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Structure functions and breakdown criteria for wave turbulence

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Abstract

We study the structure functions of wave turbulence at small separations. We show that the criteria for breakdown obtained previously by examining the uniform validity of the asymptotic closure govern how close wave turbulence stays to joint Gaussianity. A new result in the case of small separations in that the system behavior is organized by a special point in the (α, β) plane where α and β are the homogeneity parameters for the linear and nonlinear coupling coefficients. Finally, we explore how modifications of the breakdown criteria are necessary because of the strengths of non-local long-wave–short-wave interactions.

Keywords: Wave turbulence; Asymptotic closure; Gaussianity

1. Introduction

In [1,6], we developed simple criteria for the breakdown of the wave turbulence approximation for situations where the energy transfer is sufficiently local. Here we both extend these results and discover some new behaviors of the structure functions:

$$S_N(r) = \langle (u(\underline{x} + \underline{r}) - u(\underline{x}))^N \rangle, \tag{1.1}$$

when $r = |\underline{r}|$ is small. The variable $u(\underline{x}, t)$ is a spatially homogeneous, random, bounded field and we assume that, initially, its cumulants decay strongly as separations become large. It represents a wave field which, for example, one may think of as the surface elevation of the sea. To leading order, propagation of deformations of the surface obey linear dispersion relations but, over long times, energy is exchanged between waves via r wave resonances ($r = 3, 4, \dots$). This exchange is described by a kinetic equation for the wave-action spectral density whose most relevant stationary states are the Kolmogorov–Zakharov (KZ) finite flux spectra for both energy and wave-action (particles). Using these solutions, we can directly calculate the corresponding physical space averages. To examine small scale behavior, the structure functions $S_N(r)$ are most useful. The main reason is that their dependence on r for small r indicates the degree of non-smoothness of the variable $u(x)$ in realizations of the field. For example, we will find that $S_N \sim r^{\zeta_N}$, $\zeta_N < N$, which indicates that the variable (e.g. the sea surface) has lost differentiability and the values ζ_N will tell us something of how scalloped it looks. In Section 2 we examine their behavior in considerable

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detail. In Section 3, we briefly consider how the breakdown criteria change as non-local (long wave–short wave) interactions compete with the local interactions which give rise to the KZ spectra.

The new results of this paper are

1. Previously, we have shown that the expansions for the structure functions exhibit a departure from joint Gaussian statistics at small values $r < r_{\rm NL}$. The scale $r_{\rm NL}$ exactly coincides with the wave-number $k_{\rm NL}^{-1}$ we obtained for wave turbulence breakdown by examining the uniform validity of the kinetic equation's asymptotic expansion. What is new here is that we examine universality of the structure functions by calculating whether the dominant contribution to their behaviors arise from the universal Kolmogorov–Zakharov (KZ) spectra, or from the forcing terms associated with large scales. It turns out that their universality can be conveniently characterized in the (α, β) plane where α and β are numbers capturing the relative strengths of the linear and nonlinear coupling coefficients. We provide colorful road-maps in the cases of three and four wave interactions. We argue why there is a special point in the (α, β) plane about which all the different behaviors are organized. Further, we show that there is an open set of (α, β) values such that one can always find an appropriate variable for which all the structure functions are universal.
2. The analysis of wave turbulence breakdown to date has assumed that the nonlinear coupling coefficients are sufficiently local so that all the integrals multiplying the large k behaviors converge. This may not always be the case. In Section 3 we ask whether certain divergences which, depending on coupling coefficients, can emerge due to strong non-local long-wave–short-wave interactions, modify the simple breakdown criteria and find that a modification can indeed occur. We discuss this modification and, as a concrete example, look at the case for deep water gravity waves.

The wave turbulence approximation proceeds as follows. We begin with the equation:

$$\frac{\partial}{\partial t}A^s(\underline{k},t) - \mathrm{i}s\omega(\underline{k})A^s(\underline{k},t) = \varepsilon^{m-1}\sum_{m=2}^{\infty}\sum_{s_1\cdots s_m}\int L^{ss_1s_2\cdots s_m}_{\underline{k}\underline{k}_1\underline{k}_2\cdots\underline{k}_m} A^{s_1}(\underline{k}_1,t)A^{s_2}(\underline{k}_2,t)\cdots A^{s_m}(\underline{k}_m) \times \delta(\underline{k}_1+\underline{k}_2+\cdots+\underline{k}_m-\underline{k})\,\mathrm{d}\underline{k}_1\,\mathrm{d}\underline{k}_2\cdots\mathrm{d}\underline{k}_m \tag{1.2}$$

for the generalized Fourier transform εA^s_k of the canonical physical field $u^s(\underline{x})$ where $0 < \varepsilon \ll 1$. In (1.2), $\omega^s(\underline{k})$ is the dispersion relation and $L^{ss_1\cdots}_{\underline{k}\underline{k}_1\cdots}$ is the nonlinear coupling coefficient. The letter s is an integer which counts the degeneracy of the dispersion relation; here we take it to be ±1 so that $\omega^s(\underline{k}) = s\omega(\underline{k})$. The functions $\omega(\underline{k})$ and $L^{ss_1\cdots s_m}_{\underline{k}\underline{k}_1\cdots\underline{k}_m}$ are taken to be homogeneous ($\omega(\lambda\underline{k}) = \lambda^\alpha\omega(\underline{k})$; $L^{ss_1s_2}_{\lambda\underline{k}\lambda\underline{k}_1\lambda\underline{k}_2} = \lambda^\beta L^{ss_1s_2}_{\underline{k}\underline{k}_1\underline{k}_2}$;$L^{ss_1s_2s_3}_{\lambda\underline{k}\lambda\underline{k}_1\lambda\underline{k}_2\lambda\underline{k}_3} = \lambda^\gamma L^{ss_1s_2s_3}_{\underline{k}\underline{k}_1\underline{k}_2\underline{k}_3}$) of degrees α, β and γ, respectively. From (1.2) one can build the BBGKY hierarchy of equations for the Fourier cumulants, namely the Fourier transforms, $Q^{ss'\cdots s^{(N-1)}}(\underline{k},\underline{k}',\ldots,\underline{k}^{(N-1)})\delta(\underline{k}+\cdots+\underline{k}^{(N-1)})$, of physical space cumulants $R^{(N)ss'\cdots s^{(N-1)}}(\underline{r},\underline{r}',\ldots,\underline{r}^{(N-2)})$ which are defined as Nth order moments, $\langle u^s(\underline{x})u^{s'}(\underline{x}+\underline{r})\cdots u^{s^{(N-1)}}(\underline{x}+\underline{r}^{(N-2)})\rangle$, from which the appropriate combinations of products of lower order moments are subtracted so that $R^{(N)} \to 0$ as any of the separations $|\underline{r}|, |\underline{r}'|, \ldots, |\underline{r}^{(N-2)}| \to \infty$. We solve the hierarchy iteratively in powers of amplitude:

$$Q^{(N)ss'\cdots s^{(N-1)}}(\underline{k},\underline{k}',\ldots,\underline{k}^{(N-1)}) = q_0^{(N)ss'\cdots s^{(N-1)}}(\underline{k},\underline{k}',\ldots,\underline{k}^{(N-1)})\,\mathrm{e}^{\mathrm{i}(s\omega_k+s'\omega_{k'}+\cdots)t} + \varepsilon Q_1^{(N)ss'\cdots}(\underline{k},\underline{k}',\cdots)+\cdots \tag{1.3}$$

and demand that (1.3) is a uniformly valid asymptotic expansion in time. To achieve this goal, we choose the slow time behaviors of the $q_0^{(N)ss'\cdots s^{(N-1)}}(\underline{k},\cdots)$ to remove any secular growth. The resulting hierarchy of equations for the $q_0^{(N)}$ is closed. This is what we call *asymptotic closure*. Because the equations for $q_0^{(N)}$, $N > 2$, appear as expressions for $\mathrm{d}/\mathrm{d}t\,\ln q_0^{(N)}$, $\mathrm{d}/\mathrm{d}t\,\ln q_0^{ss}$ which only depend on $n^s_k = q_0^{-ss}(\underline{k})$, the asymptotic closure reduces to a kinetic equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}n_k^s = \varepsilon^2 T_2[n_k^s] + \varepsilon^4 T_4[n_k^s] + \cdots + \varepsilon^{2p} T_{2p}[n_k^s] + \cdots \tag{1.4}$$

and a frequency renormalization:

$$s\omega_k \to s\omega_k + \varepsilon^2 \Omega_2^s[n_k^s] + \cdots . \tag{1.5}$$

