EMERSON SCHOOL
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Adding Functions

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Dedicated to Connie Weber

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Benchmark

By the end of fifth grade, students should have an understanding of mathematics as the study of patterns. Functions are one context for these patterns. Students should also have an understanding of simple graphs (although this concept is not necessarily linked to functions) and an understanding that mathematics can be predictive and useful for real world situations.
This lesson makes a connection with each of the benchmark points mentioned above by emphasizing a cross-sectional view of the subject. Students analyze functions (linear only) from the point of view of patterns found in series of ordered pairs; their graphs and a few simple real world applications.

Motivation

This lesson is not about learning a new mathematical technique. Most of the techniques used here will already be familiar to the students (e.g. finding the algebraic rule that describes a series of ordered pairs; simple graphing; real world word problems). Instead, this lesson is a chance to take a step back and make connections among what is already known.
Another important consequence of this lesson is the introduction of the concept of a *mathematical object*. Numbers are the most natural examples of mathematical objects but functions are also mathematical objects. By the end of this lesson, students should understand that functions, as mathematical objects, can be added (subtracted, multiplied...) in much the same way that numbers can. The fact that this is how real mathematicians think about functions, in my experience, has proved to be a great motivating factor for the students.

Lesson Outline

Thinking Ahead:
Each student will need a few sheets of graph paper and a pencil. If a graphical program is available (such as Mathematica) this could make the delivery of the lesson and further discussion a lot of fun but it is not essential.
Delivery:
Begin with a bit of review. Write the following series of ordered pairs on the board: $(3, 6), (4, 7), (5, 8), ..., (9, 12), (10, 13)$. Ask the students to name the next ordered pair that belongs in the series. (The answer is $(11, 14)$). Next ask the students if they can identify the rule that defines the pattern of the series. Many students may say "add three" but they should be able to translate that idea into a mathematical formula such as "n+3" where "n" stands for the first number in the pair.
It's useful to remind the students at this point how they are using the idea of a function. Right now they are thinking of a function as an OPERATION that converts the first number of the ordered pair into the second number via a RULE. In this case the rule is n+3.
Try the following examples also:
A)$(2, 4), (3, 5), (4, 6), (5, 7), ..., (7, 9)$
What is the next pair in the series? $(8, 10)$
What is the rule for this pattern? n+2
B)$(2, 6), (3, 7), (4, 8), (5, 9), ..., (7, 11)$
What is the next pair in this series? $(8, 12)$
What is the rule for this pattern? n+4

Once the students seem comfortable with this way of thinking, move on to the following story problem.
Let's suppose that the series A) describes the amount of money that Jane has on each day in February. The first number in the pair is the date and the second number is the number of dollars that Jane has.

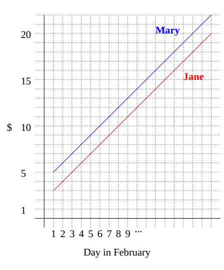

FIGURE 1: Mary and Jane's wealth in February.

Have the students plot the ordered pairs on the graph paper given. (This, I have found, is best done in small groups.) The students may need to be reminded how to draw a graph. The horizontal axis will label the days of the month. Each day corresponds to a line (not a box!). The vertical axis will be the number of dollars. Remember to leave enough space for all the points. Ordered pairs will correspond to intersections of the lines– points should be drawn on the cross-hairs, not in the boxes. This exercise could take up to 10 minutes. Once all the points have been plotted, ask the students to connect the dots as neatly as they can. What kind of graph do they see? They should all have a nice line.
Let's suppose that the series B) describes the amount of money that Mary has on each day in February. In just the same way, plot these points ON THE SAME GRAPH.
The students should now have, on a single graph, two lines. One corresponding to Jane's income (n+2) and the line above corresponds to Mary's income (n+4). Check that the students understand the meaning of the graphs by asking the following questions:
Q: How much money did Jane have on February 2nd? A: $4 This can be seen from the graph or the ordered pair (2,4).
Q: How much money did Mary have on February 2nd? A: $6.
Q: How much money did Jane start with on February 1st? A: $3 This can be seen by extending the line to the left or by the function rule: n+2 implies that Jane had $3 on Feb 1st since 1+2=3.
Q: How much money did Mary start with on February 1st? A: $5.
Q: How much money does Jane EARN each day? A: $1. From the graph, it's easy to see that after each day, Jane has one more dollar than the day before.
Q: How much money does Mary EARN each day? A: $1.

Make sure that the students understand why Mary has more money than Jane. They earn the same amount of money each day ($1 per day) but Mary started out with more money and so will continue to have more money on every day.
Ask the students if they see a relationship between the graph and the amount of money earned per day. We know that both Mary and Jane earn the same amount of money each day, so what is the same about the two lines? A: The slope. The factor in front of the "n" in both of the rules is 1.
It's important that, before moving to the next exercise, the students realize that the function rule, the graph and the real world ("wordy") interpretation are all equivalent ways of expressing the same function.
Adding Functions:
Let's suppose that Mary and Jane now want to combine their earnings.
Q: How much money will Mary and Jane have TOGETHER on ...
...February 2nd? A: $10 This can be seen from the ordered pairs. On day 2 Jane has $4 and Mary has $6. Together they have $10. These values could also have been read off from the graph.
...February 3rd? A: $12. Again, Jane has $6 and Mary has $8 so together they have $12.

