Rigorous Functional Integration with Applications to Nelson's and the Pauli-Fierz Model

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Definition (Nelson's Hamiltonian)

$$H_{\mathrm{N}} = H_{\mathrm{p}} + H_{\mathrm{f}} + H_{\mathrm{i}} \quad \text{on } L^2(\mathbb{R}^d, dx) \otimes \mathcal{F}$$

with

$$H_{\mathrm{p}} = (-(1/2)\Delta + V(x)) \otimes 1$$

$$H_{\mathrm{f}} = 1 \otimes \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk$$

$$H_{\mathrm{i}} = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} \frac{1}{\sqrt{\omega(k)}} \left(\widehat{\varrho}(k) e^{ik\cdot x} \otimes a(k) + \text{h.c.} \right), \qquad \varrho \neq \delta \quad \textit{UV cutoff}$$

- *self-adjoint* on $D(H_{\mathrm{p}}) \cap D(H_{\mathrm{f}})$ if $|\widehat{\varrho}|\omega^{-1/2}, |\widehat{\varrho}|\omega^{-1} \in L^2$
- unique, strictly positive *ground state* $\Psi \in L^2 \otimes \mathcal{F}$ if $|\widehat{\varrho}|\omega^{-3/2} \in L^2$ *IR cutoff* [Spohn 1998]

Definition (Pauli-Fierz model with spin)

$$
\begin{aligned}
H_{\mathrm{PF}} &= \frac{1}{2}(\sigma \cdot (-i\nabla \otimes 1 - eA))^2 + V \otimes 1 + 1 \otimes H_{\mathrm{f}} \\
&= \frac{1}{2}(-i\nabla \otimes 1 - eA)^2 + V \otimes 1 + 1 \otimes H_{\mathrm{f}} - \frac{e}{2}\sigma \cdot B
\end{aligned}
$$

with

$$
A_\mu(x) := \frac{1}{\sqrt{2}} \sum_{j=\pm 1} \int e_\mu(k,j) \left(\frac{\widehat{\varrho}(k)}{\sqrt{\omega(k)}} e^{ik\cdot x} \otimes a(k,j) + \text{h.c.} \right) dk
$$

$$
\sum_{j=\pm 1} e_\mu(k,j) e_\nu(k,j) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}
$$

ground state: Bach-Fröhlich-Segal 1999, Griesemer-Lieb-Loss 2001

Aims

- derive and prove qualitative behaviour of system, i.e.,

$$(\Psi, A\Psi)_{L^2 \otimes \mathcal{F}}$$

  for A of interest

- how serious are IR & UV cutoffs, i.e., does there exist a ground state in $L^2(\mathbb{R}^d, dx) \otimes \mathcal{F}$ if cutoff conditions are lifted
- how robust are qualitative properties, i.e., by removing cutoff do ground state expectations change

Method: use tools of stochastic analysis via Feynman-Kac formula

- rough paths analysis
- Lévy processes
- cluster expansion

use jointly *ground state transform* and *Wiener-Itô transform*

- $e^{-tH_{\mathrm{p}}} \implies$ *$P(\phi)_1$-process* $\mathbb{R} \ni t \mapsto X_t \in \mathbb{R}^d$

  path measure $d\mathcal{N}_t^0(X) = e^{-\int_0^t V(X_s)ds} d\mathcal{W}_t(X)$

- $e^{-tH_{\mathrm{f}}} \implies$ *Ornstein-Uhlenbeck process* $\mathbb{R} \ni t \mapsto \xi_t \in \mathcal{S}'(\mathbb{R}^d)$

  path measure $\mathcal{G}$, $\mathbb{E}_{\mathcal{G}}[\xi_t(f)] = 0$

  $$\mathbb{E}_{\mathcal{G}}[\xi_t(f)\xi_s(g)] = \int_{\mathbb{R}^d} \widehat{f}(k)\overline{\widehat{g}(k)}(2\omega(k))^{-1} e^{-\omega(k)|t-s|} dk$$

