Local scale-invariance in ageing phenomena

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Contents :

I. Ageing phenomena
physical ageing ; scaling behaviour and exponents

II. Hidden dynamical symmetries
Local scaling with $z = 2$; stochastic field-theory ; computation of response and correlation functions

III. Local scale-invariance for $z \neq 2$
Mass terms ; integrability ; test through responses and correlators in $2D$ disordered Ising model

IV. Conclusions

I. Ageing phenomena

- why do materials 'look old' after some time ?
- which (reversible) microscopic processes lead to such macroscopic effects ?
- physical ageing known since historical (or prehistorical) times
- systematic studies first in glassy systems STRUIK 78

a priori behaviour should depend on entire prehistory

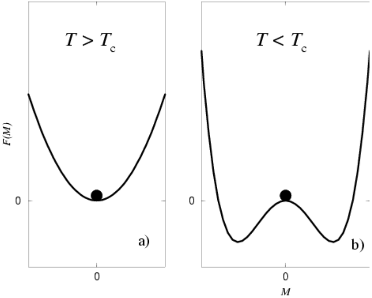
but evidence for reproducible and **universal** behaviour

- for better conceptual understanding : study ageing first in simpler systems (i.e. disordered ferromagnets)
- ageing : defining characteristics and symmetry properties :
  1. slow dynamics (i.e. non-exponential relaxation)
  2. breaking of time-translation invariance
  3. dynamical scaling
- new evidence for larger, local scaling symmetries

easier to study : ageing in simple systems without disorder
consider a simple magnet (ferromagnet, i.e. Ising model)

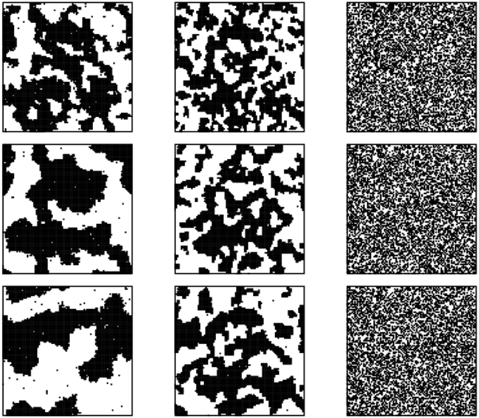
1. prepare system initially at high temperature $T \gg T_c > 0$
2. quench to temperature $T < T_c$ (or $T = T_c$)
   $\longrightarrow$ non-equilibrium state
3. fix T and observe dynamics

BRAY 94



competition :
at least 2 equivalent ground states
local fields lead to rapid local ordering
no global order, relaxation time ∞

formation of ordered domains, of linear size $L = L(t) \sim t^{1/z}$
dynamical exponent z



Snapshots of spin configurations in several $2D/3D$ Ising models quenched to $T < T_c$, for three different times $t = 25, 100, 225$.
Left : pure **Middle :** disordered **Right :** $3D$ spin glas

Scaling behaviour & exponents

single relevant time-dependent length scale $L(t) \sim t^{1/z}$

BRAY 94, JANSSEN ET AL. 92, CUGLIANDOLO & KURCHAN 90S, GODRÈCHE & LUCK 00, ...

$$\text{correlator } C(t,s;\mathbf{r}) := \langle \phi(t,\mathbf{r})\phi(s,\mathbf{0})\rangle = s^{-b} f_C\left(\frac{t}{s}, \frac{\mathbf{r}}{(t-s)^{1/z}}\right)$$

$$\text{response } R(t,s;\mathbf{r}) := \left.\frac{\delta\langle\phi(t,\mathbf{r})\rangle}{\delta h(s,\mathbf{0})}\right|_{h=0} = s^{-1-a} f_R\left(\frac{t}{s}, \frac{\mathbf{r}}{(t-s)^{1/z}}\right)$$