For time scales $\varepsilon^{2p}t = \mathrm{O}(1)$, $\varepsilon^{2p+2}t \ll 1$, $p \geq 1$, we seek solutions of the kinetic equation (1.4) truncated at order $2p$. The collision term $T_2[n_k]$ is built from three wave resonances; the terms $T_4[n_k]$ from four wave resonances and spectral gradients of the three wave resonances and so on. Here we give the expression for $T_2[n_k]$ and $\Omega_{2k}^s[n_k]$, where we have used $n_k \equiv n_k^+ = n_k^- = q_0^{(2)s-s}$:

$$\begin{aligned} T_2[n_k] = 4\pi \sum_{s_1 s_2} \int L_{\underline{k}\underline{k}_1\underline{k}_2}^{ss_1s_2} (n_k n_{k_1} n_{k_2}) \left(\frac{L_{-\underline{k}-\underline{k}_1-\underline{k}_2}^{-s-s_1-s_2}}{n_k} + \frac{L_{\underline{k}_1-\underline{k}_2\underline{k}}^{s_1-s_2s}}{n_{k_1}} + \frac{L_{\underline{k}_2\underline{k}-\underline{k}_1}^{s_2s-s_1}}{n_{k_2}} \right) \\ \times \delta(s_1\omega(\underline{k}_1) + s_2\omega(\underline{k}_2) - s\omega(\underline{k}))\delta(\underline{k}_1 + \underline{k}_2 - \underline{k})\, \mathrm{d}\underline{k}_1\, \mathrm{d}\underline{k}_2, \end{aligned} \tag{1.6}$$

$$\begin{aligned} T_2[n_k] = 4\pi \int |L_{\underline{k}\underline{k}_1\underline{k}_2}^{+++}|^2 n_k n_{k_1} n_{k_2} \Bigg(\left(\frac{1}{n_{k_1}} + \frac{1}{n_{k_2}} - \frac{1}{n_k} \right) \delta(\omega_{\underline{k}_1} + \omega_{\underline{k}_2} - \omega_{\underline{k}}) \\ + \left(\frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} - \frac{1}{n_k} \right) \delta(\omega_{\underline{k}_1} - \omega_{\underline{k}_2} - \omega_{\underline{k}}) \\ + \left(\frac{1}{n_{k_2}} - \frac{1}{n_{k_1}} - \frac{1}{n_k} \right) \delta(\omega_{\underline{k}_2} - \omega_{\underline{k}_1} - \omega_{\underline{k}}) \Bigg) \delta(\underline{k}_1 + \underline{k}_2 - \underline{k})\, \mathrm{d}\underline{k}_1\, \mathrm{d}\underline{k}_2, \end{aligned} \tag{1.7}$$

$$\begin{aligned} \Omega_{2k}^s[n_k] = 4 \sum_{s_1 s_2} \int L_{\underline{k}\underline{k}_1\underline{k}_2}^{ss_1s_2} L_{\underline{k}_1\underline{k}-\underline{k}_2}^{s_1s-s_2} n_{k_2} \left(\mathrm{PV}\left(\frac{1}{s_1\omega_{\underline{k}_1} + s_2\omega_{\underline{k}_2} - s\omega_{\underline{k}}} \right) - \mathrm{i}\pi\delta(s_1\omega_{\underline{k}_1} + s_2\omega_{\underline{k}_2} - s\omega_{\underline{k}}) \right) \\ \times \delta(\underline{k}_1 + \underline{k}_2 - \underline{k})\, \mathrm{d}\underline{k}_1\, \mathrm{d}\underline{k}_2. \end{aligned} \tag{1.8}$$

$\delta(\cdot)$ is the Dirac delta function, PV the principal value and we will often write Ω_{2k} for $\Omega_2^s[n_k]$. In Eq. (1.7), we have made use of the relation $L_{\underline{k}\underline{k}_1\underline{k}_2}^{+++} = L_{\underline{k}_1\underline{k}-\underline{k}_2}^{++-}$ which is an essential symmetry of three wave interactions. The truncated equations can be solved [8] for their finite flux Kolmogorov–Zakharov solutions. For three wave resonances:

$$n_k = C_1 P^{1/2} k^{-(\beta+d)}. \tag{1.9}$$

For cases where $T_2 = 0$, either by virtue of the resonant manifold ($\underline{k}_1 + \underline{k}_2 = \underline{k}$; $\pm\omega_{\underline{k}_1} \pm \omega_{\underline{k}_2} = \omega_{\underline{k}}$) being empty, or by $L_{\underline{k}\underline{k}_1\underline{k}_2}^{ss_1s_2}$ being zero, the kinetic equation truncated at $p = 2$ yields the finite (energy and particle) flux solutions:

$$n_k = C_2 P^{1/3} k^{-((2\gamma/3)+d)}, \tag{1.10}$$

$$n_k = C_3 Q^{1/3} k^{-((2\gamma/3)+d)+\alpha/3}. \tag{1.11}$$

Breakdown occurs when these solutions no longer keep the asymptotic expansions (1.4) and (1.5) uniformly valid in k and where, equivalently, the ratios of linear to nonlinear time scales (which should be small) approach unity. For example

$$\frac{t_{\mathrm{L}}}{t_{\mathrm{NL}}} = \frac{1}{\omega_k n_k}\frac{\partial n_k}{\partial t} \sim \frac{\varepsilon^2 T_2}{\omega_k n_k} = \varepsilon^2 P^{1/2} k^{\beta-2\alpha} I_2, \tag{1.12}$$

where I_2 is the integral

$$I_2 = 4\pi \int L^{ss_1s_2}_{\hat{\underline{k}}\hat{\underline{k}}_1\hat{\underline{k}}_2}(n_{\hat{k}}n_{\hat{k}_1}n_{\hat{k}_2})\left(\frac{L^{-s-s_1-s_2}_{-\hat{\underline{k}}-\hat{\underline{k}}_1-\hat{\underline{k}}_2}}{n_{\hat{k}}} + \frac{L^{s_1-s_2s}_{\hat{\underline{k}}_1-\hat{\underline{k}}_2\hat{\underline{k}}}}{n_{\hat{k}_1}} + \frac{L^{s_2s-s_1}_{-\hat{\underline{k}}_2\hat{\underline{k}}-\hat{\underline{k}}_1}}{n_{\hat{k}_2}}\right) \times \delta(s_1\omega_{\hat{k}_1} + s_2\omega_{\hat{k}_2} + s\omega_{\hat{k}})(\hat{k}_1)^{d-1}\,\mathrm{d}\hat{k}_1\,\mathrm{d}\Omega_1, \tag{1.13}$$

where $\hat{\underline{k}}_i = (\underline{k}_i)/(|\underline{k}|)$ has modulus $\hat{k}_i$; and $\mathrm{d}\Omega_1$ denotes the integration over angles when $\hat{\underline{k}}_1$ is expressed in d-dimensional polar coordinates. We have taken n_k to be the homogeneous Kolmogorov–Zakharov solution (1.9).

If the interactions are sufficiently local (namely the coefficients $L_{\underline{k}\underline{k}_1\underline{k}_2}$ go to zero sufficiently fast as their arguments go to zero or infinity), the integral I_2 is finite. In that case the breakdown criterion is simple because it only depends on α, β and P. Furthermore, $t_{\mathrm{L}}/t_{\mathrm{NL}}$ approaches unity at a scale $k_{\mathrm{NL}} = P^{-1/(2(\beta-2\alpha))}$, where we have incorporated ε into P ($\varepsilon^4 P \to P$). An analysis of T_4/T_2, T_6/T_4, $\Omega_{2k}/\omega_k, \ldots$ yields similar ratios. Therefore, if the coefficient integrals, I_N, are finite then, for $\beta > 2\alpha$, k_{NL} is large and wave turbulence breaks down for $k > k_{\mathrm{NL}}$; and for $\beta < 2\alpha$, k_{NL} is small and breakdown occurs for $k < k_{\mathrm{NL}}$.

For four wave interactions (when three wave resonances are not present), the corresponding ratios of linear to nonlinear time scales; $T_4/\omega_k n_k$ and the ratios of T_6/T_4, T_8/T_6, Ω^s_2/ω_k are given by $P^{2/3}k^{(2\gamma/3)-2\alpha}I_4$ and $P^{1/3}k^{(\gamma/3)-\alpha}(I_6/I_4, T_8/T_6, \ldots)$, where the I_N are integrals like in (1.13). In particular, I_4 is given by

$$I_4 = \int |G_{\hat{\underline{k}}\hat{\underline{k}}_1,\hat{\underline{k}}_2\hat{\underline{k}}_3}|^2 n_{\hat{k}}n_{\hat{k}_1}n_{\hat{k}_2}n_{\hat{k}_3}\left(\frac{1}{n_{\hat{k}}} + \frac{1}{n_{\hat{k}_1}} - \frac{1}{n_{\hat{k}_2}} - \frac{1}{n_{\hat{k}_3}}\right) \times \delta(\omega_{\hat{k}} + \omega_{\hat{k}_1} - \omega_{\hat{k}_2} - \omega_{\hat{k}_3})(\hat{k}_1\hat{k}_2)^{d-1}\,\mathrm{d}\hat{k}_1\,\mathrm{d}\hat{k}_2\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2, \tag{1.14}$$

where $k_i = (\underline{k}_i)/(|\underline{k}|)$, $i = 1, 2$ and $\underline{k}_3 = \underline{k} + \underline{k}_1 - \underline{k}_2$. $\mathrm{d}\Omega_i$ denotes integration over angles when $\hat{\underline{k}}_i$ is expressed in d-dimensional polar coordinates. (T_4 is given is Section 3 along with the definition of G.)

