Intermittency in Weakly Turbulent Surface Waves

Laura J. Biven, Alan C. Newell and Sergey V. Nazarenko

Mathematics Institute, University of Warwick, Coventry, U.K.

Abstract

When surface gradients are small, the surface waves on a deep, ideal fluid can be well described by the theory of **Weak Turbulence** (WT) for two important regimes each of which is homogeneous and isotropic: At large scales, the transport of energy is dominated by **gravity waves** whereas **surface tension waves** dominate the energy transfer at smaller scales. Although each regime is weakly nonlinear, for large energy flux values the transfer of energy from the gravity wave scales to capillary wave scales may not be weakly nonlinear. We analyze this transfer in the context of **breakdown** and **intermittency**. Our results lend further support to the work of Newell and Zakharov [4] which sites the formation of white-caps as a possible, fully nonlinear mechanism for the transfer of energy between these two weakly turbulent regimes.

What is weak turbulence ?

Given an initial, statistical description of the sea surface, weak turbulence seeks to describe the evolution of the statistics after long times. The basic components of the dynamics are dispersive waves which interact weakly. The smallness of the nonlinear interaction causes the important dynamics to occur on two, widely separated time-scales.

One finds that the dispersion governs the early dynamics, causing the sea statistics to approach a close to joint-Gaussian state on time scales of the order of $t_L \sim \frac{1}{\omega_{\mathbf{k}}}$ where $\omega_{\mathbf{k}}$ is a typical frequency. Non-trivial statistics are generated by nonlinear interactions more slowly on time scales of the order of t_{NL} where

$$\frac{t_L}{t_{NL}} \ll 1. \quad (1)$$

It should be noted that, in general, (1) is a function of the Fourier space variable k and so must be evaluated at a particular scale.

The outcome of all this weakly nonlinear mixing is a steady state characterized by a constant flux of energy from large scales (which, for example, receive energy from the wind) to small scales where viscosity acts as an energy sink. The spectrum for this constant flux steady state is known as the Kolmogorov-Zakharov (KZ) spectrum because it was derived analytically by Zakharov [6] and takes a power-law form just as Kolmogorov's famous $k^{-5/3}$ spectrum for full hydrodynamic turbulence.

The description so far is universal and quite independent of the particulars of the initial condition. We require only that the cumulants of the initial state decay with increasing separations so that their Fourier transforms exist. We also assume that the dispersion relation and interaction coefficients are homogeneous functions of the Fourier space variable k. We take $\omega_{\mathbf{k}}$ to be homogeneous of degree α. The coefficient for three wave interactions is homogeneous of degree β and the four wave interaction coefficient has degree γ. (A function $f(x)$ is homogeneous of degree β iff $f(\lambda x) = \lambda^{\beta} f(x)$ for all $\lambda > 0$).

What is Breakdown?

The notion of **breakdown of weak turbulence** comes from asking if equation (1), which underpins the mathematical analysis of weak turbulence , holds for all length scales. It turns out that, almost always, this hypothesis of the theory fails at the tails of the KZ spectrum, i.e. at small scales that may still be larger than the dissipation scale. The criterion for breakdown depends on the order of homogeneity of the interaction coefficient and dispersion relation. The criteria for breakdown are:

$$\beta > 2\alpha \quad (2)$$

$$\gamma > 3\alpha \quad (3)$$

where (2) is the criterion for turbulence dominated by three wave resonances and (3) is the criterion for four wave resonances. The marginal cases ($\beta = 2\alpha$ and $\gamma = 3\alpha$) describe a situation of complete self-similarity of the WT system. This is discussed further in [2]. The following picture emerges:

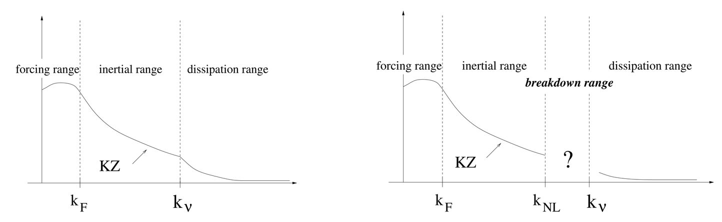

Figure 1: In the absence of breakdown the KZ spectrum is valid up to the dissipation scales. When the breakdown criterion is satisfied a breakdown region emerges ($k > k_{NL}$) where the KZ spectrum is not valid.

k_{NL} is the scale at which the linear and nonlinear timescales are comparable. When the breakdown criterion is satisfied (either (2) or (3)), the KZ spectrum fails for $k > k_{NL}$, the **breakdown region**. The dynamics among these breakdown scales are fully nonlinear and intermittent and the underlying physical processes are outside the remit of the weak turbulence theory [1]. Of course, the breakdown region may be masked by other dissipation effects (e.g. $k_\nu < k_{NL}$), however, since k_{NL} is a function of the average energy flux P, the breakdown may be realized for large values of the flux.

