

Phase diagram for interacting Bose gases

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We propose a modified form of the inversion method in terms of a self-energy expansion to access the phase diagram of the Bose-Einstein transition. The dependence of the critical temperature on the interaction parameter is calculated. This is discussed with the help of a condition for Bose-Einstein condensation in interacting systems which follows from the pole of the T matrix in the same way as from the divergence of the medium-dependent scattering length. A many-body approximation consisting of screened ladder diagrams is proposed, which describes the Monte Carlo data more appropriately. The specific results are that a non-self-consistent T matrix leads to a linear coefficient in leading order of 4.7, the screened ladder approximation to 2.3, and the self-consistent T matrix due to the effective mass to a coefficient of 1.3 close to the Monte Carlo data.

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I. INTRODUCTION

Interacting Bose gases have become a topic of current interest triggered by the experimental tabletop demonstration of Bose condensations.^{1,2} Especially, it is of great importance to know the behavior of the critical temperature dependent on the interaction strength and the density. The fluctuations beyond the mean field are especially significant since they establish the deviations of the critical temperature from the noninteracting value and might be observable by adiabatically converting down the trapping frequency.³ The measurement of the critical temperature of a trapped weakly interacting ⁸⁷Rb gas differs from the ideal gas value already by two standard deviations.⁴ Therefore, it is of great interest to understand this change even in bulk Bose systems.

The density n_0 of the noninteracting bosons with mass m and temperature T reads

$$n_0(T) = \left[\frac{mk_B T}{3\pi\hbar^2} g(\epsilon) \right]^{3/2}, \quad (1)$$

where $g(x) = \frac{3}{2}[P_{3/2}(x)]^{2/3}$ is given in terms of the polylogarithmic function

$$P_{3/2}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{tdt}}{e^{t+x}-1}$$

and we denote the negative ratio of the chemical potential to the temperature by $\epsilon = -\mu/k_B T$. The critical temperature $T_0(n)$ is given by Eq. (1) for $\epsilon \rightarrow 0$.

The critical temperature T_c of the Bose system with repulsive interaction is deviating from the free one, T_0 , and is a function of the free scattering length a_0 times the third root of the density. One usually discusses this deviation on condition that the density of the interacting system at the critical temperature, $n(a_0, T_c)$, should be equal to the density of the noninteracting Bose system $n_0(T_0)$ at the critical temperature T_0 . According to Eq. (1), we have $n_0(T_c) = (T_c/T_0)^{3/2} n_0(T_0)$ and one obtains⁵

$$x = \frac{T_c}{T_0} = \left[\frac{n_0(T_c)}{n(a_0, T_c)} \right]^{2/3}. \quad (2)$$

The change of the critical temperature $(T_c - T_0)/T_0$ dependent on the coupling strength $y_0 = |a_0|n^{1/3}$ has been found to be linear in leading order. Monte Carlo (MC) simulations and theoretical calculations by many groups have confirmed

$$\frac{T_c - T_0}{T_0} \approx c_1 y_0 + \mathcal{O}(y_0^2). \quad (3)$$

The sometimes occurring square root dependence has turned out to be an artifact of the first-order virial expansion.⁶ Although there is now an agreement about the linear slope, the actual value of c_1 varies from 0.34 to 3.8 as discussed in Refs. 6–8. The situation is different for finite Bose systems contained in a trap. The increase of interaction lowers the density in the center of the trap and reduces effectively the critical temperature⁴ such that $c_1 = -0.93$, as found in Ref. 9. Here, we want to consider only bulk Bose gases.

Let us shortly sketch the different expansion schemes. The convergence of such expansions is discussed in Ref. 10 in terms of the large- N expansion of scalar field theory. The optimized linear expansion method¹¹ yielded $c_1 = 3.06$ for two-loop contributions and $c_1 = 1.3$ for six loops.¹² Though this δ expansion works well in quantum mechanics, it fails to converge in quantum field theory.¹³ The $1/N$ expansion¹⁴ provides $c_1 = 2.33$. This seems to be in agreement with the non-self-consistent summation of bubble diagrams⁶ which gives $c_1 = \frac{8\pi}{3} \zeta(3/2)^{-4/3} \approx 2.33$. The same result has also been found with a T-matrix approximation.¹⁵ Older MC data¹⁶ show similar values such as $c_1 = 2.30 \pm 0.25$, while newer investigations^{17,18} report $c_1 \approx 1.3$. The latter result can also be found by variational perturbation theory,¹⁹ yielding $c_1 = 1.23 \pm 0.12$. The same coefficient has been obtained by exact renormalization group calculations^{20,21} which provide the momentum dependence of the self-energy at zero frequency as well. The usage of Ursell operators²² has given an even smaller value of $c_1 = 0.7$.

