Weak-Wave Turbulence: A Tragic Super-hero of Turbulence Theory

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Picture the smoke rising from a cigarette. Running your eyes up this plume, you will first see smooth, combed ribbons of smoke all rising in parallel. Then, spontaneously, the ribbons will separate, the smoky strands will begin to curl and buckle in on themselves, and finally, the smoke will no longer be in a stream at all, but will be more like a disperse cloud. This image, although beautiful and captivating, is the source of much frustration and passionate study for many mathematicians and physicists. Running your eye upwards with this rising shaft of smoke, you are watching a fluid (in this case the smoke), undergo a transition from smooth, "laminar" flow to turbulence–and turbulence is one of the most difficult, unsolved problems in mathematical physics today.

The problem is that the ways in which the smoke can twist and curl are so numerous and varied and so sensitive to subtleties in air pressure, density and speed that its behavior is impossible to predict. This sensitivity to external air conditions is what makes turbulence chaotic. On the other hand, there is a sense in which these external circumstances are irrelevant. Of course, a great gust of wind would destroy our smoke experiment, but for the most part the evolution of this stream will proceed from the silky smooth to the rough and turbulent whatever the air around it is doing–the turbulence is self-inflicted.

It is this latter property of turbulence–its *universal* nature and independence from any particular realization–that makes it possible to study. It is also what makes that study so difficult. Although it's not necessary to know in detail how the fluid interacts with its surroundings, any description of the fluid must include whatever complicated processes are responsible for turning wafting plumes into churning clouds. The same equations that describe laminar flow must also characterize the turbulent state as well as the transition between these regimes.

Models of certain types of turbulence can explain how the energy of a turbulent fluid is mixed up and shared between the different waves and whirlpools that create all that twisting and buckling. For instance, these models explain how the energy injected into water by a ship's wake is eventually changed into heat by molecular viscosity. What the model doesn't explain, however, is just how much energy the water will absorb. The drag on the ship depends on its relative speed through the water in a way which is not fully understood [**F**]. These simple models capture

This essay has been published in *Highlights of Mathematical Physics*, Eds. A. Fokas et al. AMS, 2002. The author was supported by an EPSRC PhD studentship.

the universal nature of turbulence well, but miss out on other essential components associated with the sensitivity and randomness in fluid flow.

One particular mathematical framework has tremendous relevance to turbulence; in fact, it can be viewed as a model for a new regime of turbulence. The model, called weak-wave turbulence (WWT), although limited in its scope of applicability, is consistent with an incredibly rich and varied range of fluid behavior. Furthermore, these behaviors are found in a wide range of physical contexts. WWT describes the universal aspects of turbulence, in addition, descriptions of the other, non-universal affects can be glued into the WWT model without destroying its validity. It is this last property of the WWT picture which makes it such a powerful tool for understanding full turbulence.

A total understanding and ability to predict turbulent behavior is still out of reach. What is clear, however, is that the exploration of mathematical and physical phenomena involved in the struggle for that understanding is one of the most exciting activities in mathematical physics today.

What is Turbulence?

The term *turbulence* is best defined as a mathematical phenomenon–a particular behavior which may be found in many physical systems. The smoke rising from a burning cigarette is an example of this; so is the water running over a bumpy river bed, or the surface of the ocean after a storm. Turbulence connotes a chaotic motion of particles and random fluctuations in density, speed, or pressure which change just as randomly in time.

The study of turbulence is focused around one central example– Hydrodynamic turbulence (HDT)–the Big Daddy [**NNB**] of turbulent systems. HDT refers to the turbulence in water with its rough surface, white-caps, whirlpools and tubular surfing waves. In some sense, turbulence in all other systems has been studied only as a side show. It is through an understanding of these "side shows", these less central forms of turbulence, however, that some of the most important insights into HDT will come.