If the interactions are sufficiently local so that all integrals converge, then the criteria for breakdown are simple and depend only on α, γ and P. Breakdown occurs for large (small) k if $\gamma > 3\alpha$ ($\gamma < 3\alpha$) at $k_{\mathrm{NL}} = P^{-1/(\gamma-3\alpha)}$. Therefore, in both cases, for sufficiently local interactions, the criterion for breakdown are very simple.

In many practical situations, however, the coefficients $L_{\underline{k}\underline{k}_1\underline{k}_2}$ and $L_{\underline{k}\underline{k}_1\underline{k}_2\underline{k}_3}$ for the non-local interactions between long and short waves (for three wave interactions: $\underline{k}_1$ is small while $\underline{k}$, $\underline{k}_2$ are large or $\underline{k}$ is small while $\underline{k}_1$, $\underline{k}_2$ are large; for four wave interactions: $\underline{k}_1$, $\underline{k}_3$ are small and $\underline{k}$, $\underline{k}_2$ are large) are not sufficiently small to ensure convergence. Formally, this would necessitate the introduction of a large scale cut-off k_{I} and then the breakdown wave-number k_{NL} will depend on k_{I} as well as the universal parameters P, α and β. We discuss this in Section 3. For now, we will assume that the coupling coefficients are such that interactions are local enough so that all coefficient integrals converge.

2. Structure functions

We now turn our attention to the structure functions based upon the KZ solution and the long time surviving parts of each of the cumulants. We note that, because $\mathrm{Im}\,\Omega^s_{2k} > 0$ and indeed because the zeroth order cumulant is multiplied by the exponential of a fast phase, all the initial information on cumulants of order higher than two (and of order two when $s' = s$) is lost. In what follows, we calculate only those terms which survive asymptotically and which are given by the integrals over products of n_k. We begin with the case of three wave resonances.

We intend to plot the behavior of the various structure functions in the (α, β) plane, showing where the functions behave universally and where they are dominated by the forcing. We show that, along the line $\beta = 2\alpha$, where the

KZ solution (1.9) inherits the scaling symmetries of the governing Eq. (1.2), the behavior of structure functions is either universal for all N, or dominated by the forcing for all N. The boundary between these two regimes is the point $\alpha = 2$, $\beta = 4$. This point organizes all of the intermediate cases which occur off the $\beta = 2\alpha$ line where some structure functions are universal while others are not.

We give explicit calculations for $S_2(r)$, $S_3(r)$ and $S_4(r)$ and then state the results for the higher orders. These calculations will define what we mean by the universal strip for S_N and describe the behavior of S_N in and outside of the strip. For convenience, we omit the superscript s denoting signs. Because we are dealing with KZ solutions which carry a finite amount of energy, we use as our physical variable that quantity $v(\underline{x})$ whose power spectrum is the spectral energy $\omega_k n_k$. This means that the generalized Fourier transform of v is $\omega_k^{1/2} A_k$. Formally,

$$S_2(r) = \langle (v(\underline{x}+\underline{r}) - v(\underline{x}))^2 \rangle = 2 \int \omega_k n_k (1 - \cos(\underline{k}\cdot\underline{r}))\,\mathrm{d}\underline{k}. \tag{2.1}$$

We split the integral into three regions in $\underline{k}$ space. The first, $k < k_{\mathrm{I}}$, is the region in which the non-universal forcing is applied. The second, $k_{\mathrm{I}} < k < k_{\mathrm{U}}$, is the region in which the universal KZ spectrum is obtained. The third, $k > k_{\mathrm{U}}$, is the non-universal dissipation region.

We say that the result is universal if, as $r \to 0$, the dominant contribution comes from the inertial range, $k_{\mathrm{I}} < k < k_{\mathrm{U}}$, and we can take the limits $k_{\mathrm{I}} \to 0$, $k_{\mathrm{U}} \to \infty$. In the region $k < k_{\mathrm{I}}$, $\omega_k n_k = F(k)$, some non-universal forcing which we can take to be analytic in k. Therefore, as $r \to 0$, this part of the integral scales as r^2. In the region $k_{\mathrm{U}} < k$, we assume that the solution in the dissipation region decays fast enough as a function of wave-number such that the contribution from this integral is always less than the contribution from the other regions. In the inertial region $k_{\mathrm{I}} < k < k_{\mathrm{U}}$, a simple change of variables will show that this contribution goes as $P^{1/2} r^{\beta-\alpha} I$. The integral factor I converges at small k (i.e. near k_{I} where $1 - \cos(\underline{k}\cdot\underline{r}) = \mathrm{O}(r^2)$) if $2 > \beta - \alpha$ and converges at large k (i.e. near k_{U} where $1 - \cos(\underline{k}\cdot\underline{r}) = \mathrm{O}(1)$) if $\beta - \alpha > 0$. The former condition also ensures that the inertial range, universal contribution to the scaling dominates that of the forcing interval. The latter condition is simply the condition that the KZ spectrum has finite energy capacity. Therefore, if α and β lie in the strip $0 < \beta - \alpha < 2$ in the (α, β) plane, we can say, for small r ($k_{\mathrm{U}}^{-1} \ll r \ll k_{\mathrm{I}}^{-1}$),

$$S_2(r) \sim C_1 P^{1/2} r^{\beta-\alpha} \tag{2.2}$$

is universal. For $2 < \beta - \alpha$, $S_2 \sim F^2 r^2$ is non-universal. The parameter F is a measure of the amplitude of the forcing and the width of the region over which it is applied. For example, $F^2 r^2$ will be proportional to k_{I} times some value of $F(k)(1 - \cos(kr))$ in $0 < k < k_{\mathrm{I}}$. In what follows, we use $F^N r^N$ to indicate the size of the contribution to $S_N(r)$ coming from the forcing region. Although we will not write all dimensional factors in our calculations, we follow the dependence of the structure functions on F since this will show when the breakdown scale is non-universal. We follow the dependence on the energy flux P since this is the universal parameter which we take to be small. On the line $\beta = \alpha + 2$, S_2 has universal scaling with logarithmic corrections (Fig. 1). The organizing point $\alpha = 2$, $\beta = 4$ can already be seen as the intersection between the top border of the S_2 strip and the breakdown line $\beta = 2\alpha$. Above the line $\beta = \alpha + 2$, non-universal effects contaminate the second order structure function and its scaling. The line $\beta = 2\alpha$, for small r, separates the region of wave turbulence breakdown from wave turbulence validity.

We can calculate $S_3(r)$ from the first surviving term in the asymptotic expansion for $Q^{(3)ss's''}(\underline{k}, \underline{k}', \underline{k}'')$ which is given by

$$\begin{aligned} q^{(3)ss's''}\delta(\underline{k}+\underline{k}'+\underline{k}'') = \cdots + \varepsilon^2 (L^{s-s'-s''}_{\underline{k}-\underline{k}'-\underline{k}''} n_{k'} n_{k''} + L^{s'-s''-s}_{\underline{k}'-\underline{k}''-\underline{k}} n_{k''} n_k + L^{s''-s-s'}_{\underline{k}''-\underline{k}-\underline{k}'} n_k n_{k'}) \\ \times \left(\pi\delta(s\omega_k + s'\omega_{k'} + s''\omega_{k''}) + \mathrm{iPV}\left(\frac{1}{s\omega_k + s'\omega_{k'} + s''\omega_{k''}}\right)\right)\delta(\underline{k}+\underline{k}'+\underline{k}'') + \cdots, \end{aligned} \tag{2.3}$$

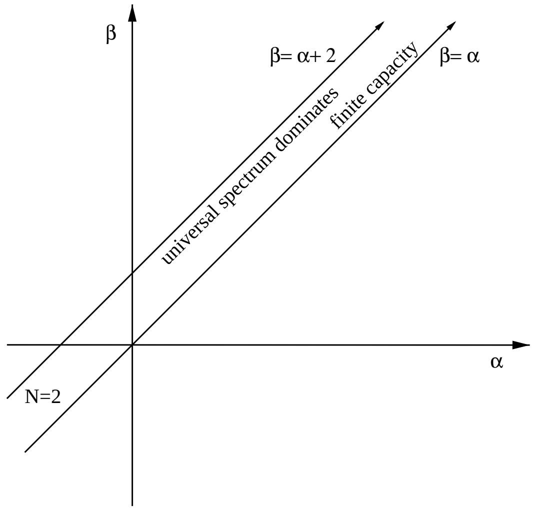

Fig. 1. Strip in which S_2 is universal.

where PV denotes the principal value. This is, of course, only the first term in an asymptotic series. The ratios of successive terms will be of the same order as we will find when we calculate the deviation from joint Gaussianity. The exception will occur when non-local, long-wave–short-wave interactions are important. We return to this point in Section 3.