Though the weak coupling behavior of the critical temperature can be considered as settled, there is considerably less known about the strong coupling behavior. In Ref. 23, a nonlinear behavior of the critical temperature was proposed; first the interaction leads to an increase of the critical tem-

perature up to a maximal one. A further increase of interaction strength or correlations then decreases the critical temperature until the Bose condensation vanishes at a maximal coupling parameter y_0^{crit} . One finds the interesting physics that too strong interaction prevents the Bose condensation.

The variational perturbation method developed in Ref. 24 and advocated in Ref. 23 allows one to sum up higher-order diagrams in order to account for strong coupling effects. In this paper, we present an alternative method well known as inversion method to obtain from the ladder summation diagrams a higher-order perturbation result by inversion. The motivation to propose yet another theoretical method besides the already mentioned Monte Carlo simulations and the variational perturbation method is twofold. First, the strong coupling limit is a problem of theoretical challenge which is not yet solved. Different schemes favor different summations of diagrams and therefore weigh different processes differently. We think that different paths and comparing the results between them might be enlightening for understanding the physical mechanisms. The second motivation is a pure methodological one. The proposed inversion method has already led to remarkable results in other fields. Therefore, we think that it is interesting to try how far this method can be applied fruitfully to the highly nontrivial problem of interacting Bose gases. We expect from this method theoretical insights into the systematics of higher-order processes which have to be considered for the description of strongly coupled systems showing spontaneous symmetry breaking. Here, the inversion method seems to have its advantages since it allows explicitly for symmetry breaking. This is, however, beyond the scope of the present work.

Below, we first discuss the method to extract the phase diagram within the standard ladder approximation solving the Feynman-Galitzky form of the Bethe-Salpeter equation. This will be performed with the help of the separable potential which is then used in the limit of the contact interaction. This way is mathematically convenient since we can renormalize the coupling strength to the physical scattering length avoiding divergences. Then, we apply the inversion method starting from the ladder approximation. This leads us to a twofold limit of the resulting series, one of which shows the nontrivial behavior in the phase diagram due to symmetry breaking. We compare this result with the Monte Carlo data and other theoretical approaches in the literature. In Sec. V, we discuss improvements, mainly the summation of ring diagrams connected with ladder diagrams which will modify the linear slope but will not change the scaled nonlinear curve. Possible further improvements are outlined in the summary.

II. INVERSION METHOD

The inversion method we use has been discussed thoroughly in the centennial review paper of Fukuda *et al.*²⁵ We consider a system which can undergo a symmetry breaking due to a small external perturbation such that a phase transition is possible. The corresponding order parameter ϕ shall depend on an external source parameter S . We can expand the order parameter in terms of the coupling g to the source via

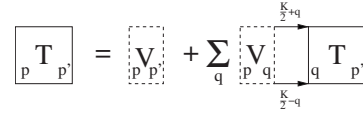


FIG. 1. The diagrams of the T-matrix approximation where the difference momenta are indicated as well as the momenta of the two intermediate propagators.

$$\phi = f(S) = \sum_{n=0}^{\infty} g^n f_n(S), \quad (4)$$

which corresponds to a perturbation series. Since we can perform only a calculation to finite order in g , a vanishing S means also a vanishing order parameter ϕ and we are not able to describe the phase transition as a proper spontaneously broken symmetry. The situation can be improved if Eq. (4) is inverted,

$$S = h(\phi) = \sum_{n=0}^{\infty} g^n h_n(\phi). \quad (5)$$

For any expansion with a finite number of f_n , we can calculate the inverted coefficients h_n by introducing Eq. (5) into Eq. (4) and comparing the powers of g to obtain²⁵

$$h_0(\phi) = f_0^{-1}(\phi),$$

$$h_1(\phi) = - \left. \frac{f_1(S)}{f_0'(S)} \right|_{S=h_0(\phi)}, \quad (6)$$

and so on. If we now set $S=0$ in the inverted series, the nontrivial solution $\phi \neq 0$ is obtainable though with finite-order calculation of the h_n in Eq. (5). The finite truncation in the inverted series (5) corresponds to an infinite series summation in Eq. (4). For further details and successful applications of this method, see Ref. 25.