- $H_{\mathrm{i}} \mapsto (\xi_t * \varrho)(x)$

Theorem (Feynman-Kac-Nelson)

$$(F, e^{-t\tilde{H}_{\mathrm{N}}} G)_{L^2} = \int \overline{F(X_0, \xi_0)} G(X_t, \xi_t) \underbrace{e^{-\int_0^t (\xi_t * \varrho)(X_s) ds} d(\mathcal{N}_t^0 \times \mathcal{G})}_{\mathcal{P}_t \,=\, \textit{path measure int. syst.}}$$

Structure of path measure

mixture of *Gaussian* and *Gibbsian*

$$\mathcal{P}_T(\cdot) = \int \mathcal{P}_T(\cdot|X) d\mathcal{N}_T(X) \quad \text{on} \quad C\left([-T,T], \mathbb{R}^d \times \mathcal{S}'(\mathbb{R}^d)\right)$$

with

$$d\mathcal{N}_T = \frac{1}{Z_T} e^{-\int_{-T}^T \int_{-T}^T W(X_t - X_s, t-s) dt ds} d\mathcal{N}_T^0$$

$$Z_T = \int e^{-\int_{-T}^T (\xi_t * \varrho)(X_t) dt} d(\mathcal{N}_T^0 \times \mathcal{G}) = \int e^{-\int_{-T}^T \int_{-T}^T W(X_t - X_s, t-s) dt ds} d\mathcal{N}_T^0$$

$$W(x,t) = -\frac{1}{4} \int_{\mathbb{R}^d} \frac{|\widehat{\varrho}(k)|^2}{\omega(k)} \cos(k \cdot x) e^{-\omega(k)|t|} dk$$

Theorem

$$\exists\, \mathcal{N} = \lim_{T\to\infty} \mathcal{N}_T \implies \exists\, \mathcal{P} = \lim_{T\to\infty} \mathcal{P}_T$$

Theorem

if $e = \int \varrho(x)dx$ *small and* $V(x) \simeq |x|^{2a}$, $a > 1$, *then*

(1) $\exists\, \mathcal{N} = \lim_{T_n \to \infty} \mathcal{N}_{T_n}$

(2) $\mathcal{N}$ *uniquely supported on* $C(\mathbb{R}, \mathbb{R}^d)$

(3) $\mathcal{N}$*-a.s.* $|X_t| \;\leq\; C(\log(|t|+1))^{1/(a+1)} + Q(X)$

(4) $\exists\, C, \gamma > 0$ *s.t.* $\forall F, G$ *bounded*

$$\mathrm{cov}_{\mathcal{N}}(F(X_s), G(X_t)) \;\leq\; C\, \frac{\sup |F| \sup |G|}{|s-t|^\gamma + 1}$$

(5) $\forall T > 0$, $\mathrm{N}_T \ll \mathrm{N}^0$ *and* $d\mathrm{N}_T/d\mathrm{N}^0 \to d\mathrm{N}/d\mathrm{N}^0$

L-Minlos 2001, Betz-L 2003

Fock space quantization

- $\mathcal{H}^0 := L^2(\mathbb{R}^d \times \mathcal{S}'(\mathbb{R}^d), d\mathsf{P}^0) \simeq L^2(\mathbb{R}^d, dx) \otimes \mathcal{F}$
- $\tilde{H}_{\mathrm{N}} \simeq H_{\mathrm{N}}$
- $\Phi \leftrightarrow \Psi$

Euclidean quantization

- $\mathcal{H} := L^2(\mathbb{R}^d \times \mathcal{S}'(\mathbb{R}^d), d\mathsf{P})$
- $\mathbb{E}_{\mathcal{P}}[F(X_0, \xi_0)G(X_t, \xi_t)] = (\bar{F}, T_t G)_{\mathcal{H}}$

  $T_t = e^{-tH_{\mathrm{euc}}}$ — *Euclidean Hamiltonian*, s.a., semibd
- 1 is g.s. of H_{euc}; unique since T_t positivity improving

Theorem

$$\mathsf{P} \ll \mathsf{P}^0 \implies \Phi^2 = \frac{d\mathsf{P}}{d\mathsf{P}^0}$$