The third order structure function is given by

$$S_3(r) = \int (\omega_{\underline{k}}\omega_{\underline{k}'}\omega_{\underline{k}''})^{1/2} q^{(3)}(\underline{k}, \underline{k}', \underline{k}'')\delta(\underline{k} + \underline{k}' + \underline{k}'') \times 2\mathrm{i}(\sin(\underline{r}\cdot(\underline{k}+\underline{k}')) - \sin(\underline{r}\cdot\underline{k}) - \sin(\underline{r}\cdot\underline{k}'))\,\mathrm{d}\underline{k}\,\mathrm{d}\underline{k}'\,\mathrm{d}\underline{k}'' \tag{2.4}$$

which is identically zero due to the fact that the sin functions are odd functions of the angles between the $\underline{k}$ vectors and $\underline{r}$ and because we have chosen the variable v to preserve the isotropy of the system. In these cases, the dispersion relation depends on $|\underline{k}|$ and the coefficients, $L_{\underline{k}\underline{k}_1\underline{k}_2}$, are even functions of the wave vector angles. The same holds for all odd structure functions. The main thrust of the breakdown result, however, does not depend on this property.

The fourth order structure function:

$$S_4(r) = \langle (v(\underline{x}+\underline{r}) - v(\underline{x}))^4\rangle = \int \sqrt{\omega_k\omega_{k_1}\omega_{k_2}\omega_{k_3}}\langle A_{\underline{k}}A_{\underline{k}_1}A_{\underline{k}_2}A_{\underline{k}_3}\rangle[1-\mathrm{e}^{\mathrm{i}\underline{k}\cdot\underline{r}}][1-\mathrm{e}^{\mathrm{i}\underline{k}_1\cdot\underline{r}}][1-\mathrm{e}^{\mathrm{i}\underline{k}_2\cdot\underline{r}}][1-\mathrm{e}^{\mathrm{i}\underline{k}_3\cdot\underline{r}}]\,\mathrm{d}\underline{k}\,\mathrm{d}\underline{k}_1\,\mathrm{d}\underline{k}_2\,\mathrm{d}\underline{k}_3$$

splits into two parts just as the fourth order moment splits into its fourth order part plus products of second order cumulants:

$$S_4(r) = 3S_2^2(r) + \mathfrak{S}_4(r), \tag{2.5}$$

where $\mathfrak{S}_4$ is calculated from the first term of the surviving fourth order cumulant which is proportional to a product of three n_k's. We find that $\mathfrak{S}_4$ is universal if α, β lie in the strip $0 < \beta < 4$. Outside of this strip, $\mathfrak{S}_4$ is non-universal. On the line $\beta = 4$ which defines the top of the strip, $\mathfrak{S}_4$ will have the universal scaling with logarithmic corrections. Furthermore, this line intersects the analogous line for the S_2 strip ($\beta = \alpha + 2$) and the breakdown line ($\beta = 2\alpha$) at the special point ($\alpha = 2, \beta = 4$) (Fig. 2). This pattern will continue for all strips.

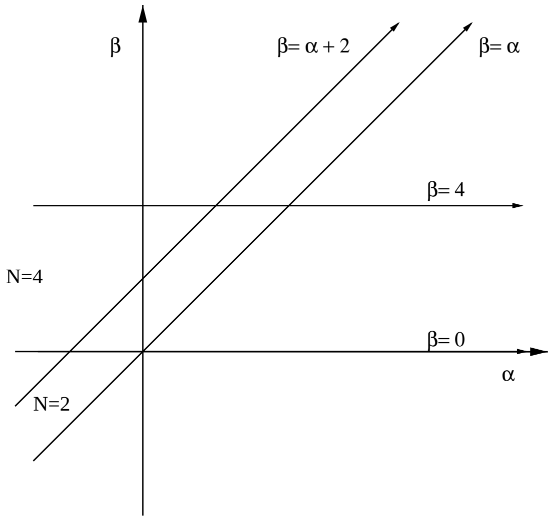

Fig. 2. Strip in which $\mathfrak{S}_4$ is universal. S_4 is universal in the intersection of the two strips.

For α, β inside the $\mathfrak{S}_4$ strip and for small r:

$$S_4(r) = C_1^2\,\mathrm{Pr}^{2\beta-2\alpha} + C_1^3 P^{3/2} r^{\beta}. \tag{2.6}$$

For a joint Gaussian field, S_4 would be simply $3S_2^2(r)$ and thus the remainder, $\mathfrak{S}_4$, measures deviations from joint Gaussianity. Observe that

$$\frac{\mathfrak{S}_4}{S_2^2} \sim P^{1/2} r^{2\alpha-\beta}. \tag{2.7}$$

Wave turbulence requires that the fields remain close to joint Gaussian at all scales. The right hand side of (2.7) fails to do this at small scales if $\beta > 2\alpha$ and indeed the length scale, r_{NL}, for which $P^{1/2}r^{2\alpha-\beta}$ becomes of order unity is precisely k_{NL}^{-1} where k_{NL} was defined in the introduction.

The strips, together with the breakdown line determine the behavior of the structure functions and the presence of breakdown. For instance, we find that the Nth order structure function can be written as a series

$$S_N(r) = \mathfrak{S}_N + \cdots + C_0 (S_2)^{N/2}.$$

$\mathfrak{S}_N$ is universal in the strip for $0 < \beta + (N/2-2)\alpha < N$. The intersection $\bigcap(0 < \beta + (N/2-2)\alpha < N)$ of all the strips is the black and white parallelogram, which we call the universal rhombus, shown in Fig. 3 ($\alpha < \beta < \alpha + 2$, $0 < \alpha \le 2$). Its diagonal is $\beta = 2\alpha$. Inside this parallelogram, we have

$$\frac{S_N}{(S_2)^{N/2}} = C_0 + \sum_{1}^{(N/2)-1} C_{N-2s} (P^{1/2} r^{2\alpha-\beta})^s. \tag{2.8}$$

Depending on the value of the exponent, $2\alpha-\beta$, there are three possible scenarios at small scales: (i) $2\alpha-\beta > 0$ and the ratio of structure functions can be approximated by the Gaussian value of C_0; $S_N/(S_2)^{N/2} \sim C_0$. Furthermore, this approximation gets better as $r \to 0$ so wave turbulence looks more and more Gaussian at small scales. (ii) $2\alpha - \beta = 0$. In this case, the ratio of non-Gaussian to Gaussian parts of the structure functions is independent of r and universally small. Wave turbulence has complete self-similarity with respect to k_{NL}. The KZ spectrum exactly

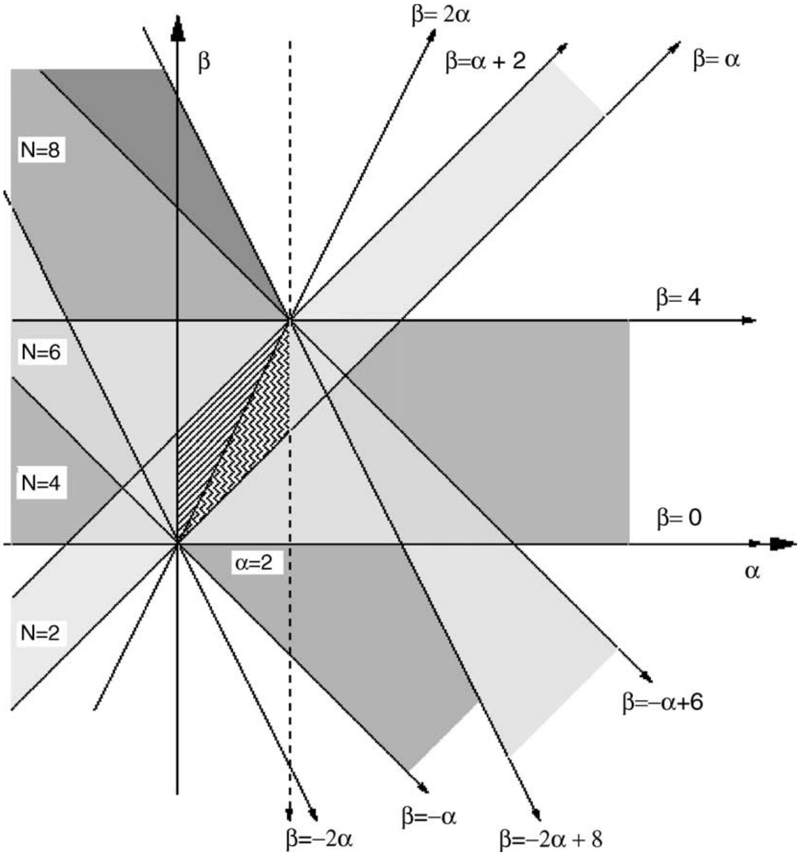

Fig. 3. Strips for the three wave energy spectrum: each strip is the condition for some $\mathfrak{S}_N$ to be universal. The black and white parallelogram shows the region in which all S_N are universal. The breakdown line is $\beta = 2\alpha$ and breakdown at small scales occurs for $\beta > 2\alpha$.

inherits the scaling properties of the original equation (1.2) and wave turbulence is uniformly valid over all scales. (iii) $2\alpha - \beta < 0$ in which case the r-dependent corrections to the Gaussian constant grow as $r \to 0$. At small scales, structures appear which are outside of the remit of weak turbulence theory. In this case, we identify a breakdown region at small scales as we have discussed. Looking at Fig. 3, the universal rhombus contains two triangles. In the triangle $\beta > 2\alpha$, we still have breakdown for $k > k_{\mathrm{NL}}$. In the triangle $\beta < 2\alpha$, the wave turbulence approximation is applicable for all small r (large k) and is universal.