We will use the ladder summation to establish a sum for the scattering length and will invert this series with respect to the self-energy in order to obtain a nonlinear phase transition curve, i.e., the dependence of the critical temperature on the coupling strength.

III. MANY-BODY T-MATRIX APPROXIMATION AND BETHE-SALPETER EQUATION

The T matrix depends on the frequency ω/\hbar and the center-of-mass momentum K and reads, according to Fig. 1,

$$T_{pp'}(\omega, K) = V_{pp'} + \sum_q V_{pq} \frac{1 + f_{K/2-q} + f_{K/2+q}}{\omega - \epsilon_{K/2-q} - \epsilon_{K/2+q} + i\eta} T_{qp'}(\omega, K), \quad (7)$$

where we indicate the difference momenta of the incoming and the outgoing channel by subscript, the energy dispersion is assumed to be $\epsilon_p = p^2/2m$, and the Bose functions are denoted by f_k .

This Bethe-Salpeter equation can be solved by a separable potential $V_{pp'} = \lambda g_p g_{p'}$ with a coupling constant λ and a

Yamaguchi form factor $g_p=1/(p^2+\beta^2)$, where β describes the finite range of the potential. One obtains

$$T_{pp'}(\omega, K) = \frac{\lambda g_p g_{p'}}{1 - \lambda J_g(\omega, K)},$$

$$J_g(\omega, K) = \sum_q g_q^2 \frac{1 + f_{K/2-q} + f_{K/2+q}}{\omega - \epsilon_{K/2-q} - \epsilon_{K/2+q} + i\eta}. \quad (8)$$

The scattering phase

$$\tan \delta = \frac{\text{Im } T}{\text{Re } T} \Big|_{p=p', K=0} \approx ak + \mathcal{O}(k^2) \quad (9)$$

$\omega = K^2/4m + p^2/m$

is linked to the scattering length by the small momentum expansion, $p = \hbar k$. From Eq. (8), we obtain²⁶

$$a = - \frac{\frac{\pi \hbar \lambda}{2\beta^4} (1 + 2f_0)}{\frac{2\pi^2 \hbar^3}{m} + \lambda \int_0^\infty d\bar{p} \frac{1 + 2f_{\bar{p}}}{(\beta^2 + \bar{p}^2)^2}}. \quad (10)$$

Now we can perform the limit to the contact potential using a β -dependent coupling constant $\lambda(\beta)$ in such a way that the free two-particle scattering length a_0 in the absence of many-body effects, $f_k \rightarrow 0$, is reproduced.²⁶ From Eq. (10), we have

$$\lambda(a_0, \beta) = - \frac{\frac{2\pi^2 \hbar^3}{m}}{\frac{\pi \hbar}{2a_0 \beta^4} + \int_0^\infty d\bar{p} \frac{1}{(\beta^2 + \bar{p}^2)^2}}, \quad (11)$$

which we use to eliminate λ in Eq. (10) to obtain $a(a_0, \beta)$. Performing now the limit $\beta \rightarrow \infty$, we obtain a well defined expression for the scattering length with many-body effects,

$$\frac{a}{a_0} = \frac{1 + 2f_0}{1 - \frac{4a_0}{\pi \hbar} \int_0^\infty d\bar{p} f_{\bar{p}}}. \quad (12)$$

This scattering length depends on the chemical potential and on the temperature, $a(\mu, T)$, via the Bose function f_p . This renormalization procedure is completely equivalent to a momentum cutoff used in Ref. 27. Higher-order partial wave contact interactions have been discussed in Ref. 28, which are important for constructing pseudopotentials.²⁹ For further relations of the T matrices in different dimensions, see Ref. 30.