Theorem

B s.a. in $L^2(\mathsf{G})$, $\mathbb{E}_{\mathcal{N}}\left[\|B : e^{\xi(f_X^{\pm})} :\|^2_{L^2(\mathsf{G})}\right] < \infty$, $g \in L^\infty(\mathbb{R}^d)$

$$(\Phi, g \otimes B\,\Phi)_{\mathcal{H}^0} = \mathbb{E}_{\mathcal{N}}\left[M_B\, g(X_0)\, e^{2\int_{-\infty}^0 \int_0^\infty W(X_s - X_t, s-t)dsdt}\right]$$

with

$$M_B = \left(: e^{\xi(q_X^-)} :, B : e^{\xi(q_X^+)} :\right)_{L^2(\mathsf{G})}$$

$$q_X^{\pm} = -\int_{\mathbb{R}^\pm} \widehat{\varrho}(k) \cos(k \cdot X_t) e^{-\omega(k)|t|} dt$$

Betz, Hiroshima, Minlos, L, Spohn 2002

with

$$\pi_n : \mathcal{F} \to \mathcal{F}^{(n)}, \quad p_n := (\Phi, 1 \otimes \pi_n \Phi)_{\mathcal{H}^0}$$

$$I := \int_{\mathbb{R}} |\widehat{\varrho}|^2 \omega^{-3} dk, \quad w := \int_{-\infty}^{0} \int_{0}^{\infty} W(X_s - X_t, s-t) ds dt$$

Corollary

(1) $\quad p_n = (1/n!)\, \mathbb{E}_{\mathcal{N}} \left[(-2w)^n \, e^{2w} \right]$

(2) $\quad p_n \le (I^n/n!)\, e^I$

(3) $\quad (\Phi, e^{\alpha N} \Phi)_{\mathcal{H}^0} < \infty, \ \forall \alpha > 0$

for massive bosons $\quad p_n \ge (D^n/n!)\, e^{-I}, \ D < I$

with

$$\chi := \psi^2 \int \Phi(x,\xi)^2 d\mathcal{G}(\xi), \quad V_\omega := \widehat{-1/\omega^2}$$

Corollary

(1) $(\Phi, \xi(k)\Phi)_{\mathcal{H}^0} = \dfrac{\widehat{\varrho}(k)}{(2\pi)^{d/2}\omega(k)^2}$

(2) $(\Phi, \xi(x)\Phi)_{L^2(\mathsf{P}^0)} = (\chi * V_\omega * \varrho)(x) \to \dfrac{1}{|x|}$, *as* $|x| \to \infty$

(3) *field fluctuations increase on coupling particle to field*

Lemma (Diamagnetic Inequality)

with $f, g \in \mathcal{H}^0$

$$(f, e^{-t\tilde{H}_{\mathrm{N}}} g)_{\mathcal{H}^0} \;\leq\; e^{tI} \left(\|f\|_{L^2(\mathsf{G})}, e^{-t\tilde{H}_{\mathrm{P}}} \|g\|_{L^2(\mathsf{G})} \right)_{L^2(\mathsf{N}^0)}$$

Theorem

with $V(x) \simeq |x|^{2a}$, $\exists\, C_1, C_2 > 0$

$$\chi(x) \;\leq\; C_1\, e^{-C_2 |x|^{a+1}}$$

if total charge small, then $\exists\, C_3, C_4 > 0$

$$\chi(x) \;\geq\; C_3\, e^{-C_4 |x|^{a+1}}$$

Definition (IR divergence)

$\tilde{H}_{\mathrm{N}}$ is IR divergent if has no ground state in $\mathcal{H}^0$

Theorem (Characterization of IR divergence)

suppose $\mathcal{N}$ exists; then: $\tilde{H}_{\mathrm{N}}$ *is IR divergent* $\iff \mathsf{P} \perp \mathsf{P}^0$

L-Minlos-Spohn 2002a

Theorem (3D IR divergence)

no IR assumption; if $d = 3$, then

(1) *particle charge small* $\Longrightarrow$ $\tilde{H}_{\mathrm{N}}$ *IR divergent*

(2) $\varrho \geq 0$, $\lim_{|x|\to\infty} V(x) = \infty$ $\Longrightarrow$ $\tilde{H}_{\mathrm{N}}$ *IR divergent*

Theorem (Higher-dimensional IR regularity)

if $d \geq 4$, then $\tilde{H}_{\mathrm{N}}$ has unique g.s. $\Phi \in \mathcal{H}^0$ and $\Phi^2 = dP/dP^0$