Throughout the (α, β) plane, the occurrence of breakdown is governed by the line $\beta = 2\alpha$. For $\beta > 2\alpha$, the ratio of time scales becomes of the order of unity at $k \sim k_{\mathrm{NL}}$ and asymptotic closures of the weak turbulence theory fail. Furthermore, when universal, the real space structure function can illustrate the breakdown by measuring strong corrections to Gaussianity. Inside the universal rhombus where all S_N are universal, corrections get proportionally larger at smaller scales:

$$\frac{\mathfrak{S}_N}{(S_2)^{N/2}} \sim (P^{1/2} r^{-\beta+2\alpha})^{(N/2)-1} \tag{2.9}$$

and we define r_{NL} as k_{NL}^{-1} as before.

Outside this rhombus, the general features of breakdown remain. However, the breakdown scale may have a dependence on non-universal effects. As examples, we consider the regions 1,5 and 7 of Fig. 4, where we have

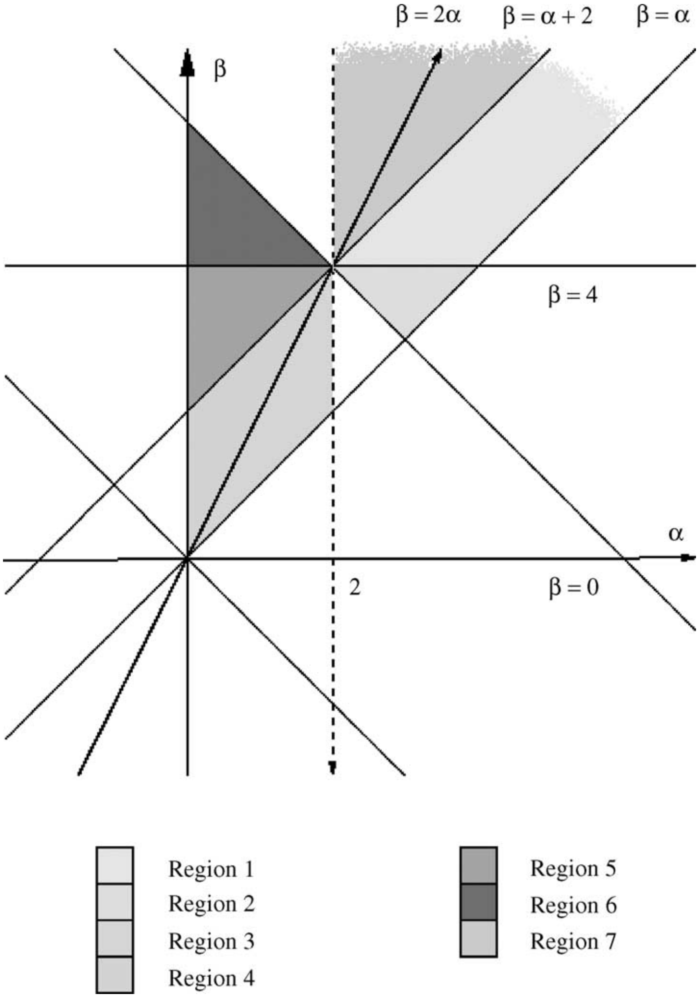

Fig. 4. Sectors for three wave energy case: shows all the regions for which we can take $k_U \to \infty$. The forcing plays a role in all regions except 3 and 4. For example, both S_2 and S_4 are universal in region 2; only S_2 is universal in region 1.

singled out the area in the (α, β) plane corresponding systems which have capacity to absorb only a finite amount of energy at small scales.

In region 1, only S_2 is universal and scales as $P^{1/2}r^{\beta-\alpha}$, while all S_N, $N > 2$ scale as $F^N r^N$. (For convenience, we can assume that there is only one universal dimensional parameter in the system which we take to be P [2].) The normalized corrections to Gaussianity are

$$\frac{\mathfrak{S}_N}{(S_2)^{N/2}} = F^N (P^{-1/2} r^{2-\beta+\alpha})^{N/2}. \tag{2.10}$$

Since $2 + \alpha - \beta > 0$ for the entire region, there is no breakdown at small scales. Similar expressions are obtained for the ratio of $\mathfrak{S}_N/(S_2)^{N/2}$ in regions 2,... for N sufficiently high such that $\mathfrak{S}_N$ feels the forcing.

Now consider region 5 for which $\alpha + 2 < \beta < 4$ and $\alpha < 2$. In this region, $S_2 \sim F^2 r^2$ is non-universal, while $\mathfrak{S}_N \sim P^{(1/2)(N-1)} r^{\beta+(N/2-2)\alpha}$ are universal for all $N > 2$. In this case, the ratio of structure functions gives

$$\frac{\mathfrak{S}_N}{(S_2)^{N/2}} = (P^{-1/2} r^{\beta-2\alpha})(PF^{-2} r^{\alpha-2})^{N/2}. \tag{2.11}$$

Since $\beta > 2\alpha$ in this region, the first bracket suppresses universal breakdown at small scales. In its place, we see a non-universal breakdown for $r < (P^{-1}F^2)^{1/(\alpha-2)}$ because $\alpha < 2$. We call the breakdown *non-universal* because the breakdown scale depends on the non-universal parameters collected in F.

In region 7, all the structure functions are dominated by the forcing at small scales. $\mathfrak{S}_N \sim r^N$ for all N. Weak turbulence is irrelevant for the small scale picture.

Before briefly discussing how similar calculations can be done for four wave interactions, we comment on the usefulness of the strip diagram. On the $\beta = 2\alpha$ line, all normalized corrections to Gaussianity are scale-independent; a property which we call quasi-Gaussianity. The behavior on this line of the S_N is either universal for all N or non-universal for all N. The border between these two regions is the point $\alpha = 2$, $\beta = 4$ which is common to all (top) strip borders.

The position of the universal rhombus and all strips depends on the field v used to form the structure functions. We chose for our calculations that quantity whose second order moment has as its Fourier transform the spectral energy—a natural choice for trying to capture the behavior due to the finite energy flux KZ spectrum. If we had chosen a different, perhaps more easily measurable, variable such as the surface velocities for capillary waves ($\langle u_{\underline{x}+\underline{r}} u_{\underline{x}}\rangle \sim \int k^{5/2} n_k \, e^{i\underline{k}\underline{r}}$), we would have found the stripes shown in Fig. 5(a).

We note that, for α, β inside the strip defined by $2\alpha - 2 < \beta < 2\alpha + 2$ which is centered about the breakdown line, one can always find an appropriate v for which all the S_N on the KZ spectrum are universal. We call this region the *Universal High Way* for the (α, β) plane road map.

We now briefly describe how a very similar analysis can be done for the case of four wave resonances.

The second order structure function for the case of four wave resonances can be defined just as in Eq. (2.1). In this case, the KZ finite energy flux solution is given by $P^{1/3} k^{-(2/3)\gamma - d}$, where we again allow P to absorb the small parameter $\varepsilon^{12} P \to P$. The arguments following (2.1) have a direct analog in the four wave case. We find that $S_2(r)$ scales universally as

$$S_2(r) \sim P^{1/3} r^{(2/3)\gamma-\alpha} \tag{2.12}$$

provided α and γ lie within the strip defined by

$$0 < \tfrac{2}{3}\gamma - \alpha < 2. \tag{2.13}$$

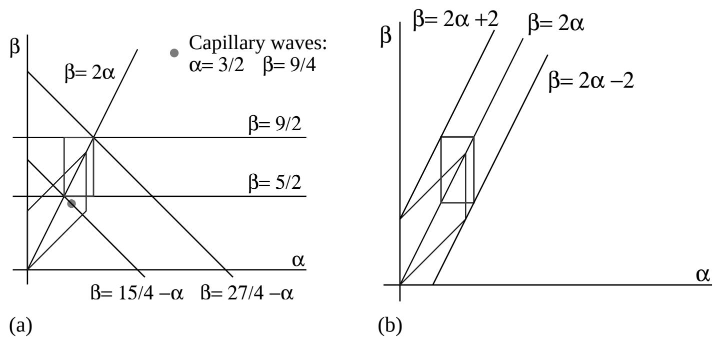

Fig. 5. (a) The strip diagram using surface velocities as the field variable. The red parallelogram is the universal rhombus in this case; the blue parallelogram is the universal rhombus from our previous calculations. (b) The Universal High Way.