In Ref. 26, expression (12) was discussed and it was found that this medium-dependent scattering length diverges at the Bose-Einstein transition characterized by the critical density and temperature. We will employ Eq. (12) as a good measure of approaching the Bose-Einstein condensate. With the help of Eq. (1), we can rewrite Eq. (12) into an equation for the dimensionless interaction parameter, $y = an^{1/3}$, versus the free one, $y_0 = a_0 n_0^{1/3}$,

$$\frac{y}{y_0} = \frac{\coth \frac{\epsilon_\Sigma}{2}}{1 + 4y_0 g'(\epsilon_\Sigma)}, \quad (13)$$

on the condition $n(a_0, T_c) = n_0(T_0)$. The ratio of the temperatures $x = T_c/T_0$ for interacting to noninteracting Bose systems follows from Eqs. (1) and (2) as

$$x = \frac{g(0)}{g(\epsilon_\Sigma)}, \quad (14)$$

where $\epsilon_\Sigma = \epsilon + \Sigma(\epsilon_\Sigma, y_0)/k_B T$ denotes the functional dependence of the quasiparticle energy in the argument of the distribution function on the self-energy.

IV. BOSE-EINSTEIN CONDENSATION BORDER LINE

A. T-matrix approximation

Our aim is to extract the phase diagram of a_0 in terms of $x = T_c/T_0$. A simple elimination of ϵ in the system of equations (13) and (14) would not lead to an answer since the ladder approximation is a weak coupling theory³¹ and cannot lead to the proper description of strong coupling. The simple elimination $\epsilon = \epsilon(y) \rightarrow x = x[\epsilon(y)]$ would correspond to the direct series (4). We will present how the strong coupling limit can be reached in the next section. Here, we derive a proper condition for the Bose-Einstein condensation border line. This will be performed in two ways, first by the observation of a diverging medium-dependent scattering length and second by the condition of the appearance of a Bose pole in the T matrix. Both conditions will lead to the same criterion.

To start with, we take into account the condition of being as near as possible to the phase separation line. We use the fact that the medium-dependent scattering length diverges at this phase separation line such that we obtain for critical $\epsilon_\Sigma^{\text{crit}} \rightarrow \tilde{\epsilon}$ from Eqs. (13) and (14)

$$y_0 = \frac{y}{\coth \frac{\epsilon_\Sigma}{2} - 4y g'(\epsilon_\Sigma)} \rightarrow - \frac{1}{4g'(\tilde{\epsilon})} \equiv y_0^c,$$

$$x = \frac{g(0)}{g(\tilde{\epsilon})} \quad (15)$$

as the system of equations which gives the separation line in the phase diagram. The meaning of $\tilde{\epsilon} = -\mu(a_0, T_c)/k_B T_c + \Sigma/k_B T_c$ is the critical chemical potential and the self-energy for the interacting Bose gas divided by the critical temperature.

The same condition (15) for the formation of a condensate can be found by calculating the binding pole of the T matrix (8). With the renormalization (11), the bosonic T matrix (8) for contact interaction reads

$$T_{pp'}(\omega, Q) = -\frac{4\pi\hbar^2 a_0}{m} \frac{1}{1 + J(\omega, Q)}. \quad (16)$$

In the center-of-mass frame, one finds

$$J(\omega, 0) = -\frac{4a_0}{\pi\hbar} \int_0^\infty dq \frac{q^2 f_q}{q^2 - m\omega} + \frac{a_0}{\hbar} \begin{cases} \sqrt{-\omega m} & \text{for } \omega \leq 0 \\ -i\sqrt{\omega m}(1 + 2f_{\omega/2}) & \text{for } \omega > 0. \end{cases} \quad (17)$$

In the dilute limit, $f_q \rightarrow 0$, the free two-particle phase shift of Eq. (9) starts from zero for positive a_0 and from π for negative scattering length, indicating a bound state according to the Levinson theorem.³² For $a_0 < 0$, we have therefore a bound state pole at $J(-E_b, 0) = -1$ with the binding energy $E_b = \hbar^2 / ma_0^2$. With the sign of the scattering length in Eq. (9), we follow the convention of Ref. 32 which has an opposite sign compared to that in Ref. 33. Generally, odd numbers of bound states correspond to negative scattering lengths and even numbers to positive scattering lengths.