No fluctuation-dissipation theorem : $R(t,s;\mathbf{r}) \neq T\partial C(t,s;\mathbf{r})/\partial s$

values of exponents : equilibrium correlator → classes **S** and **L**

$$C_{\rm eq}(\mathbf{r}) \sim \left\{ \begin{array}{l} \exp(-|\mathbf{r}|/\xi) \\ |\mathbf{r}|^{-(d-2+\eta)} \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \text{class } \mathbf{S} \\ \text{class } \mathbf{L} \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} a = 1/z \\ a = (d-2+\eta)/z \end{array} \right.$$

if $T < T_c$: $z = 2$ and $b = 0$ if $T = T_c$: $z = z_c$ and $b = a$

for $y \to \infty$: $f_{C,R}(y,\mathbf{0}) \sim y^{-\lambda_{C,R}/z}$, $\lambda_{C,R}$ independent exponents

Question : **general** arguments to find form of scaling functions ?

II. Hidden dynamical symmetries

Consider possible symmetries of Langevin equations (model A)

Hohenberg-Halperin 77

$$2\mathcal{M}\frac{\partial\phi}{\partial t} = \Delta\phi - \frac{\delta\mathcal{V}[\phi]}{\delta\phi} + \eta$$

non-conserved order-parameter $\phi(t, \mathbf{r})$, centreded noise η :

$$\langle\eta(t, \mathbf{r})\eta(t', \mathbf{r}')\rangle = 2T\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$$

fully disordered initial conditions (centered gaussian noise)

Question : extended dynamical scaling for given $z \neq 1$? MH 92, 94, 02

motivation :

1. conformal invariance in equilibrium critical phenomena, $z = 1$
2. Schrödinger-invariance of simple diffusion, $z = 2$

Lie 1881, Niederer 72, Hagen 71, Kastrup 68

$$t \mapsto \frac{\alpha t}{\gamma t + \delta} \ , \ \mathbf{r} \mapsto \frac{\mathcal{R}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta} \ , \ \alpha\delta = 1$$

Lie algebra $\mathfrak{age}_d \subset \mathfrak{sch}_d$, projective representations

Stochastic field-theory

Langevin equations do **not** have non-trivial dynamical symmetries !
compare results of deterministic symmetries to stochastic models ?
go to stochastic field-theory, action JANSSEN, DE DOMINICIS,...70S-80S

$$\mathcal{J}[\phi,\widetilde{\phi}] = \underbrace{\int \widetilde{\phi}(2\mathcal{M}\partial_t - \Delta)\phi + \widetilde{\phi}\mathcal{V}'[\phi]}_{\mathcal{J}_0[\phi,\widetilde{\phi}]\ :\ \text{deterministic}} \underbrace{-T\int \widetilde{\phi}^2 - \int \widetilde{\phi}_{t=0} C_{init} \widetilde{\phi}_{t=0}}_{+\ \mathcal{J}_b[\widetilde{\phi}]\ :\ \text{noise}}$$

$\widetilde{\phi}$: response field ; $C(t,s) = \langle \phi(t)\phi(s)\rangle,\ R(t,s) = \langle \phi(t)\widetilde{\phi}(s)\rangle$

averages : $\langle A\rangle_0 := \int \mathcal{D}\phi\mathcal{D}\widetilde{\phi}\, A[\phi,\widetilde{\phi}]\exp(-\mathcal{J}_0[\phi,\widetilde{\phi}])$

masses :

$$\boxed{\mathcal{M}_\phi = -\mathcal{M}_{\widetilde{\phi}}}$$

Theorem : IF $\mathcal{J}_0$ is Galilei- and spatially translation-invariant,
then Bargman superselection rules hold true :

$$\left\langle \phi_1 \cdots \phi_n \widetilde{\phi}_1 \cdots \widetilde{\phi}_m \right\rangle_0 \sim \delta_{n,m} \qquad (1)$$

computation of a response function

$$\begin{aligned} R(t,s) &= \left\langle \phi(t)\widetilde{\phi}(s) \right\rangle = \left\langle \phi(t)\widetilde{\phi}(s) e^{-\mathcal{J}_b[\widetilde{\phi}]} \right\rangle_0 \\ &= \left\langle \phi(t)\widetilde{\phi}(s) \right\rangle_0 = R_0(t,s) \end{aligned}$$