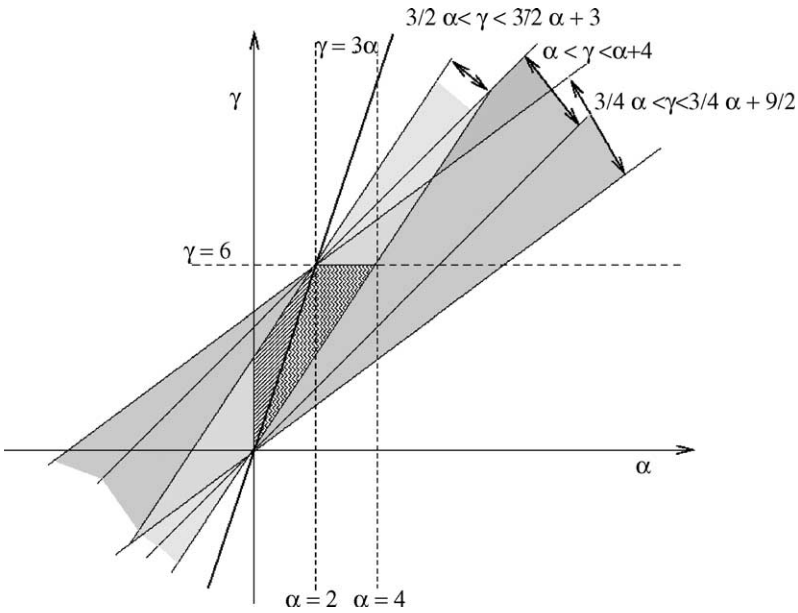

Fig. 6. Strips for the four wave energy spectrum: each strip is the condition for some $\mathfrak{S}_N$ to scale universally. All S_N are universal in the black and white region.

In general, we will find that the term in S_N due to the N order moment has universal scaling:

$$\mathfrak{S}_N(r) \sim P^{(1/3)(N-1)} r^{(1/3)((N/2)+1)\gamma-\alpha} \tag{2.14}$$

provided

$$0 < \frac{1}{3}\left(\frac{N}{2}+1\right)\gamma - \alpha < N. \tag{2.15}$$

Again, the lower bound ensures that the Fourier integral in the definition of $\mathfrak{S}_N$ is independent of the high wave-number cut-off, while the lower bound ensures independence with respect to the lower cut-off.

Fig. 6 shows the first three strips for $N = 2, 4, 6$ and their intersections. All structure functions will scale universally at small scales in $(0 < 1/3(N/2+1)\gamma - \alpha < N)$ (with the added condition that $0 < \alpha$). This region is shown in black and white in Fig. 6 ($(3/2)\alpha + 3 < \gamma < (3/2)\alpha$; $0 < \gamma < 6$ and $0 < \alpha < 4$), and as regions 3 and 4 in Fig. 7.

Within this (truncated) parallelogram, the ratio of structure functions gives the following corrections to Gaussianity:

$$\frac{\mathfrak{S}_N}{(S_2)^{N/2}} = (P^{1/3} r^{\alpha-(1/3)\gamma})^{((N/2)-1)}. \tag{2.16}$$

Breakdown at small scales, $r < r_{4\mathrm{NL}} = P^{-1/(3\alpha-\gamma)}$, occurs for $\gamma > 3\alpha$. The breakdown line, $\gamma = 3\alpha$ bisects the rhombus of universal scaling.

The point $(\alpha = 2, \gamma = 6)$ is the four wave analogue of $(\alpha = 2, \beta = 4)$ from the three wave case. It is the unique point where all structure functions are self similar with respect to k_I and k_{NL}. Along the line $\gamma = 3\alpha$, this point separates the regions where all structure functions are universal, and where they are all dominated by the forcing.

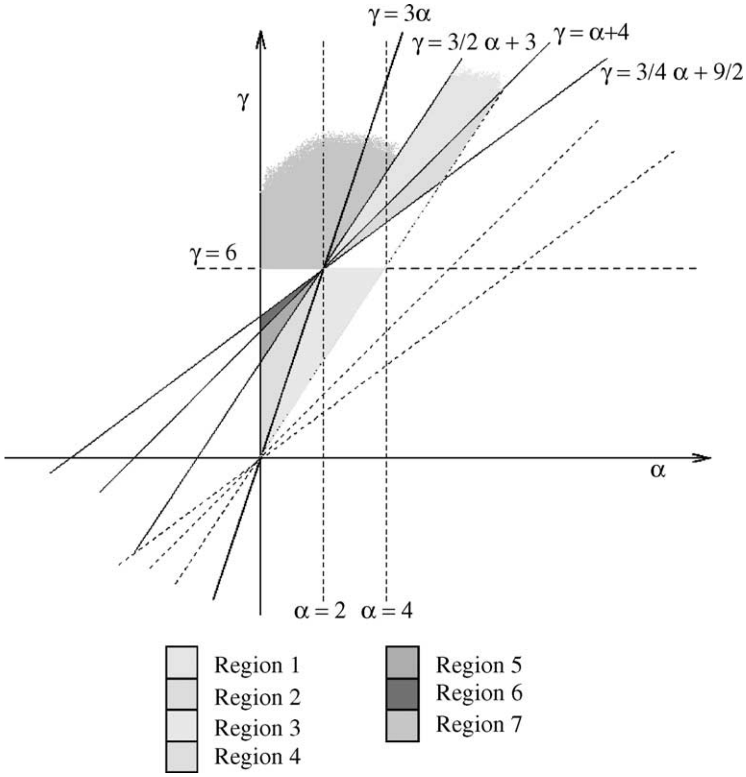

Fig. 7. Sectors for the four wave energy case. The figure shows the sections around (2, 6) for which we can take $k_{\mathrm{U}} \to \infty$. Different regions display different scaling behavior.

Just as in the three wave case, we can consider the sectors surrounding this point for which only some $\mathfrak{S}_N$ are universal. We consider the regions identified in Fig. 7, all of which correspond to finite energy capacity.

Region 1, defined by $\alpha + 4 < \gamma < (3/2)\alpha + 3$, is characterized by $S_2 \sim P^{1/3} r^{(2/3)\gamma - \alpha}$ universal at small scales while $\mathfrak{S}_N \sim F^N r^N$, $N > 2$ non-universal where again, F^N represents a collection of non-universal parameters. For $N > 2$, the ratio of structure functions gives

$$\frac{\mathfrak{S}_N}{(S_2)^{N/2}} = F^N (P^{-1/2} r^{3+(3/2)\alpha-\gamma})^{N/3}. \tag{2.17}$$

Since $\gamma < (3/2)\alpha + 3$, these corrections get smaller at small scales and there is no breakdown. Similar results hold for regions $2, 3, \dots, 4$.

Consider region 5 given by $(3/2)\alpha + 3 < \gamma < \alpha + 4$, $0 < \alpha < 2$, which lies above the breakdown region. The second order structure function scales with the non-universal forcing while the structure functions of order $N > 2$ scale universally at small scales. We have $S_2 \sim F^2 r^2$ non-universal and $\mathfrak{S}_{N>2} \sim P^{(1/3)(N-1)} r^{(1/3)(N/2+1)\gamma-\alpha}$ universal. The ratio of structure functions gives

$$\frac{\mathfrak{S}_N}{(S_2)^{N/2}} = (P^{-1/3} r^{(1/3)\gamma-\alpha})(P^2 F^{-6} r^{\gamma-6})^{N/6}. \tag{2.18}$$

Again, universal breakdown in the first bracket is suppressed. A non-universal breakdown occurs for $r < (P^2 F^{-6})^{-1/(\gamma-6)}$ since $\gamma < 6$.

In region 7, the small scale behavior of all structure functions is totally determined by the forcing. $\mathfrak{S}_N \sim r^N$ for all N.

We have so far examined how the structure functions behave on the finite flux energy spectra of weak turbulence where we have taken structure functions to mean moments of Δv_r, where the power spectrum of v is the spectral energy. If we had chosen v in a different way (still evaluating it on the KZ energy spectrum), the universal rhombus and all the strips would be shifted in the (α, γ) plane. We note that it is always possible to find a v such that the universal rhombus includes the point (α, β) provided this point lies in the universal high way defined by $3\alpha - 6 < \gamma < 3\alpha + 6$.