We are interested here in the Bose condensation, that means in a pole for $a_0 > 0$. In this case, there is no bound state but a Bose-condensation pole possible at $J(-E_b, 0) = -1$. We can estimate

$$J(\omega < 0, 0) > J(0, 0) = -\frac{4a_0}{\pi\hbar} \int_0^\infty dq f_q = 4y_0 g'(\epsilon). \quad (18)$$

The T matrix (16) has therefore a Bose-condensation pole if $J(0, 0) \leq -1$, which means in view of Eq. (18) that

$$y_0 \geq -\frac{1}{4g'(\epsilon)}. \quad (19)$$

This is the same condition as Eq. (15) and one can see that the existence of a pole in the T matrix is equivalent to the existence of a Bose condensation. In other words, this pole represents the medium dependence of the Bose condensation in T matrix approximation.

Let us shortly discuss the physical content of the pole condition of the T matrix. The result according to Eq. (17) is plotted in Fig. 2 versus the dimensionless interaction parameter. For $\epsilon = -\mu/k_B T \rightarrow 0$, the curve starts at zero, which means that any small interaction leads to a positive Bose-condensation energy while for noninteracting systems the condensation energy remains at zero. For the critical $\epsilon = 0$, the curve approaches $E_b = 1.011(2k_B T_c)$ for large interactions, which shows that a further increase of the Bose-condensation energy is not possible. This gives already a hint that a too strong interaction is not increasing Bose condensation further. In fact, we will find in the next section within the strongcoupling summation that the Bose condensation will decrease with increasing coupling parameter. Here, in the weak coupling limit it stays constant.

A further interesting observation is that for lower densities $\epsilon > 0$, there appears a minimal coupling parameter above which only a positive Bose-condensation energy is possible. Below this minimal interaction parameter, we have a strong imaginary part of the pole (17), indicating an unstable state

with a finite lifetime. This minimal interaction parameter as a function of ϵ is seen in the inset of Fig. 2. In fact, it rises exponentially such that for a given interaction strength we have an upper limit in $\epsilon = -\mu/T$ up to which Bose-Einstein condensation is possible.

B. Inverse self-energy expansion

We can describe the strong coupling limit by the proper inversion method as demonstrated in the following. Analogous to the inversion method discussed in Sec. II, we identify now the order parameter ϕ with x , the parameter S with $\bar{\epsilon}$, and the expansion parameter g with y_0 .

The separation line of Bose-Einstein condensation appears for vanishing $\bar{\epsilon}$. Inverting the small $\bar{\epsilon}$ expansion of the function $x(\bar{\epsilon})$ in the second line of Eq. (15), the resulting $\bar{\epsilon}(x)$ expansion corresponds to the finite series (5). Inserting this into the first line of Eq. (15) gives $y_0^c(x)$ which again is inverted, leading to

$$x = 1 + c_1 y_0^c + c_2 (y_0^c)^2 + c_3 (y_0^c)^3 + \dots, \quad (20)$$

with

$$c_1 = \frac{16\pi}{3\zeta\left(\frac{3}{2}\right)^{4/3}} = 4.657, \\ c_2 = \frac{32\pi \left[14\pi + 9\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right) \right]}{9\zeta\left(\frac{3}{2}\right)^{8/3}} = 8.325, \\ c_3 = \frac{512\pi \left[88\pi^2 + 108\pi\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right) + 27\zeta\left(\frac{1}{2}\right)^2 \zeta\left(\frac{3}{2}\right)^2 \right]}{81\zeta\left(\frac{3}{2}\right)^4} \\ = -14.032, \quad (21)$$

where we have derived the expansion up to eighth order in $\bar{\epsilon}$ in order to achieve sufficient convergence. It has to be remarked that the series possesses diverging terms with alternating signs which nevertheless converge to two limiting curves.

The results (20) are plotted in Fig. 3. One of the two limiting curves (dotted line), called second solution, corresponds to the naive elimination $x = x[\epsilon(y_0^c)]$ discussed above. The thick dashed line shows the result due to the inversion method, which goes beyond the level of the T-matrix summation of diagrams. We see that the transition curve bends

