. Rev. A **75**, 023619 (2007)

  Falco, Pelster, and Graham, Phys. Rev. A **75**, 063619 (2007)

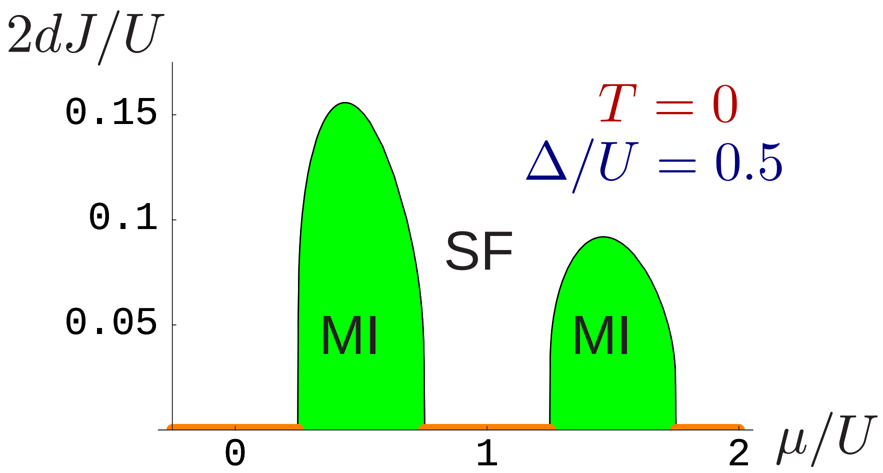
Disordered Bosons in Lattice

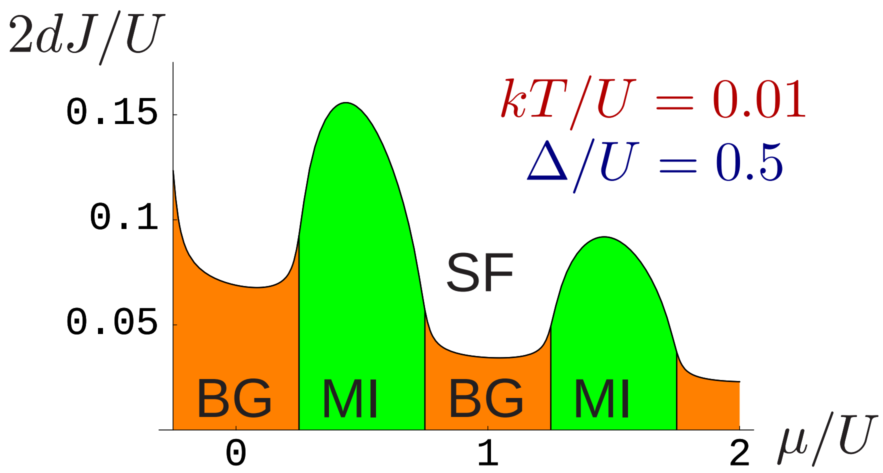
Bose-Hubbard Hamilton Operator:

$$\hat{H}_{BH} = -J \sum_{<\mathrm{i},\mathrm{j}>} \hat{a}_\mathrm{i}^\dagger \hat{a}_\mathrm{j} + \frac{U}{2}\sum_\mathrm{i} \hat{a}_\mathrm{i}^\dagger \hat{a}_\mathrm{i}^\dagger \hat{a}_\mathrm{i} \hat{a}_\mathrm{i} + \sum_\mathrm{i} (\epsilon_\mathrm{i} - \mu)\, \hat{a}_\mathrm{i}^\dagger \hat{a}_\mathrm{i}$$

$$\overline{\bullet} = \prod_\mathrm{i} \int_{-\infty}^{+\infty} \bullet\, p(\epsilon_\mathrm{i})\, d\epsilon_\mathrm{i} \;, \qquad p(\epsilon_\mathrm{i}) = \begin{cases} 1/\Delta & ; \; \epsilon_\mathrm{i} \in [-\Delta/2, +\Delta/2] \\ 0 & ; \; \text{otherwise} \end{cases}$$

Mean-Field Phase Diagram:





Krutitsky, Pelster, and Graham, New J. Phys. **8**, 187 (2006)