Bargman eq. (1) $\Longrightarrow$ response function does not depend on noise !
left side : computed in stochastic models
right side : local scale-symmetry of deterministic equation
application to ageing : $\mathfrak{age}_d$-covariant two-point response function

$$R(t,s;\mathbf{r}) = r_0 s^{-1-a} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/z} \left(\frac{t}{s}-1\right)^{-1-a'} \exp\left(-\frac{\mathcal{M}}{2}\frac{\mathbf{r}^2}{t-s}\right)$$

find $C(t,s) = \langle \phi(t)\phi(s)\rangle = \langle \phi(t)\phi(s)e^{-\mathcal{J}_b[\widetilde{\phi}]}\rangle_0$ from Bargman rule
'initial' and 'thermal' contributions ; contain four-point responses
explicit tests in Ising/Potts models H & P 03, MH et al. 04, Lorenz & Janke 07

III. Local scale-invariance for $z \neq 2$

Extend to $z \neq 1, 2$ by generators with mass terms (for $d = 1$) :

MH 02 ; Baumann & MH 07

$$
\begin{aligned}
Y_{1-1/z} &:= -t\partial_r - \mu z r \nabla_{\mathbf{r}}^{2-z} - \gamma z(2-z)\partial_r \nabla_{\mathbf{r}}^{-z} && \text{Galilei} \\
X_1 &:= -t^2\partial_t - \frac{2}{z} t r \partial_r - \frac{2(x+\xi)}{z} t - \mu r^2 \nabla_{\mathbf{r}}^{2-z} && \text{special} \\
& \quad -2\gamma(2-z) r \partial_r \nabla_{\mathbf{r}}^{-z} - \gamma(2-z)(1-z)\nabla_{\mathbf{r}}^{-z}
\end{aligned}
$$

- depend on two parameters γ, μ and on two dimensions x, ξ
- contains fractional derivative ($\widehat{f}$: Fourier transform)

$$
\nabla_{\mathbf{r}}^{\alpha} f(\mathbf{r}) := \mathrm{i}^{\alpha} \int_{\mathbb{R}^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{\alpha} e^{\mathrm{i}\mathbf{r}\cdot\mathbf{k}} \widehat{f}(\mathbf{k})
$$

- some properties : $\nabla_{\mathbf{r}}^{\alpha}\nabla_{\mathbf{r}}^{\beta} = \nabla_{\mathbf{r}}^{\alpha+\beta}$, $[\nabla_{\mathbf{r}}^{\alpha}, r_i] = \alpha\partial_{r_i}\nabla_{\mathbf{r}}^{\alpha-2}$
  $\nabla_{\mathbf{r}}^{\alpha} \exp(\mathrm{i}\mathbf{q}\cdot\mathbf{r}) = \mathrm{i}^{\alpha}|\mathbf{q}|^{\alpha} \exp(\mathrm{i}\mathbf{q}\cdot\mathbf{r})$

Fact 1 : simple algebraic structure :

$$[X_n, X_{n'}] = (n - n')X_{n+n'} \quad , \quad [X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}$$

$\rightarrow$ Generate Y_m from $Y_{-1/z} = -\partial_r$.

Fact 2 : LSI-invariant Schrödinger operator :

$$\mathcal{S} := -\mu\partial_t + z^{-2}\nabla_{\mathbf{r}}^z$$

Let $x_0 + \xi = 1 - 2/z + (2-z)\gamma/\mu$. Then $[\mathcal{S}, Y_m] = 0$ and

$$[\mathcal{S}, X_0] = -\mathcal{S} \quad , \quad [\mathcal{S}, X_1] = -2t\mathcal{S} + \frac{2\mu}{z}(x - x_0)$$

$\Longrightarrow$ $\mathcal{S}\phi = 0$ is lsi-**invariant** equation, if $x_\phi = x_0$.