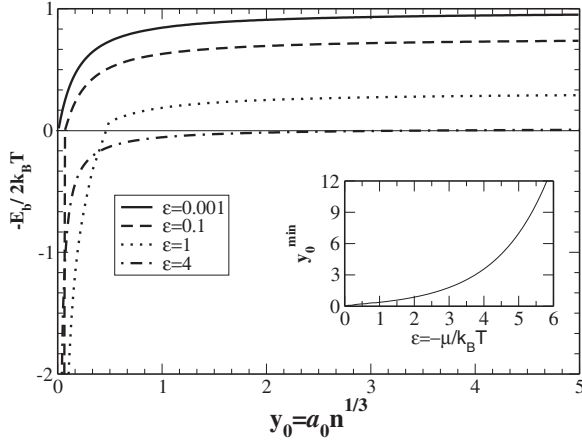


FIG. 2. The condition for appearance of a Bose-condensation pole, $J(-E_b, 0) = -1$, according to Eq. (17). The Bose-condensation energy is plotted versus the dimensionless interaction parameter for different ratios of the chemical potential to the temperature, $\epsilon = -\mu/k_B T$. The inset shows the minimal interaction parameter as a function of ϵ .

back and a maximal scattering length occurs below which Bose condensation can happen. It is instructive to also compare the absolute values of the maximal critical parameters where the Bose condensation vanishes. These are presented in Table I. Our maximal critical temperature is too high. In this respect, the variational improved perturbation method²³ (thin dashed line in Fig. 3) is nearer to the MC data. This will be improved in the next section.

Let us stress again that this inversion procedure is not a simple expansion of $x(y_0^c)$ in powers of y_0^c but primarily an expansion in $\tilde{\epsilon}(x)$. In fact, we had to expand up to second order in $\tilde{\epsilon}$ in order to obtain the third-order coefficient in Eq. (20), to third order in $\tilde{\epsilon}$ in order to obtain the fourth-order coefficient in Eq. (20), etc. This is the reason why we do not get a logarithmic term for the second term as obtained perturbatively in second order in Ref. 36.

V. FURTHER IMPROVEMENTS

A. Toward self-consistency

Though Fig. 3 describes the data quite well, it is mainly due to the scaling of the a axis to 1. It is more instructive to

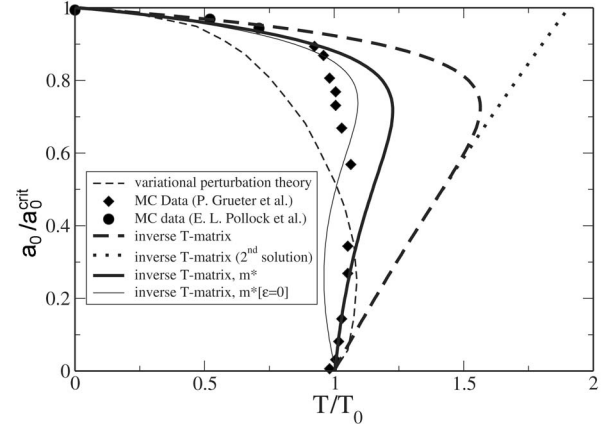


FIG. 3. The phase diagram of the scaled dimensionless scattering length versus the critical temperature. The MC data are from Refs. 34 and 35 and the variational improved perturbation theory from Refs. 8 and 23.

compare the leading order c_1 explicitly. The non-self-consistent T matrix leads to an increase which is too large by nearly a factor of 2 compared with that in Refs. 6 and 14–16. We can then improve this result by the self-consistent T matrix. This is achieved if we replace the free dispersion $p^2/2m$ in the arguments of the Bose function in Eqs. (1), (8), and (12) by the quasiparticle one ϵ_p which is a solution of the self-consistent equation,

$$\epsilon_p = \frac{p^2}{2m} + \text{Re} \Sigma(\omega, p)|_{\omega=\epsilon_p}. \quad (22)$$

The retarded self-energy reads³⁷

$$\begin{aligned} \Sigma(\omega, k) = & \sum_p T_{(k-p)/2, (k-p)/2}(\omega + \epsilon_p, k + p) f_p \\ & + \sum_{pq} \frac{|T_{(k-p)/2, (k-p)/2-q}(\epsilon_{k-q} + \epsilon_{p+q}, k + p)|^2 f_{k-q} f_{p+q}}{\epsilon_{k-q} + \epsilon_{p+q} - \epsilon_p - \omega + i\eta}, \end{aligned} \quad (23)$$

with the retarded renormalized T matrix (8) in the limit of contact interaction,

$$T_{(k-p)/2, (k-p)/2-q}(\omega, k + p) = - \frac{4\pi\hbar^2 a_0/m}{1 + 4\pi\hbar^2 a_0 \sum_{p'} \left(\frac{1}{p'^2} - \frac{1 + 2f_{p/2+p'}}{p'^2 + p^2/4 - (p-k)^2/2 - m\omega - i\eta} \right)}. \quad (24)$$