3. Non-universal corrections to the breakdown criteria

Zakharov [9] has pointed out to us that in some situations, the coupling strengths of long-wave–short-wave interactions may not be small enough to guarantee convergence of all the integrals which arise. A particular example is the four wave interaction of gravity waves. In this case, $T_2 = 0$ and

$$\Omega_{2k} = \int G_{\underline{k}\underline{k}_1,\underline{k}\underline{k}_1} n_{k_1}\,\mathrm{d}\underline{k}_1, \tag{3.1}$$

$$\begin{aligned}\Omega_{4k} = \int_\Delta |G_{\underline{k}\underline{k}_1,\underline{k}_2\underline{k}_3}|^2 n_{k_1} n_{k_2} n_{k_3} \\ \times \left(\frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}}\right)\left(-\mathrm{i}\pi\delta(\omega + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) + \mathrm{PV}\frac{1}{\omega + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}}\right) \\ \times \delta(\underline{k} + \underline{k}_1 - \underline{k}_2 - \underline{k}_3)\,\mathrm{d}\underline{k}_1\,\mathrm{d}\underline{k}_2\,\mathrm{d}\underline{k}_3,\end{aligned} \tag{3.2}$$

$$\begin{aligned}T_4[n_k] = \int_\Delta |G_{\underline{k}\underline{k}_1,\underline{k}_2\underline{k}_3}|^2 n_k n_{k_1} n_{k_2} n_{k_3} \left(\frac{1}{n_k} + \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}}\right) \\ \times \delta(\omega + \omega_1 - \omega_2 - \omega_3)\delta(\underline{k} + \underline{k}_1 - \underline{k}_2 - \underline{k}_3)\,\mathrm{d}\underline{k}_1\,\mathrm{d}\underline{k}_2\,\mathrm{d}\underline{k}_3.\end{aligned} \tag{3.3}$$

Note that T_4 has the form $F[n_{k_1}, n_{k_2}, n_{k_3}] - n_k \,\mathrm{Im}\,\Omega_{4k}$ so that $(\mathrm{Im}\,\Omega_{4k})^{-1}$ is a relaxation time scale for the redistribution of energy. In equations (3.1)–(3.3), the coefficient $G_{\underline{k}\underline{k}_1,\underline{k}_2\underline{k}_3}$ is $L^{++--}_{\underline{k}\underline{k}_1,\underline{k}_2\underline{k}_3}$ from which quadratic products of $L^{+s_1s_2}_{\underline{k}\underline{k}_1\underline{k}_2}$ are subtracted. The integration region Δ in Eqs. (3.2) and (3.3), after angle averaging, is that region of the $\omega_{k_1}, \omega_{k_2}$ plane defined by $\omega_{k_2} > 0$, $\omega_{k_3} > 0$ and $\omega_{k_1} = \omega_{k_2} + \omega_{k_3} - \omega_k > 0$. The neighborhood of $\omega_{k_1} = \omega_{k_2} = \omega_{k_3} = \omega_k$ is that of ultra-local interactions while the neighborhood of $\omega_{k_1} = \omega_{k_3} = 0$, $\omega_{k_2} = \omega_k$ or $\omega_{k_1} = \omega_{k_2} = 0$, $\omega_{k_3} = \omega_k$ correspond to long-wave–short-wave interactions for which the group velocity of the short wave at k in the direction of $\underline{k}_1 - \underline{k}_3$ is equal to the phase velocity of the long-wave second harmonic $\exp(\mathrm{i}(\underline{k}_1 - \underline{k}_3)\underline{x} - \mathrm{i}(\omega_{\underline{k}_1} - \omega_{\underline{k}_3})t)$.

For gravity waves, $\omega_k \sim (gk)^{1/2}$, $\gamma = 3$ and $d = 2$ so that Eq. (1.10) becomes $n_k = C_2 P^{1/3} k^{-4}$. The coefficient $G_{\underline{k}\underline{k}_1\underline{k}_2\underline{k}_3}$ behaves as kk_1^2 for k_2 close to k and $k_1, k_3 \ll k, k_2$. A little calculation shows that Ω_{2k} diverges logarithmically near $k_1 = 0$ and is given by $2\pi C_2 P^{1/3} k \ln(k_\mathrm{I}/k)$, where k_I is some infrared cut-off at the forcing scale. The frequency correction $\mathrm{Im}\,\Omega_{4k}$ (due to the term containing $n_{k_1} n_{k_3}$) is proportional to $(P^{2/3}k^{3/2})/g^{1/2}(k_\mathrm{I}/k)^{-1/2}$. The ratios Ω_{2k}/ω_k and $\mathrm{Im}\,\Omega_{4k}/\omega_k$ are $(P^{1/3}k^{1/2}/g^{1/2})\ \ln(k_\mathrm{I}/k)$ and $(P^{2/3}k/g)(k_\mathrm{I}/k)^{-1/2}$ in contrast to $(P^{1/3}k^{1/2}/g^{1/2})$ and $P^{2/3}k/g$ if there were no divergences. As a consequence of the extra factor, the revised breakdown wave number, k_LS, is still large but less than $k_\mathrm{NL} = gP^{-2/3}$ calculated before. The collision integral, T_4, on the other hand, has no such divergence because of the cancellation between the terms $n_{k_1} n_{k_2} n_{k_3}$ and $-n_k n_{k_1} n_{k_3}$ in the integrand.

The open questions are: what is the consequence of the fact that $\mathrm{Im}\,\Omega_{4k}/\omega_k$ becomes of order unity at $k_\mathrm{LS} = g^{2/3} k_\mathrm{I}^{1/3} P^{-4/9}$ rather than at $k_\mathrm{NL} = gP^{-2/3}$, the point at which the ratio $T_4/\omega_k n_k$ becomes unity? Is the ratio T_8/T_4 of order $P^{2/3}k/g$ or $(P^{2/3}k/g)(k/k_\mathrm{I})^{1/2}$ reflecting the ratio Ω_{4k}/ω_k? Can there be worse divergences?

To attempt to answer these questions, we have carried out calculations in order to determine the long time behaviors of the next corrections $T_6[n_k]$ and $T_8[n_k]$ in the collision integral asymptotic expansion, Eq. (1.4). In $T_8[n_k]$, there arise two types of secular terms. The first involve interactions which are genuine compositions of four wave interactions (e.g. $\underline{k}, \underline{k}_1 \to \underline{k}_4, \underline{k}_5 \to \underline{k}_6, \underline{k}_7 \to \underline{k}_2, \underline{k}_3$ which comes from the one loop correction to the fourth order cumulant) that lead to non-trivial energy exchange. The ratio of these terms in T_8/T_4 is $P^{2/3}k/g$. The second involve modal interactions which are simply the next terms in the expansion of the Dirac delta function $\delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})$ in T_4 with the frequency ω_k replaced by its renormalized value; namely

$$\omega_k \to \omega_k + \Omega_{2k} + \Omega_{4k} + \cdots \tag{3.4}$$

with Ω_{2k}, Ω_{4k} given by Eqs. (3.1) and (3.2), respectively.

Accordingly, T_6 and T_8 contain (respectively) the combinations:

$$(\Omega_{2k} + \Omega_{2k_1} - \Omega_{2k_2} - \Omega_{2k_3})\delta'(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})$$

and

$$(\Omega_{4k} + \Omega_{4k_1} - \Omega_{4k_2} - \Omega_{4k_3})\delta'(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}),$$
$$\tfrac{1}{2}(\Omega_{2k} + \Omega_{2k_1} - \Omega_{2k_2} - \Omega_{2k_3})^2\delta''(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \tag{3.5}$$

in their integrands in place of the $\delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})$ appearing in T_4. There are two observations we can now make. The first is that the ratios T_6/T_4 and T_8/T_4 will be at least of orders $(P^{1/3}k^{1/2}/g^{1/2})\ \ln(k/k_{\mathrm{I}})$ and $(P^{2/3}k/g)(k/k_{\mathrm{I}})^{1/2}$, respectively, as the combinations $\Omega_{jk} + \Omega_{jk_1} - \Omega_{jk_2} - \Omega_{jk_3}$, $j = 1, 2$ will not in general vanish on the resonant manifold. On the other hand, it will vanish on that part of the manifold where k_1, k_3 are small and k, k_2 are large. Therefore, there will be no worse divergences which might otherwise arise because the derivative delta function will act as $\delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})(\partial/\partial k_3)$ (one can show that $\delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \simeq \delta(\omega_{k_1} - \omega_{k_3} + \mathrm{O}(\underline{k}_1 - \underline{k}_3)) \simeq \delta(g^{1/2}(k_1 - k_3)/(k_1^{1/2} + k_3^{1/2})))$. The potential divergence resulting from the term $n_1 n_3(\partial n_2/\partial k_2)(\partial k_2/\partial k_3)$ is killed by the vanishing of $\Omega_{jk} + \Omega_{jk_1} - \Omega_{jk_2} - \Omega_{jk_3}$ near $k_1, k_3 = 0$.