Fact 3 : non-trivial conservation laws :
iterated commutator with $G := Y_{1-1/z}$, $\mathrm{ad}\,_G. = [., G]$

$$M_\ell := (\mathrm{ad}_G)^{2\ell+1}\, Y_{-1/z} = a_\ell \mu^{2\ell+1} \nabla_{\mathbf{r}}^{(2\ell+1)(1-z)+1}$$

For $z = 2$, $a_\ell = 0$ if $\ell \geq 1$. For a n-point function $F^{(n)} = \langle \phi_1 \ldots \phi_n \rangle$, $M_\ell F^{(n)} = 0$ gives in momentum space

$$\left(\sum_{i=1}^{n} \mu_i^{2\ell-1} |\mathbf{k}_i|^{2\ell-(2\ell-1)z} \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) = 0$$

$$\left(\sum_{i=1}^{n} \mathbf{k}_i \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) = 0$$

$\Longrightarrow$ momentum conservation & conservation of $|\mathbf{k}|^\alpha$!
analogous to relativistic factorisable scattering   Zamolodchikov² 79, 89

Consequence : a $\mathfrak{lsi}$-covariant $2n$-point function $F^{(2n)}$ is only non-zero, if the 'masses' μ_i can be arranged in pairs $(\mu_i, \mu_{\sigma(i)})$ with $i = 1, \ldots, n$ such that $\boxed{\mu_i = -\mu_{\sigma(i)}}$.

generalized Galilei-invariance with $z \neq 2 \Longrightarrow$ 'integrability'

Corollary 1 : Bargman rule : $\langle \phi_1 \ldots \phi_n \widetilde{\phi}_1 \ldots \widetilde{\phi}_m \rangle_0 \sim \delta_{n,m}$

Corollary 2 : treat (linear) stochastic equations with $\mathfrak{lsi}$-invariant deterministic part, reduction formulæ

Corollary 3 : response function noise-independent

$$
\begin{aligned}
R(t, s; \mathbf{r}) &= R(t, s) \mathcal{F}^{(\mu_1, \gamma_1)}(|\mathbf{r}|(t-s)^{-1/z}) \\
R(t, s) &= r_0 \, s^{-a} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/z} \left(\frac{t}{s} - 1\right)^{-1-a'} \\
\mathcal{F}^{(\mu, \gamma)}(\mathbf{u}) &= \int_{\mathbb{R}^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{\gamma} \exp\left(\mathrm{i}\mathbf{u} \cdot \mathbf{k} - \mu |\mathbf{k}|^z\right)
\end{aligned}
$$

Corollary 4 :

Correlators obtained from factorised 4-point responses.

How to test the foundations of LSI

describes dynamic symmetries of **deterministic part** of Langevin eq.
local scaling theory is built on :

a) simple scaling – domain sizes $L(t) \sim t^{1/z}$
b) invariance under Möbius transformation $t \mapsto t/(\gamma t + \delta)$
c) Galilei-invariance generalised to $z \neq 2$

together with spatial translation-invariance
$\Longrightarrow$ extended Bargman rules
$\Longrightarrow$ factorisation of $2n$-point functions

Möbius transformation	autoresponse $R(t,s)$
generalised Galilei-invariance	space-time response $R(t,s;\mathbf{r})$
factorisation	two-time correlation function

Tests of LSI for $z \neq 2$:

- spherical model with conserved order-parameter, $T = T_c$, $z = 4$ Baumann & MH 06
- Mullins-Herring model for surface growth, $z = 4$

  Röthlein, Baumann, Pleimling 06
- spherical model with long-ranged interactions, $T \leq T_c$, $0 < z = \sigma < 2$ Cannas et al. 01 ; Baumann, Dutta, MH 07
- $2D$ **Ising model with disorder**, $T < T_c$ (non-frustrated)

  Hamiltonian $\mathcal{H} = -\sum_{i,j} J_{ij}\sigma_i\sigma_j$

  uniform disorder $J_{ij} \in [1 - \epsilon/2, 1 + \epsilon/2] \Longrightarrow T_c(\epsilon) \approx T_c(0)$