Corollary

- $\tilde{H}_{\mathrm{N}}$ *IR divergent* $\Longleftrightarrow$ H_{N} *has no ground state in* $L^2(\mathbb{R}^d) \otimes \mathcal{F}$
- *in 3D* $\mathcal{H}^0 \not\simeq \mathcal{H}$ *resp.* $\tilde{H}_{\mathrm{N}} \not\simeq H_{\mathrm{euc}}$
- *in* $> 3D$ $\mathcal{H}^0 \simeq \mathcal{H}$ *resp.* $\tilde{H}_{\mathrm{N}} \simeq H_{\mathrm{euc}}$

Theorem

with h s.t. $\widehat{h} \in \mathbb{R}$, even, bd and $\widehat{h}(0) = 1$

$$
\begin{aligned}
H_{\mathrm{N}}^{\mathrm{ren}} &= \int_{\mathbb{R}^3} dk \frac{\widehat{\varrho}(k)}{\sqrt{2\omega(k)}} \left(e^{ik\cdot x} - \widehat{h}(k) \right) \otimes a(k) + h.c. \\
&\quad - \int_{\mathbb{R}^3} dk \frac{|\widehat{\varrho}(k)|^2}{\omega(k)^2} \widehat{h}(k) \left(e^{ik\cdot x} - \widehat{h}(k) \right) \otimes 1 \\
&\quad + H_{\mathrm{p}} \otimes 1 + 1 \otimes H_{\mathrm{f}}
\end{aligned}
$$

is unitary equivalent with H_{euc} and has a unique strictly positive ground state

L-Minlos-Spohn 2002b

point charge limit $\varrho(x) \to \delta(x)$

scaling charge distribution $\widehat{\varrho}_\Lambda(k) = \widehat{\varrho}(k/\Lambda)$

UV limit $\Lambda \to \infty$

ground state energy (perturbatively)

$$E_\Lambda = -e^2 \int \frac{|\widehat{\varrho}_\Lambda(k)|^2}{2\omega(k)} \frac{1}{\omega(k) + |k|^2/2} dk + O(e^4) \ \sim \ -\log\Lambda$$

Gross transform

$$T_\Lambda = -e \int \frac{|\widehat{\varrho}_\Lambda(k)|}{\sqrt{2\omega(k)}} \frac{1}{\omega(k) + |k|^2/2} \left(e^{ik\cdot x} \otimes a(k) + \text{h.c.} \right) dk$$

Gross transformed Nelson Hamiltonian

$$e^T H_\mathrm{N} e^{-T} = \text{terms dependent on a vector potential} + E_\Lambda$$

Theorem (UV renormalized Hamiltonian)

$$H_\mathrm{N}^\mathrm{UV} := \lim_{\Lambda\to\infty} (e^{T_\Lambda} H_\mathrm{N} e^{-T_\Lambda} - E_\Lambda)$$

exists, and for small e is self-adjoint and bd below

Nelson 1964

Regularized functional interaction

interaction (ill defined) $W(x,t) = -\int_{\mathbb{R}^3} 1/(2|k|)\cos(k\cdot x)e^{-|k||t|}dk$

Definition (regularized interaction)

$$W^\varepsilon(x,t) = -\frac{1}{2}\int_{\mathbb{R}^3}\frac{1}{2|k|}e^{-\varepsilon|k|^2}\cos(k\cdot x)e^{-|k||t|}dk, \quad \varepsilon > 0$$

Lemma

$\forall\varepsilon > 0,\ \forall[-T,T]\subset\mathbb{R}$

$$W_T^\varepsilon(X) = 2\int_{-T}^{T}dt\int_{-T}^{t}\nabla\varphi_\varepsilon(X_t - X_s, t-s)dX_s$$
$$+4T\varphi_\varepsilon(0,0) - 2\int_{-T}^{T}\varphi_\varepsilon(X_t - X_{-T}, t+T)dt$$

with $\varphi_\varepsilon(x,t) = -\int_{\mathbb{R}^3}(2|k|(|k| + |k|^2/2))^{-1}e^{-\varepsilon|k|^2}\cos(k\cdot x)e^{-|k||t|}dk$

Definition (renormalized regular pair potential)

$$\widetilde{W}_T^\varepsilon(X) = W_T^\varepsilon(X) - 4T\varphi_\varepsilon(0,0)$$