Physically, the origin of these non-local effects is the interaction between the given wave vectors $\underline{k}$ and a nearby wave vector $\underline{k}_2 = \underline{k} + \underline{k}_1 - \underline{k}_3$ and a family of long waves which are the first superharmonic $\exp(\mathrm{i}(\underline{k}_1 - \underline{k}_3)\underline{x} - \mathrm{i}(\omega_{k_1} - \omega_{k_3})t)$ of the long waves $\underline{k}_1, \underline{k}_3$. The resonance criterion is that $\omega_{k_1} - \omega_{k_3} = \omega(\underline{k} + \underline{k}_1 - \underline{k}_3) - \omega(\underline{k}) \simeq |\underline{k}_1 - \underline{k}_3|\omega_k' \cos\theta$, where ω_k' is the group velocity of the short-wave packet and θ is the angle between $\underline{k}$ and the long-wave superharmonic wave vector $\underline{k}_1 - \underline{k}_3$. For short enough waves $k > k_{\mathrm{LS}}(> k_{\mathrm{NL}})$, the short-wave packets remain attached (phase locked) to the long wave for a sufficiently long time for energy exchange to take place. The resonance does not have to be exact. All wave vectors $\underline{k}, \underline{k}_1, \underline{k}_2 = \underline{k} + \underline{k}_1 - \underline{k}_2, \underline{k}_3$ in a finite window of width Im Ω_k around the resonant locus will participate.

Where are such effects manifested in the structure functions? We conjecture, but have not yet completely proved, that they arise when we calculate the relative magnitudes of successive surviving terms in each of the cumulants. For example, in $\mathfrak{S}_4$, the ratio of the next surviving term to the one we have calculated in (2.14) for $N = 4$ will exhibit the same divergence that appears in Im Ω_{4k}/ω_k except that it will be manifested in physical space and lead to a revision of the breakdown scale from $r_{\mathrm{NL}} = k_{\mathrm{NL}}^{-1}$ to $r_{\mathrm{LS}} = k_{\mathrm{LS}}^{-1}$. It is important to emphasize, therefore, that the results of Section 2 only obtain when coupling coefficients are sufficiently local.

Our conclusion is then that the wave-number range of validity of wave turbulence may be further shortened by the effects of non-local, long-wave–short-wave interactions. How should we interpret this behavior? The imaginary part of Ω_{4k} (and higher order corrections) can be interpreted as a broadening of the resonant manifold. Quartets of wave vectors $\underline{k}$, $\underline{k}_1\underline{k}_2\underline{k}_3$ in an (Im Ω_{4k}) neighborhood of the resonant manifold (shifted from the one based on the linear dispersion relation because of real frequency corrections) play a role in the energy exchange to and from wave-numbers k greater than $k_{\rm LS}$ (where $(P^{2/3}k/g)(k/k_{\rm I})^{1/2}$ is unity) given by $(gP^{-2/3})^{2/3}k_{\rm I}^{1/3}$ or $k_{\rm NL}^{2/3}k_{\rm I}^{1/3}$ for gravity water waves. For the MMT model [5], there is a stronger long-wave–short-wave coupling as $|G_{\underline{k}\underline{k}_1\underline{k}_2\underline{k}_3}|^2 \sim k^3 k_1^{3/2} k_3^{3/2}$. This leads to a $k_{\rm LS}$ of $k_{\rm NL}^{2/5}k_{\rm I}^{3/5}$ and a much shorter wave-number range for which the wave turbulence approximation is valid.

The evidence from experiments, numerical and physical, suggest that the KZ spectrum $C_2 P^{1/3} k^{-4}$ remains valid all the way either to the surface tension scale, $K_0 = \sqrt{g/\sigma}$, where $\sigma = S/\rho_{\rm w}$, S is surface tension and $\rho_{\rm w}$ is the density of water, or to $k_{\rm NL} = gP^{-2/3}$, whichever is less [7,4]. $k_{\rm NL}$, as we have noted in [1], is precisely the point that the KZ spectrum intersects the Phillips spectrum, $n_k = C_3 g^{1/2} k^{-9/2}$, which corresponds to derivative discontinuities in the ocean surface, nearly whitecaps. The criterion for whitecaps is thus $k_{\rm NL} < K_0$ which translates into $P > P_{\rm c} = (\sigma g)^{3/4}$ or a wind speed of $V = (\rho_{\rm w}/\rho_{\rm a})^{1/2} P^{1/3}$ ($\rho_{\rm a}$ is the density of air) greater than 6 m/s [7]. (One gets exactly the same criterion if one calculates $k_{\rm NL}$ for surface tension waves which exchange energy by three wave resonances and compares $k_{\rm NL}$ to K_0.) On the other hand, there is no evidence for the KZ spectrum in the MMT experiments.

Similar calculations apply to three wave interactions. However, in most situations (and definitely if the system contains only one additional dimensional parameter, e.g. surface tension waves [2]) $\beta < 2\alpha$ and so breakdown occurs at low values of k. In either case, the long wave–short wave, non-local interaction which might cause the multiplying integrals to diverge involve a small k and large k_1, k_2 close to one another. The resonant condition becomes $\underline{k} \cdot \nabla_{\underline{k}_2}\omega = \omega_{\underline{k}}$. However, for large k_2, the integrals would appear, in almost all cases, to converge.

To date we have left out the question of how the cut-off wave-number $k_{\rm I}$ is chosen. For wind generated gravity waves, one might choose $k_{\rm I}$ to be the wave-number at which energy is injected. This is imprecise as energy is transferred over a large range of scales. Moreover, from four wave resonances with the collision integral (3.3), there is a secondary inverse cascade of wave-action (particles) from short waves to long waves. If we choose $k_{\rm I}$ to be the front $k_*(t)$ of the long-wave spectrum, namely the point to which wave-action has been transferred after time t by wave–wave interactions, we also have to take into account that the particle flux spectrum is less steep and therefore the long-wave–short-wave wave-number $k_{\rm LS}$ is closer to $k_{\rm NL}$. For water waves, the ratio Im Ω_{4k}/ω_k is $(g^{1/3}Q^{2/3}k^{3/2}/g)k_{\rm I}^{-1/6} \equiv (P^{2/3}k/g)(k/\bar{k})^{1/3}(k/k_{\rm I})^{1/6}$, where we have written the particle flux Q as $P/\bar{\omega}$ and $\bar{\omega} = \sqrt{g\bar{k}}$, where $\bar{k}$ is some injection wave-number which is not necessarily small. We now see that $k_{\rm LS} = (gP^{-2/3})^{2/3}\bar{k}^{2/9}k_{\rm I}^{1/9}$. It is not unreasonable to take $\bar{k} \sim k_{\rm NL} \ll k_{\rm I}$. If indeed we take $\bar{k}$ to be of the same order as (but numerically less than) $k_{\rm NL}$, then $k_{\rm LS} = k_{\rm NL}(k_{\rm I}/k_{\rm NL})^{1/9}$. The dependence on the cut-off is very weak. This may well account for the fact that the KZ spectrum is observed over such a large range.

Definite conclusions concerning the validity of the wave turbulence approximations are further inhibited by the manner in which the KZ spectrum is realized. In finite capacity situations for which $\int \omega_k n_k \, {\rm d}\underline{k} \sim \int k^{\alpha-2\gamma/3-1}\,{\rm d}k < \infty$ (for water waves and MMT $\alpha = 1/2, \gamma = 3$), a companion paper in this volume [3] points out that the stationary spectrum is achieved in a non-trivial manner. For a finitely supported initial power spectrum, the energy spreads to higher wave-numbers at an accelerating rate behind a moving front $k_* = (t_* - t)^b$, where $b = -(2(x - x_0) - (2\gamma - 3\alpha)/3)$, where $x_0 = 2\gamma/3 + {\rm d}$. In the wake of this front lies a steeper than KZ spectrum $n_k \sim k^{-x}$, where $x = 2\gamma/3 + d + (2\gamma - 3\alpha)/12$ which, for water waves and MMT is 4(3/8). Only when $t > t_*$, after the spectrum reaches $k = k_{\rm d}$, the small scale at which dissipation occurs, can a finite flux P of energy begin to set up the KZ spectrum. It does so as the wake of a backward traveling front which invades the region of steeper slope. One might ask how it manages to achieve the KZ spectrum if either $k_{\rm NL}$ or $k_{\rm LS}$ is smaller than $k_{\rm d}$ and the backward moving front starts out in the fully nonlinear region.

We know no answer for this at present. While the results presented here have added some new insights to the delicate questions concerning the applicability of wave turbulence, it is clear that the subject is very much alive and that there are still many open and intriguing challenges.

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