Writing Theorems

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Benchmark:

This lesson is intended for a 10^{th} grade audience.

High school mathematics curricula must educate students on the subject and techniques of mathematics as well as the skills needed to communicate mathematical ideas clearly and effectively. The ability to read and write theorems is a skill used daily by all mathematicians and theorems are, perhaps, the most widely used medium for communicating mathematical ideas.

Prerequisite skills for being able to read, appreciate and write theorems are outlined in the Science Literacy Benchmarks [1]. For example, by the end of 12^{th} grade, students should understand how to make a convincing logical argument; they should know how to break a complicated argument into smaller steps and they should understand that the truth value of a statement can (usually) be assessed using logic.

This lesson puts less emphasis on the logical structure of theorems, which many younger students can find intimidating, and more emphasis on using theorems as a way to communicate mathematical ideas.

Purpose:

To be able to read, write simple theorems;
To be able to communicate mathematical ideas with words and complete sentences rather than just mathematical symbols and numbers;
To see the connections between the expression of mathematical ideas and the expression of ideas in an essay or poetry, for example.

Motivation:

This lesson is about communicating mathematical ideas through words. Many high school students believe that mathematical ideas can only be expressed with symbols, numbers, equations and graphs. This limited understanding of mathematics hinders their ability to express their own mathematical ideas as well as make connections between mathematics and other subjects.

This lesson builds on mathematical understanding and techniques already familiar to the students (in this case, we build on an understanding of quadratic functions and the concept of the average rate of change of a function between two points, but any foundation of previously learned material would work). As part of this lesson, students are given an introduction to reading and writing theorems

[1] www.sciencenetlinks.com

using techniques more commonly found in a writing class. The students are then asked to write a theorem of their own. The tactics employed in writing a theorem are similar (if not identical) to the tactics a tenth grader would use in writing an essay. The students begin with an idea– a thesis statement. They then write a draft theorem which is subsequently edited repeatedly until a polished, clear and correct theorem results. As an optional component of this lesson, students may be asked to prove their own theorem, or a classmate's theorem.

The goals of this lesson are two-fold. The ability to read and write theorems is a vital skill for any mathematician. Many students find the jargon and style of theorems difficult to penetrate. This lesson aims to provide an introduction to theorems which alleviates some of this frustration by emphasizing the similarities between theorems and other styles of writing, rather than the differences. An early appreciation of the nature of theorems will set the stage for further and richer mathematical literacy.

Another goal of this lesson is to break down some of the barriers between subjects. The modular nature of the high school curriculum unfortunately inhibits connections being made between subjects. Mathematics is particularly at risk of this sort of isolation and students rarely bring the skills learned in other subjects to bear on their mathematics studies. This lesson gives at least one example of how mathematics does make a connection with the writing skills learned in other classes.

Lesson Outline:

Thinking Ahead:

I gave this lesson to my tenth grade class after we had studied parabolas, quadratic equations and the concept of the average rate of change of a function between two points. These ideas provide the context for this lesson on reading and writing theorems. It should be noted, however, that almost any other mathematical context would serve the purpose equally well. Students could be asked to write theorems about linear functions, geometry, trigonometric functions etc. Indeed, it would be interesting to revisit this lesson at various stages of learning to see how the students' understanding of theorems evolves.

This lesson may not be suitable for a single, hour-long period. I had originally planned for this lesson to take two periods (one for writing the theorems and another for discussing the proofs). I was so pleased with my students' enthusiasm for theorems and the interesting discussions that this lesson generated, however, that I let the lesson continue over three periods.

Delivery:

Begin the lesson by giving the students the following prompts: "Think about how you would write a poem about how you get to school in the morning." and "Think about how you would write an essay about how you get to school in the morning." Discuss with the class how the poem and the essay, although addressing the same topic, would look very different and may even contain different information. The poem would pay attention to details and imagery, while the essay might pay more attention to facts and chronological order. The language in each case would be different also. The poem would be more descriptive and the essay would be more formal. The students should feel comfortable with this discussion and they should realize that the discussion could be broadened to compare and contrast many different styles of writing.

Introduce the concept of a theorem– the mathematical equivalent of a poem. Prompt the students again: "Think about how you would write a theorem about how you get to school in the morning." Some of my students were surprised to think that a theorem could be written about a non-mathematical topic.

At this point, some discussion about what constitutes a theorem is helpful. I defined a theorem as *any true statement which can be proved.* This definition of a theorem rules out false statements and definitions or axioms (which cannot be proved) but certainly leaves room for statements which are not obviously mathematical.

Discuss the style of writing used in theorems. Knowing the style of a theorem can help to understand its meaning. Every theorem takes the basic form of "if..., then...". The wording might change slightly, but this is always the format. For example, the word "if" might be replaced with "given..." or "suppose that..." or "let...". (If this seems unnecessarily restrictive to some students, remind them of the huge variety of ideas that can be expressed in the form of a haiku, for example– a poem where the order and number of syllables is rigidly fixed.)

Go through one or two examples of theorems with the class. I chose the following examples:

Theorem (1). *The average rate of change of a linear function, between any two points, is equal to the slope of that function.*

Analyze the theorem with the class in three stages:
1) for content
2) for style
3) for logical coherence.

Read the theorem out loud and ask a student to restate the theorem in his/her own words. Take care that they preserve the order of implication and mention all parts of the hypothesis. My students had difficulty with this. Most students re-expressed the theorem almost exactly as it was written on the board. Instead, I offered my own "translation" of the theorem into different styles. As a conversation, this theorem might sound like this:

"Hey, I've got some news. I have this linear function here. I don't know that much about it– I don't even know its slope or y-intercept, but I do know that it's definitely linear. Anyway, let me tell you one true thing about this function: if you select any two points on the line and then calculate the average rate of change between those two points, the number you get will be the same as the slope of the line."