The goal of turbulence research may, at first glance, seem ridiculously limited. Researchers do not even bother to ask questions like "If I know what the surface of the ocean looks like now, what will it look like in one hour's time?" or even "Where will the next ten foot wave occur in the ocean?". These questions are too difficult and too dependent on the external conditions in the same way that the precise point at which the smoke from the cigarette began to buckle is too sensitive on the conditions of the surrounding air. Instead, mathematicians and physicists content themselves with questions like "How many ten foot waves will there be in the ocean in one hour?" and "How much of the ocean's energy will be contained in these waves?". In other words, researchers ask questions about the *statistical* properties of turbulence.[1]

[1]Engineers, of course, may not be satisfied with these limited questions. For the designer of a ventilation system, it may be very important to know if a strong flurry of air will occur inside the ducts near the driving fan, say, or further along the ducts toward the room vents. These sorts of questions are not addressed by most turbulence research although certainly make up a substantial part of the more general fluid dynamics research. Importantly, the approach to these types of questions tends to assume that the flow in the ducts is *not* turbulent, but rather assumes that these strong wind patterns are aberrations to a more smooth and laminar background flow. Another approach involves intense computational simulations which is not possible for most problems.

Viewing turbulence in a statistical framework helps greatly to give turbulence research a well-defined mathematical formulation, but it does not make the answers to questions about turbulence any easier to find. In a sense, it just makes the questions easier to ask. Rather than having to ask "For this particular cigarette burning this type of tobacco with some specific and detailed air conditions, when will the break into turbulence begin?" researchers can ask "For the general type of turbulence arising from the free flow of smoke upwards, what are the relevant averaged properties of the flow that determine the onset of turbulence?". Looking at the statistics of the turbulence taps into its universal nature.

The problem remains, however, of how to determine the statistical properties of turbulence and in particular, how to determine their evolutions in time. The question "If I know what the ocean looks like now, what will it look like in an hour?" has become "If I know some statistical properties of the ocean now, what will these statistical properties be in one hour?". The second question makes more sense than the first, but may not be any easier to answer.

Why is Turbulence Difficult?

Statistics is the language of probabilities. Anything can be described in this language; for instance, a book at rest on a table. The probability that the book will be in the same spot on the desk after one hour is one– i.e. 100% certain. This language was developed, however, to deal with more complicated situations where the probabilities are not so obvious. To continue the example, consider a (hypothetical) book on a table who's position is not well defined or is rapidly changing. The speed of water at a particular point in the ocean, say, can be rapidly changing *and* ill-defined (turbulence theory takes into account many possible external conditions or realizations of the ocean dynamics), so statistics is a natural language in which to talk about turbulence.

Amazingly, all the statistical information about a system is contained in one curve called the probability distribution function (pdf). This curve is a graph from which one can read out probabilities. For example, to find the probability of measuring the speed of water at a particular point in the ocean to be five meters per second, read up from the "5" mark along one axis to the height of the curve–this is the desired probability. In some cases, this curve can be described exactly. In other cases, including the turbulence case, this curve must be described with a series of mathematical quantities, called moments, each giving more and more detail about the curve's shape. So, for instance, the first moment might say "the curve has one large maximum around the point zero". The next moment could say "the width of this bulge is three" and so on with moments describing the flatness and convexity etc. It's much like describing a best friend by first saying "my friend is female" and then "she is 23 years old" and so on describing her hair color, height, occupation etc. The unfortunate aspect of the pdf is that, just like with best friends, no matter how many descriptive properties are given, it's never enough. An infinite number of moments are needed to define the pdf at a given time [2]. This in itself is not an insurmountable problem. Even the first few moments would describe the pdf well enough for any theory; but the theory doesn't end here.

The goal of any statistical approach is to find a description of the pdf (however crude) which can be evolved to give an equally good description at later times. For

[2]Even this infinite list may not be enough to describe the pdf uniquely.

turbulence, the goal is to be able to find this list of moments describing the pdf for all times. That is, to find the evolution of these moments. Unfortunately, in the case of turbulence, the evolution of each moment depends on the others. In order to know how the first moment evolves it's necessary to know about the second and the third moments, say. The physical reason for this has to do with the fact that turbulence requires interaction between the different waves and whirlpools. To continue the analogy with a best friend: in order to know what color hair your friend will have in ten years, it's necessary to know her occupation; and in order to know how her occupation will change over this time, it's necessary to know if she owns any pets and so on. An infinite hierarchy of equations, each depending on the next, emerges. Solving this hierarchy is like trying to build a pyramid from the top down–there is no safe starting point. Mathematicians call this the *closure* problem because the door to the number of equations you need to solve can never be closed.

Although the closure problem is an inextricable property of HDT, progress in turbulence research has been made. In 1941, A.N. Kolmogorov put forward a few simple and salient assumptions about HDT which resulted in the first quantitative results in turbulence theory. These elegant claims speak to the universal nature of turbulence on the basis of physical insight. This rather axiomatic approach circumvented the need for all the moment-hierarchy formalism; although Kolmogorov's ideas can be expressed in this statistical language.