Ask the students to write a series of ordered pairs for the amount of money that Mary and Jane have TOGETHER on the 1st, 2nd, 3rd, 4th and 5th days of February. $(1, 8), (2, 10), (3, 12), (4, 14), (5, 16)...$
Ask the students to plot this new series of points on the SAME GRAPH as before. This may take a few minutes. In the end the students should have a third line, higher than the other two, which also rises more steeply.
Q: Looking at the ordered pairs, what is the RULE for the new series? A: 2n+6. Some students may know this as a "two step" function because it involves both multiplication and addition.
Ask the students if they could have guessed this new rule just by knowing the rules for the amount of money Jane and Mary have separately. In other words, ask them if they see a relation between the three rules we have identified:

Jane: n+2; Mary: n+4
Jane and Mary TOGETHER: 2n+6.

In my experience, this is the magical moment of the lesson. After a few minutes, light bulbs start going off all over the room. Rather than share the answer immediately, I encourage those students who have seen the relationship to share it only with their neighbors. The answer, of course, is that

$$(n+2) + (n+4) = 2n+6.$$

Jane's money PLUS Mary's money IS their combined wealth. This relationship can be seen also from the graphs.

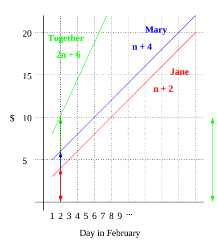

FIGURE 2: Mary and Jane's combined wealth in February

A little summary is in order at this point:
This lesson has shown that functions are not JUST a pattern in a series of ordered pairs; and not JUST an algebraic rule; and not JUST a graph and not JUST a solution to a word problem– a function can be ALL of these things. It is very much our choice how we choose to represent a problem.
We've also seen that functions can be added in much the same way that we are used to adding numbers. In this sense, functions are a new type of mathematical object and it's often useful to think of them in this way.

Assessment

The following assignment is short and exactly analogous to the problem worked in class. Even better, it is self-grading!
Assignment:
John and Ryan are each pouring water into an empty well. John pours in 2 buckets of water each hour and Ryan pours in 3 buckets per hour.
Part 1) Write a series of ordered pairs for the number of buckets John has poured after 1,2,3,4 and 5 hours. Call this series J. Write another series of ordered pairs for the number of buckets Ryan has poured after 1,2,3,4,and 5 hours and call this series R.
Using just these series of ordered pairs, write down a new series for the number of buckets of water in the well after 1,2,3,4 and 5 hours. Call this series T.
Part 2) Write down theFUNCTION RULE for series J. Write down the FUNCTION RULE for series R. Looking just at series T, write down it's FUNCTION RULE. How are these rules related?
Part 3) Using graph paper, plot the points (ordered pairs) of series J. Connect the points using a ruler. On the same graph, plot the points from series R and connect these points using a ruler. What are the slopes of each lines? What do the slopes mean? Finally, still on the same graph, plot the points of series T and connect these points using a ruler. What is the slope of this line? How could you have guessed this slope without having to draw it out? (Look again at part 2 for help.)
The answers from all three parts should agree. Check your own work.
Answers:
Part 1)
J: (1,2),(2,4),(3,6),(4,8),(5,10)
R: (1,3),(2,6),(3,9),(4,12),(5,15)
T: (1,5),(2,10),(3,15),(4,20),(5,25)
Part 2)
J: 2n
R: 3n
T: 5n = 2n + 3n
Part 3)

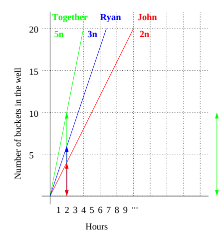

FIGURE 3: Number of buckets poured into the well.

Line J has slope 2.
Line R has slope 3.
Line T has slope 5. Since the slope is just the factor in front of "n" in the function rule, the slope 5 could have been determined from the function rule in part 2.
The meaning of the slope of line J is the rate (number of buckets of water per hour) that John pours into the well. The slope of line R is the rate at which Ryan pours water into the well. The slope of line T is the rate at which the amount of water in the well increases.

Acknowledgments

This lesson was presented during a visit to Connie Weber's 5th/6th grade class at Emerson School in 2003. It is impossible to over estimate the thanks I owe to Connie who was my own teacher years ago and is now my friend and mentor. The environment that Connie creates in her class is the setting of my most enjoyable memories as a student and as a teacher. Emerson students of all ages are unique for the high level of individual attention that they receive and the broad ranging emphasis on creativity that their education nurtures. I would like to thank Connie Weber and Emerson School for allowing me to share in some of their magic.

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