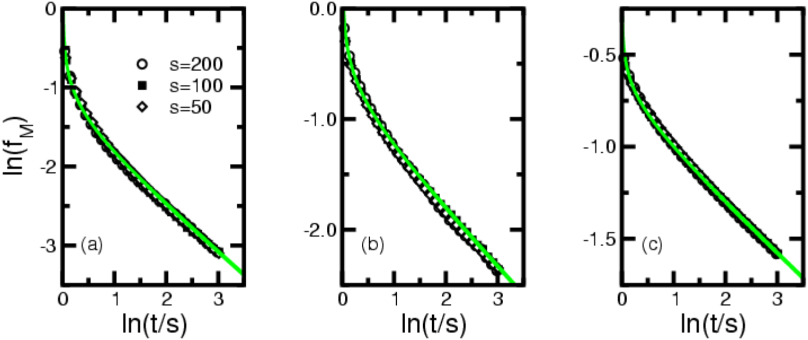
  disorder defects 'pin' domain walls $\Longrightarrow$ thermal activation

  if logarithmic barrier heights $\Longrightarrow$ $\boxed{z = 2 + \epsilon/T}$.

  Paul, Puri & Rieger 04

change contrôle parameters to vary z

Practical tests of LSI, I : autoresponse



(a) $\epsilon = 0.5,\ T = 0.6$ (b) $\epsilon = 1,\ T = 1$ (c) $\epsilon = 2,\ T = 0.6$

Thermoremanent susceptibility

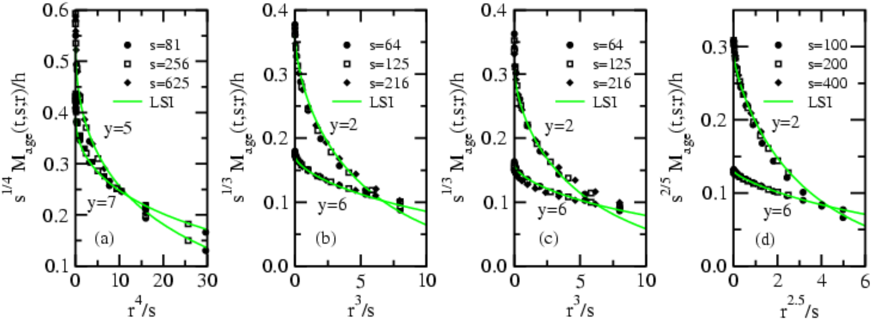
$\chi_{\rm TRM}(t,s) = \int_0^s \mathrm{d}u\, R(t,u) = s^{-a} f_M(t/s) + \mathrm{O}(s^{-\lambda_R/z})$

Full **curve** : LSI-prediction, with $a = a' = 1/z$.

Confirm $z = 2 + \epsilon/T$: agreement with PAUL, PURI & RIEGER 04.

MH & PLEIMLING, Europhys. Lett. **76**, 561 (2006).

Practical tests of LSI, II : space-time response



(a) $\epsilon = 2,\ T = 1$ (b) $\epsilon = T = 1$ (c) $\epsilon = T = 0.5$ (d) $\epsilon = 0.5,\ T = 1$

$\chi_{\rm TRM}(t,s;\mathbf{r}) = \int_0^s \mathrm{d}u\, R(t,u;\mathbf{r}) = s^{-a} r_0 f_M(t/s, \mathbf{r}s^{-1/z}) + \mathrm{O}(t^{-\lambda_R/z})$

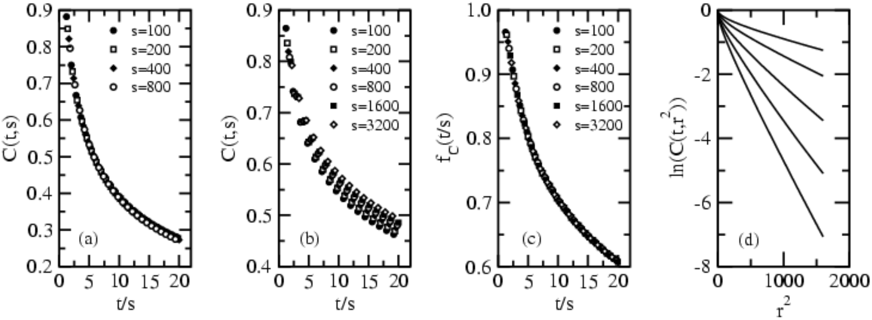
Scaling function f_M only depends on ratio $\epsilon/T \Longrightarrow$ universality

Full **curve** : LSI-prediction, with $y = t/s$ fixed and $a = a' = 1/z$.

first test of 'Galilei-invariance' for $z \neq 2$ in a non-linear model

BAUMANN, MH & PLEIMLING, arXiv :0709.3228.