Lemma

(1) *$\widetilde{W}_T^\varepsilon(X)$ converges to a random variable $\widetilde{W}_T^0(X)$ as $\varepsilon \to 0$*

(2) *with $\psi(x,t) = \int_{\mathbb{R}^3}(|k|(|k|+|k|^2/2)^2)^{-1}\cos(k\cdot x)(e^{-|k||t|}-1)$*

$$\begin{aligned}\widetilde{W}_T^0(X) &= -2\int_{-T}^T dX_s \int_s^T \nabla^2\psi(X_t - X_s, t-s)dX_t \\ &+ 2\int_{-T}^T \nabla\psi(X_T - X_s, T-s)dX_s - 2\psi(X_T - X_{-T}, 2T) \\ &+ 2\int_{-T}^T \nabla\psi(X_t - X_{-T}, t+T)dX_t\end{aligned}$$

Lemma

e small $\Longrightarrow$ $\widetilde{W}^0_I(X)$ *exponentially integrable wrt Brownian bridge*

Theorem (renormalized Gibbs measure)

$\forall T > 0$, *small* $e > 0$, $x, y \in \mathbb{R}^3$

$$\begin{aligned}
\exists\ d\mu_T(X|x,y) &:= w - \lim_{\varepsilon \to 0} d\mu^\varepsilon_T(X|x,y) \\
&:= w - \lim_{\varepsilon \to 0} \frac{e^{-W^\varepsilon_T(X) - \int_{-T}^T V(X_s)ds}}{Z^\varepsilon_T(x,y)} d\mathcal{W}^{x,y}_T(X) \\
&= \frac{e^{-\widetilde{W}^0_T(X) - \int_{-T}^T V(X_s)ds}}{\widetilde{Z}^0_T(x,y)} d\mathcal{W}^{x,y}_T(X).
\end{aligned}$$

Gubinelli-L 2007a&b

Theorem

$$(F, e^{-tH_{\mathrm{PF}}}G) = \lim_{\varepsilon\to 0} e^t \sum_{\sigma=\pm 1} \mathbb{E}^{x,\sigma} \int dx \left[e^{-\int_0^t V(B_s)ds} \int_{\mathcal{Q}} \overline{J_0 F(B_0,\sigma_0)} e^{U_t(\varepsilon)} J_t G(B_t,\sigma_t) d\mathcal{P}^0 \right]$$

with

$$U_t(\varepsilon) = -ie\sum_{\mu=1}^3 \int_0^t \mathcal{A}_\mu(j_s\lambda(\cdot - B_s))dB_s^\mu - \int_0^t H_{\mathrm{on}}(B_s,\sigma_s,s)ds$$
$$+ \int_0^{t+} \log\left(-H_{\mathrm{off}}(B_s,-\sigma_{s-},s) - \varepsilon\psi_\varepsilon(H_{\mathrm{off}}(B_s,-\sigma_{s-},s))\right) dN_s$$

$$\sigma_t = \sigma(-1)^{N_t},\ \sigma = \pm 1$$

$$H_{\mathrm{on}}(x,\sigma,s) = -\frac{e}{2}\sigma\mathcal{B}_3(j_s\lambda(\cdot - x))$$

$$H_{\mathrm{off}}(x,-\sigma,s) = -\frac{e}{2}(\mathcal{B}_1(j_s\lambda(\cdot - x)) - i\sigma\mathcal{B}_2(j_s\lambda(\cdot - x))).$$

Consequences

Corollary (Energy comparison inequality)

$$\max \left\{ \begin{array}{l} E(0, \sqrt{\mathcal{B}_1^2 + \mathcal{B}_2^2}, 0, \mathcal{B}_3) \\ E(0, \sqrt{\mathcal{B}_3^2 + \mathcal{B}_1^2}, 0, \mathcal{B}_2) \\ E(0, \sqrt{\mathcal{B}_2^2 + \mathcal{B}_3^2}, 0, \mathcal{B}_1) \end{array} \right\} \leq E(\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3).$$

Corollary (Boson sector decay)

$$\textit{small enough } \alpha > 0 \implies (\Psi, e^{\alpha N}\Psi) < \infty$$

Hiroshima-L 2007a&b