Another leap forward in the understanding of turbulence came in the 1960's when V.E. Zakharov discovered that a particular model of wave systems could match the behavior that Kolmogorov predicted for turbulence. This model is now known as WWT since in this context, it describes a limited regime of turbulence– a sort of approximation to HDT. What makes WWT so useful is that its mathematical description allows for a resolution to the closure problem. WWT constitutes a regime of turbulence whose statistical properties can be explicitly calculated.

Kolmogorov's Model

Think again about the turbulent ocean with all its undulations and twisting currents. The shape and structure of the ocean is so complicated and evolves so mysteriously that it may seem difficult to identify any particular shapes at all. It is one of the wonders of mathematical science that a system as complicated as the turbulent ocean can be described by a few simple observations. The best model of HDT to date is incredibly primitive and yet it captures the essence of the universal aspect of turbulence– the way in which, despite all its unique and detailed structure, one turbulent ocean looks much like any other.

The first component of the model for HDT was put forward by Richardson in 1922 [**R**]. He suggested that of all the shapes and structures in the ocean, whirlpools are the ones responsible for turbulence. Furthermore, each whirlpool has a size determined by its diameter. He suggested that in turbulence, large whirlpools break up to form smaller ones. Think of the oar of a row-boat cutting through the water's surface. One whirlpool rolls off each edge of the oar; these in turn spin off into smaller (less noticeable) whirlpools and so on.

The next pertinent observations came from Kolmogorov in 1941. These observations can be summarized with the following consistent assumptions. [3] The first says that size does not matter. The process of large whirlpools breaking up into smaller, daughter whirlpools is exactly the same as these daughter whirlpools breaking up into the granddaughters. The generations of this process are indistinguishable. The consequences of this assumption are profound. Take a photo of the ocean surface. The picture shows waves upon waves and whirlpools among whirlpools, but it is impossible to determine their size. It could be a photo of a spot one inch long on each side, or one mile long. (Of course, if you manage to get a camera small enough to photograph individual molecules, or large enough to pick out the curvature of the earth then the assumption will fail. It is amazing still, that this idea remains true for such a large range of sizes.) Mathematically, this property is called *self-similarity* and the shape of the ocean surface is called *fractal.*

The second assumption is not so visually obvious and requires more physical intuition. Kolmogorov proposed that the statistical, universal dynamics of the turbulent ocean is determined by only a few quantities. That is, at least for the range of whirlpool sizes where self-similarity applies, it's not necessary to know about all the breaking up and twisting of each whirlpool. The statistics will be completely determined by only two values: the average rate of energy dissipation (loss) and the average size of the whirlpools.

A further assumption is that the statistical properties of turbulent flow will eventually stop changing. After some initial adjustments, these mysterious and random fluctuations in the ocean will conspire to have statistical properties which remain constant. At first glance, this assumption seems extremely bold since there is no reason to assume that the waves and fluctuations in the ocean can conspire to anything. It is this assumption, however, that seeks to explain why one turbulent ocean is much like another– although they may be doing different things at every point, they are statistically the same.

Putting all of these ideas together, Kolmogorov was able to derive a mathematical equation for the second moment describing the pdf for turbulence. The solution to this equation says that energy in the ocean, which is contained in the whirlpools, is passed from large whirlpools to smaller ones. Energy is carried down through the generations of whirlpool break-up and is finally released when the whirlpools get too small. Hence, the energy from a ship's wake is carried away by eddies in the water which successively break into even smaller eddies, carrying the energy to smaller structures all the while.

Despite the simplicity of Kolmogorov's model, it does give an accurate qualitative description of turbulence. The *cascade* of energy down to small structures is observed in real turbulent flows. What is not accurately described by this model is the precise manner in which this cascade occurs. In particular, real turbulent flows are not perfectly self-similar. There are real turbulent processes, not captured in Kolmogorov's model, which spoil self-similarity and the precise cascade prediction.

This discrepancy between real turbulence and the Kolmogorov model is known as *intermittency.* The events in real turbulence which create deviations from Kolmogorov theory are thought to occur "intermittently", or sporadically in the fluid. They are events which depend on individual circumstances, they may be

[3] The assumptions listed here are not exactly those sited by Kolmogorov but are consistent with his ideas. The list presented here is similar to those described by Frisch (1995).

non-universal, and therefore cannot be predicted by the averaged quantities used Kolmogorov's theory.