Practical tests of LSI, III : autocorrelation



(a) $\epsilon = 0.5,\ T = 1$, (b,c) $\epsilon = 2,\ T = 1$ (d) $t = [200, 300, 500, 1000, 2000]$

No simple scaling with $y = t/s$ for $z \gtrsim 4$! P, S & R 06 ; H & P 06

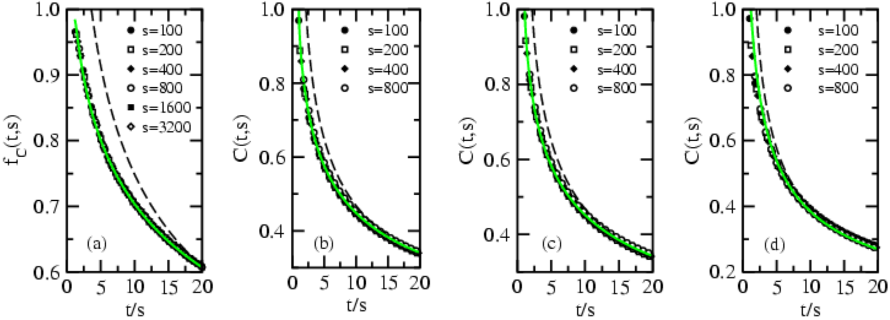
↘ indication for **'superageing'** ? Paul, Schehr, Rieger 07

1. subtract finite-time correction, $C(t,s) = f_C(t/s) - s^{-b'} g_C(t/s)$
2. then scaling of $C(t,s)$ according to **simple ageing** with $y = t/s$

* Scaling function f_C only depends on ratio $\epsilon/T \Longrightarrow$ universality

* ageing sets in at late time scale $\tau = t - s \sim s^{\zeta}$ Zippold, Kühn, Horner 00

use $C(s+\tau, s; \mathbf{r}) \sim \exp(-\nu \mathbf{r}^2 s^{-2/z})$ generalised from Ohta,Jasnow,Kawasaki 82



(a) $\epsilon = 2,\ T = 1$ (b) $\epsilon = T = 1$ (c) $\epsilon = T = 0.5$ (d) $\epsilon = 0.5,\ T = 1$
Dashed line : LSI with fully disordered initial correlator
Full **curve** : LSI prediction

$$f_C(y) = c_2 y^\rho \int_{\mathbb{R}^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{2\beta} \exp\left(-\alpha |\mathbf{k}|^z (y-1) - \frac{\mathbf{k}^2}{4\nu} \right)$$

with $\beta = \lambda_C - \lambda_R$, $\rho = (2\beta + d - \lambda_C)/z$. Used 'initial' correlator $C(s+\tau, s; \mathbf{r}) \sim \exp(-\nu \mathbf{r}^2 s^{-2/z})$: asymptotic, enough for $z > 2$

Baumann, MH & Pleimling, arXiv :0709.3228.

IV. Conclusions

1. look for extensions of dynamical scaling in ageing systems

   recently, scaling derived for phase-ordering ARENZON ET AL. 07
2. here : **hypothesis** of generalised Galilei-invariance
3. leads to Bargman rule if $z = 2$
   and further to 'integrability' if $z \neq 1, 2$.
4. **hidden** dynamical symmetry of deterministic part of (linear) Langevin equations
5. Tests : derive two-time response and correlation functions
6. LSI exactly proven for linear Langevin equations
   very good numerical evidence for non-linear systems

Some questions (the list could/should be extended) :

- how to physically justify Galilei-invariance ?
- how to extend to non-linear equations ? first attempts STOIMENOV & MH 05
- choice of the type of fractional derivative ?
- what is the algebraic (non-Lie !) structure of LSI ?
- treatment of master equations with LSI ?