Since the medium-dependent scattering length (12) is determined by the small k expansion and the poles of the different occurring Bose functions appear for small $\tilde{\epsilon}$ and consequently small momenta, we consider the quasiparticle fea-

tures only at small momenta. One should note that we are still in the gaslike limit for the quasiparticle dispersion as can be seen from its quadratic behavior. Further improvements require the consideration of the condensate explicitly³⁸ lead-

TABLE I. The maximal critical temperature, the critical coupling parameter where the Bose condensation vanishes, and the linear coefficient from the critical temperature according to Fig. 3 for the three different approximations used in the text.

	$(T_c/T_0)_{\max}$	y_0^{crit}	c_1
T matrix	1.56	0.16	4.66
T matrix ($m^*(\epsilon)$)	1.22	0.16	1.32
T matrix ($m^*(0)$)	1.07	0.16	-2.13
Screened T matrix	1.56	0.32	2.33
MC data ^a	1.06		0.34±0.08
MC data ^b			2.30±0.25
MC data ^c			1.32±0.02
MC data ^d			1.29±0.05

^aReference 34.

^bReference 16.

^cReference 17.

^dReference 18.

ing to linear phononlike dispersions³⁹ up to roton minima at higher momenta.⁴⁰ For the present level, we restrict our investigation to the quasiparticle renormalization at small momenta and quadratic dispersion.

The one-particle dispersion relation takes the form

$$\begin{aligned} \epsilon_p &\approx \frac{p^2}{2m} + \text{Re} \Sigma(0,0) + \left(\frac{p^2}{2m} \frac{\partial}{\partial p'^2} + \epsilon_p \frac{\partial}{\partial \omega'} \right) \\ &\quad \times \text{Re} \Sigma(\omega', p') \Big|_{p'=\omega'=0} + \mathcal{O}(p^3) \\ &\equiv \frac{p^2}{2m^*} + \alpha \text{Re} \Sigma(0,0) + \mathcal{O}(p^3), \end{aligned} \quad (25)$$

with

$$\alpha = \frac{1}{1 - \frac{\partial}{\partial \omega'} \text{Re} \Sigma(\omega', 0) \Big|_{\omega'=0}} \quad (26)$$

and the effective mass

$$\begin{aligned} \frac{m}{m^*(\epsilon, y_0)} &= \alpha \left(1 + \frac{\partial}{\partial p'^2} \text{Re} \Sigma(\omega', p') \Big|_{\omega'=p'=0} \right) \\ &= \left[1 - \left(\frac{\partial}{\partial p'^2} + \frac{\partial}{\partial \omega'} \right) \text{Re} \Sigma(\omega', p') \Big|_{\omega'=p'=0} \right]^{-1}. \end{aligned} \quad (27)$$

The second expression is the self-consistent one taking into account that the effective mass appears in all expressions of the self-energy. The constant self-energy shift $\alpha \text{Re} \Sigma(0,0)$ can be included into the definition of the chemical potential in the same way as the mean-field contribution which does not lead to any change in the critical temperature. All what is left from self-consistency is the effective mass.

It is not difficult to see from Eqs. (1) and (2) that in the second line of Eq. (15) an additional factor m/m^* has to appear,

$$\begin{aligned} y_0^c &= -\frac{1}{4g'(\bar{\epsilon})}, \\ x &= \frac{g(0)}{g(\bar{\epsilon})} \frac{m}{m^*(\bar{\epsilon}, y_0^c)}. \end{aligned} \quad (28)$$

For our present discussion, we have used only the first part of the self-energy (23). The system of equations (28) is solved and the resulting curve approaches the MC data, as can be seen in Fig. 3 (thick solid line). Especially, the maximal critical temperature is nearer to the MC data, as can be seen in Table I. Now the effective mass depends on $\bar{\epsilon}$. This leads to a delicate numerical balance as illustrated by the curve for the effective mass at fixed $\bar{\epsilon}=0$, which gives the best overall agreement with the data points but shows a negative c_1 . Calculations with higher-order approximation of the self-energy are therefore certainly necessary to achieve a sufficient convergence.