The unfortunate short-coming of Kolmogorov theory is that it does not identify the physical mechanisms for the energy cascade. Richardson's picture of whirlpool break-up is consistent with Kolmogorov theory, but Kolmogorov's ideas do not specifically demand it. If Kolmogorov theory cannot explain how the flow transports energy, it surely cannot point to how the flow may be intermittent. The question of how to resolve the intermittency problem in the context of Kolmogorov theory is ongoing.

Weak-Wave Turbulence

Imagine two pebbles dropped into a pond. Each pebble creates circular ripples which grow outwards from their center. The ripples from one pebble pass through the ripples from the other almost entirely unnoticed and unaffected. The waves do not interact. If this pond scenario is the simplest type of wave system, and HDT is the most complicated, then the picture of weak-wave turbulence (WWT) is somewhere in between. In fact, WWT bridges the gap between these two regimes– it can be thought of as a modification of either extreme. One image for WWT is that of the turbulent ocean without all the white-caps and whirlpools and tubular surfing waves so that just a rough surface is left. Another image is of the surface of the pond after many pebbles of all different sizes have been dropped. Allowing these ripples to interact slightly (rather than just pass through one another) describes the dynamics in WWT. The study of WWT begs the question, "How well can HDT be approximated by a system not much more complicated than the pebble picture?"

WWT is not a model of full HDT; but this lack of direct applicability to full turbulence is compensated for by mathematical tractability and a clearer understanding of the physical processes involved. Instead, WWT models a different regime of turbulence which, surprisingly, has many characteristics in common with full turbulence at least on a universal level. Furthermore, because it can be solved mathematically, this model can be used to understand and predict properties of HDT that Kolmogorov's picture cannot address. WWT shares mathematical tractability and resolution of the closure problem with the pebble picture while maintaining the essential features of turbulence.

There are two broad classes of wave and whirlpool interactions in turbulence: The first, called *linear* interaction, tries to make the ocean look more and more random– a situation with a trivial statistical description. The second type, called *nonlinear* interaction, creates structure in turbulence and leads to non-trivial statistics. Roughly speaking, the ripples in the pond only interact linearly, while HDT is a mixture of linear and nonlinear interactions. The defining feature of WWT is that both linear and nonlinear processes occur, but the linear processes happen much more quickly than the nonlinear ones. These processes must fight for control over the statistical evolution of the system. In HDT, the nonlinearity always dominates. In WWT, however, although the nonlinear processes will eventually control the statistics, their effect takes longer to appear.

Nonlinear interactions between turbulent structures correspond directly to the interdependence between moments in the moment hierarchy. Hence, slow nonlinear interaction between these structures means that, at least for some initial (but substantial) length of time, the moment equations are independent. The equation which gives the evolution of the second order moment of the WWT pdf is independent of all other moments. This equation is called the *kinetic equation.* Although the second order moment on its own is a rather crude description of the full pdf, it is enough to give a great deal of insight into the physical processes in WWT. In particular, it is the second order moment that describes how the energy is arranged among the waves and whirlpools at a particular time; and it is the kinetic equation that tells how this energy is exchanged between them over time.

The power of the WWT model comes from the fact that the kinetic equation has solutions of the type found by Kolmogorov for HDT. We started with the understanding that WWT was a very different sort of turbulence from HDT, but calculations paint a picture of WWT which is, in many ways, similar to HDT. The WWT kinetic equation has solutions which describe a cascade of energy from large scale structures (waves) to small scale structures. What's more, the form of the kinetic equation prescribes the physical mechanism for this transfer of energy. Waves on the surface of the ocean run into each other and scatter and interact. If the sizes of the colliding waves happen to satisfy a particular condition determined by the kinetic equation then, as well as scatter, the waves will exchange energy. These energy exchanges produce an over all effect within the fluid– energy is transferred, on average, from larger waves to smaller ones. This situation is completely analogous to the Richardson picture of large eddies breaking up and passing energy on to smaller eddies. The difference with WWT is that the mechanism for this transfer is not just consistent with the Kolmogorov cascade of energy, it is explicitly identified in the kinetic equation.