Surface Waves and Whitecaps

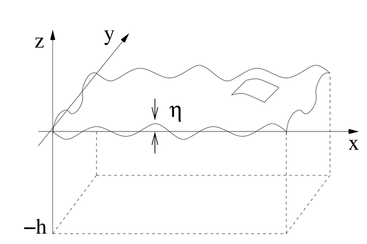

Figure 2: Surface Waves

Now consider the waves on the surface of an ideal fluid of infinite depth and the turbulence in which these waves participate. In the absence of any disturbances, the sea surface rests on the $z = 0$ plane. The undisturbed depth is h (we will later take $h \to \infty$) and $\eta(x, y, t)$ is the height of the disturbance at a point (x, y). (See figure 2.) The motion of the fluid is described by the irrotational Euler equations with appropriate boundary conditions. It turns out that these boundary equations amount to a Hamiltonian system with coordinates η, the surface height and ϕ, the velocity potential at the surface [5]. Taking the average surface slope, $\nabla\eta$ as a small parameter, this Hamiltonian can be expanded in an asymptotic expansion. The linear term gives the dispersion of the surface waves as:

$$\omega_{\mathbf{k}}^2 = gk + \sigma k^3 \quad (4)$$

where we have already assumed the fluid to be of infinite depth.

Surface waves generated at small scales (on the order of centimeters) develop according to the dispersion relation $\omega_{\mathbf{k}}^2 = \sigma k^3$ based on the surface tension, σ. These are capillary waves. Longer waves are governed by the gravity wave dispersion relation, $\omega_{\mathbf{k}}^2 = gk$, where g is the gravitational acceleration. These are gravity waves. The scale which divides these two regimes is given by

$$k_0 = \sqrt{\frac{g}{\sigma}} \quad (5)$$

where the two terms in the unified dispersion relation, (4), are balanced.

Newell and Zakharov [4] have speculated that the occurrence of *whitecaps* (derivative discontinuities in the fluid surface) are provoked by a super critical flux of energy in the gravity wave direct cascade. This super critical flux creates a non-universal, non-weak turbulence transition region between the gravity and capillary wave regions at scales for which whitecaps are likely to appear. The transition spectrum is assumed to be connected to the Phillips spectrum which characterizes a surface dominated by derivative discontinuities (whitecaps).

In [4], the authors show that a necessary condition for the Phillips spectrum (k^{-4}) to meet the gravity wave spectrum at a wave number less than k_0 is that the flux P is larger than a critical flux $P_c \propto (g\sigma)^{3/4}$. Similarly, this is the condition for the Phillips spectrum to meet the capillary wave spectrum at a wave number greater than k_0. Furthermore, the condition $P > P_c$ can be translated into a condition on the wind speed, $V_{wind} > V_c \equiv 6\frac{m}{s}$. It is empirically known that whitecaps begin to appear at this critical wind speed. The implication is that for super critical flux values, the transition from gravity to capillary waves may proceed via the creation of whitecaps.

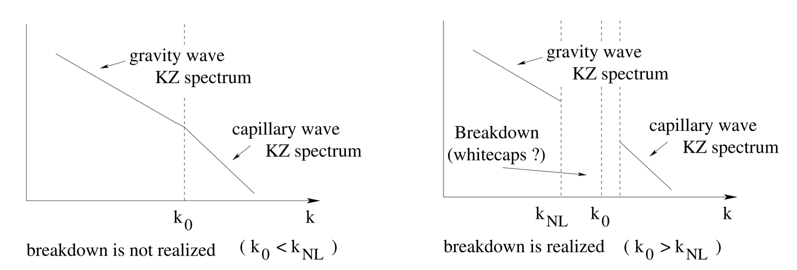

Figure 3: The gravity wave and surface tension wave spectra with a crossover regime which is not weakly nonlinear.

This picture can be reformulated using our new understanding of weak turbulence breakdown[3]. Evaluating k_{NL} in the gravity wave energy flux spectra readily shows that the condition for $k_{NL} \sim P^{-2/3} g < k_0$ is precisely the condition $P > P_c$ as before. The reasoning that Newell and Zakharov offered for the appearance of whitecaps associates these singular structures with the breakdown of weak turbulence . If whitecaps are involved in the transfer of energy from gravity wave scales to a capillary wave process, then the transition is necessarily outside the remit of weak turbulence theory.

References

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