B. Screened ladder diagrams

We can now further improve the many-body approximation by using a ring summation of diagrams for the potential of the T matrix. This corresponds to the screened ladder approximation such that the bare potential in the T matrix is replaced by a screened one.⁴¹ In order to maintain separability of the T matrix, we consider this screened potential in separable form as a result of the screening,

$$\tilde{V}_{pp'} = V_{pp'} - \sum_{\bar{p}} \tilde{V}_{p\bar{p}} \frac{f_{p/2+\bar{p}} - f_{p/2-\bar{p}}}{\epsilon_{p/2+\bar{p}} - \epsilon_{p/2-\bar{p}} - \omega + i\eta} V_{\bar{p}p'}, \quad (29)$$

which is solved yielding

$$\tilde{V}_{pp'} = \frac{\lambda g_p g_{p'}}{1 + \lambda \Pi(\omega, p)},$$

$$\Pi(\omega, p) = \sum_{\bar{p}} g_{\bar{p}}^2 \frac{f_{p/2+\bar{p}} - f_{p/2-\bar{p}}}{\epsilon_{p/2+\bar{p}} - \epsilon_{p/2-\bar{p}} - \omega + i\eta}. \quad (30)$$

We see that the screening corresponds to a scaling of the coupling constant in the T matrix (7),

$$\lambda \rightarrow \frac{\lambda}{1 + \lambda \Pi(\omega, p)}, \quad (31)$$

and the same solution as Eq. (8) appears but with a new momentum- and energy-dependent coupling. We can now follow the same procedure as outlined above in using a β -dependent coupling constant to perform the contact potential limit $\beta \rightarrow \infty$. For this aim, we need the small momentum expansion

$$\lim_{\beta \rightarrow \infty} \Pi \left(\frac{p^2}{2m}, p \right) = - \frac{m}{2\pi^2 \hbar^3 \beta^4} \int_0^\infty dp f_p + i \frac{mpf_0}{8\pi \hbar^3 \beta^4} + \mathcal{O}(p^2). \quad (32)$$

The resulting formula analogous to Eq. (12) now reads

$$\frac{a}{a_0} = \frac{1 + 3f_0}{1 - \frac{2a_0}{\pi \hbar} \int_0^\infty d\bar{p} f_{\bar{p}}}, \quad (33)$$

which leads to

$$y_0^{cs} = - \frac{1}{2g'(\bar{\epsilon})},$$

$$x = \frac{g(0)}{g(\bar{\epsilon})} \quad (34)$$

instead of Eq. (15). Therefore, an additional factor 1/2 appears in the linear coefficient (21),

$$c_1^s = \frac{8\pi}{3\zeta\left(\frac{3}{2}\right)^{4/3}} = 2.33, \quad (35)$$

a result in remarkably good agreement with the results of Refs. 6, 14, and 15 and the MC data.¹⁶ The critical values are found in Table I. The screened ladder approximation leads to twice the maximal critical coupling strength but the same maximal critical temperature compared to the T-matrix approximation. We note that further improvements should be possible again by considering the self-consistency, i.e., the effective mass.

VI. SUMMARY

We have discussed interacting Bose systems interaction. The appearance of the Bose-Einstein condensation becomes dependent on the interaction parameter. We derive the same condition for Bose-Einstein condensation both from the pole of the T matrix and from the observation that the medium-dependent scattering length is diverging at the Bose condensation. This condition is used to produce a perturbation series of higher order than the original T-matrix summation adopting the well known method of inversion. With the help of this method, we obtain a strong coupling result out of the weak coupling T-matrix summation. We are able to describe the nonlinear behavior of the critical temperature. The critical temperature rises linearly with the interaction parameter in leading order, reaching a maximal value and is reduced for higher interaction strength. At an upper specific interaction parameter, Bose condensation is not possible anymore. The system is too correlated for allowing condensation.

We improve the original weak coupling series further by using screened vertices which reproduce the leading order coefficient of the MC data. Further improvements toward the nonlinear curve of the MC data and the variational perturbation theory are possible by self-consistency. Here, we present as a first step an effective mass calculation inside the T matrix and show that the inversion method leads then to a better description of the data.

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