From our initial understanding, WWT applies only to a restricted class of problems (ones where the nonlinear processes act over long times in comparison to linear ones– something more like the pebble picture); but WWT has surpassed its limitations to become a champion of turbulence theory. It has championed, too, over technical obstacles in defeating the closure problem. WWT combines an exactly calculable mathematical model with the richness of Kolmogorov turbulence.

WWT is not the end of the line for turbulence research, however. The question of how to incorporate intermittency effects remains. The next section will show that the question of intermittency in WWT is related to the question of how the cascades of energy affect the validity of the WWT assumptions.

Weak-Wave Turbulence: a Tragic Super-hero

As a model of HDT, WWT seems nearly perfect. It correctly predicts the qualitative behavior of the universal aspects of HDT derived by Kolmogorov while giving these ideas a rigorous mathematical framework. Like every hero, however, this hero of turbulence theory is not without its Achilles heel. Inherent in the WWT model is a susceptibility to failure which is also its key to success. Furthermore, the source of this danger is contained within the theory itself [**NNB**].

Ideally, the resonant scattering interactions of waves prescribed by the WWT theory would be enough to transport energy from large to small sized structures until the structures got so small that molecular friction took over and changed the energy into heat. For certain types of systems, however, these wave interactions may not be sufficient. For instance, if the rate at which energy must cascade through the different sizes of structures is too much for the WWT processes, then more complicated, nonlinear processes must take up the job. In terms of the WWT model, this corresponds to an increase in the number of nonlinear interactions. (Looking closer, the picture looks less like ripples in a pond and more like a real rough sea.) As a result, the gap in between the time when linear and nonlinear effects are felt by the statistics decreases. This decrease is enough to invalidate the assumptions of the WWT theory. In other words, these WWT systems necessarily create conditions under which its own theory no longer holds.

These complicated, nonlinear processes have a profound effect on the basic characteristics of the system. They collapse the gap between the linear and nonlinear times (a pre-requisite for WWT) and break down statistical self-similarity of the flow (a sign of Kolmogorov turbulence). That is, a break down of the conditions which uphold the WWT theory coincides with a break down in the assumptions of the Kolmogorov theory. This coincidence points to a physical explanation: perhaps the nonlinear processes which take over from WWT are also the processes responsible for intermittency.

WWT as a model of universal turbulence is doomed [**BNN**]. Its very nature necessitates its downfall. The interactions of WWT are not always efficient enough to prevent strongly nonlinear processes from taking over. But WWT is analogous to Kolmogorov's picture of HDT: both are characterized by a cascade of energy from large structures (waves or whirlpools) to smaller ones. Perhaps the strongly nonlinear processes which plague WWT are similar to the intermittent processes which spoil Kolmogorov HDT. Through this extended analogy, the break-down of WWT indirectly invites a better understanding of intermittency.

In some specific cases of turbulence, a match between breakdown of WWT and the onset of intermittent structures has been made more rigorous. In these cases, the nature of the intermittent events is supposed to be know. For instance, sharp crests on the ocean surface are thought to be one type of intermittent event. These random, sporadic crests dramatically affect the statistics of the water surface around them, but they do not destroy the WWT picture in between the crests. In most cases, however, the precise physical manifestation of these intermittent events is not known and so the relationship between the break down of WWT and intermittency is less obvious.

Weak-Wave Turbulence and Beyond

Picture again the smoky plumes of a cigarette, or the tumbling waves and whirlpools of the ocean. Both are examples of turbulent fluid flow. There are aspects of these flows which are similar and universal– like the buckling and curling of streamlines and the statistical evenness of their general appearance. There are also aspects of these flows which are very particular to the precise physics involved. The smoke may twist itself into thin filaments before dispersing into a hazy cloud, while the ocean may develop white-caps and sharp corners to its profile. The challenge to the study of turbulence is to create a model which can describe both the universal and unique behavior of these flows. WWT may be such a theory.

It is clear that WWT can describe the same sort of universal behavior for turbulence that Kolmogorov predicted for HDT. But WWT is staging its own demise. It invites strongly nonlinear events to participate in the distribution of energy in the system. The price is that the small structures no longer obey the rules of WWT. To look at it in another way: The breakdown of WWT creates a much richer view of turbulence; on which incorporates WWT with strongly nonlinearity, and one that parallels not only Kolmogorov's picture of HDT, but also the intermittency of real turbulence. Although WWT is already a hero of turbulence theory for overcoming the closure problem; it is the break down of WWT, its tragic flaw, that may bring its most exciting predictions yet.

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