Make sure that the students understand the content and meaning of the theorem before proceeding to discuss its style.

Since each theorem must be of the form: "if... then...", where are the if/then parts of this theorem? Let the students try to figure this out by themselves or by discussing in small groups. It's helpful to rewrite the theorem using the results of this work:

Theorem (1). *If we have a linear function, and we select any two points on that line, then, the average rate of change between those two points will have the same value as the slope of the line.*

Finally, make sure that the students believe the theorem. (In my class, the students should have seen this information presented in a different way in a previous lesson.)

Return to the task of writing a theorem about getting to school. One possible theorem is

Theorem (2). *If the day is Monday, then I walk to school.*

This fits the definition and style of a theorem. It could easily be true (we don't know), furthermore, we might even be able to prove the statement if, for instance, all public transportation stops on Mondays and the only way to get to school is to walk. No mathematics is involved, but it still fits our definition of a theorem.

(At this point, a discussion about the proof of Theorem (1) could happen. The proof relies on writing down the general form for a linear equation, $y = mx+b$, and two general points, x_1 and x_2, say, and then calculating the average rate of change of y between these two points. The result is the slope m, which proves the theorem.)

As another example, my class discussed the following theorem in a similar way.

Theorem (3). *Given a parabola, $y = ax^2 + bx + c$, where $a > 0$, the average rate of change between the points $x = -b/2a$ and any point greater than x, is positive.*

Once the class has a sense of the style and purpose of theorems, ask them to invent a theorem of their own. I narrowed the scope for my students and asked that their theorem have something to do with parabolas and average rates of change. I also allowed the students to work in small groups of two or three.

Advise the students to approach this task in stages: first come up with an idea which is true: what the theorem will be about. This task is analogous to identifying a thesis for an essay. Next, try to write this idea down as a theorem making the if and then parts of the theorem explicit. Remind them that this should feel very much like the preparation for and editing of an essay. Walk through the room several times so that you have a chance to help each group with each of these stages. The result is a situation where students write drafts of a theorem which you then proof read and offer suggestions about.

I list here some of the difficulties my students had at this stage: The first difficulty was something akin to writer's block. The students had trouble thinking of statements which would be true in some general, or nontrivial sense. To this, I reminded the students of our definition of a theorem. Any true statement would suffice. Some of these students went on to write theorems of this nature:

Theorem (4). *Given a parabola of the form $y = x^2$, the average rate of change between $x = -2$ and $x = 2$ is zero.*

Although this statement is, in itself, rather trivial, it fulfills the expectations of the assignment and shows that the student understands how to write a theorem. Some of these students were able to generalize this statement to the following theorems:

Theorem (5). *Given a parabola of the form $y = x^2$, the average rate of change between two points, $-x$ and x, is zero.*

Theorem (6). *Given a parabola of the form $y = ax^2 + bx + c$, the average rate of change between the two x-intercepts of the parabola is zero.*

The second difficulty my students faced was a lack of rigor in expressing their ideas. For instance, one group of students wrote the following theorem:

Theorem (7). *Given a parabola, $y = ax^2 + bx + c$, the average rate of change between $x = -b/2a$ and another point to the right of x, is negative.*

The students failed to realize that this statement is only true if we also require the coefficient, a, to be less than zero. For these students, the assignment of writing a theorem about parabolas actually gave them a more subtle and deep understanding of the subject than they would have achieved otherwise.

The most amazing and rewarding part of this lesson is what happened just as the students became comfortable with the assignment and the style of writing used in writing theorems. It was during my second or third visit with each group that the students' enthusiasm and curiosity began to take root. By this stage, most groups already had a good draft of a theorem but the ideas expressed in the theorem were more or less trivial. With very little prompting from me, the students developed a desire to write more general, abstract and adventurous theorems.

One group said that they wanted to write a theorem which expressed the fact that the average rate of change of a parabola increases as you go away from the minimum point. After some discussions with me and among themselves, and several drafts and editing sessions, the students arrived at the following theorem:

Theorem (8). *Given a parabola of the form $y = ax^2 + bx + c$ with $a > 0$, and given three points, $x = -b/2a$, $x_1 > x$ and $x_2 > x_1 > x$, the average rate of change of y between x and x_2 is greater than the average rate of change of y between x_1 and x.*

Another student (who was working independently) wanted to write a theorem which expressed a formula he had derived for the average rate of change of $y = x^2$ between consecutive integers. Several discussions and edits resulted in this theorem:

Theorem (9). *Given the function $y = x^2$, the average rate of change of y between the points $x > 0$ and $x + 1$ is $2x + 1$.*

Assessment:

One of the advantages of this assignment is that its level of difficulty and sophistication is determined by the student. In my own class, I did not award marks based on the complexity or usefulness of the theorem. Indeed, all students who wrote a theorem of any kind were awarded full marks. In addition, I "published" a booklet of all the theorems and proofs and distributed it to the class. These theorems are now available for us to use at any point throughout the year. The usefulness and quality of the theorems will be determined by the class in time.

Acknowledgments:

This lesson was given as part of a tenth grade Algebra and Trigonometry course at Bard High School Early College (BHSEC) in New York in 2004. The success of this lesson is largely determined by the students: their ability to work together and their ability to focus and enjoy learning. It has been an honor and a pleasure for me to work with such dedicated, curious and enthusiastic students at BHSEC.

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