

SINGLE VORTICES AND VIRIAL THEOREMS IN BOSE-EINSTEIN CONDENSATES

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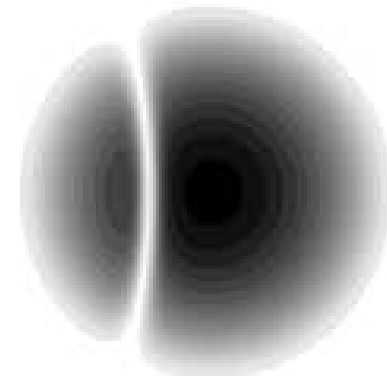
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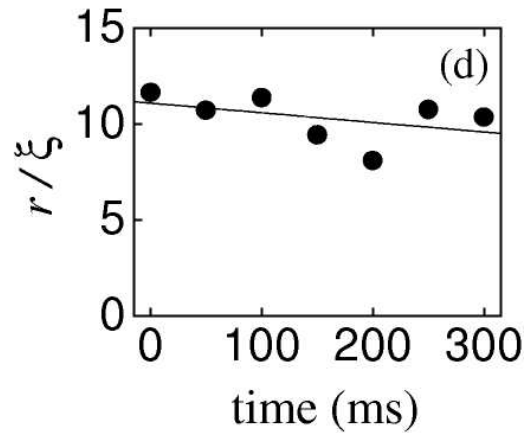
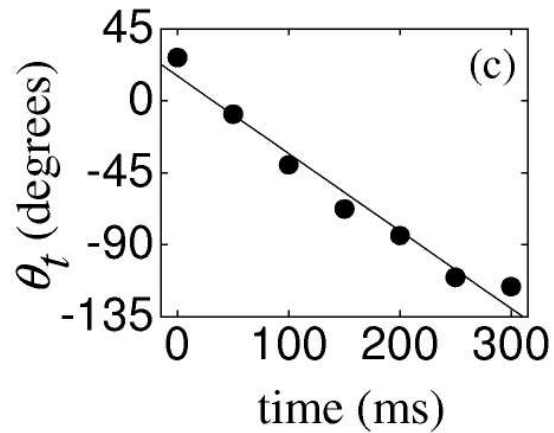
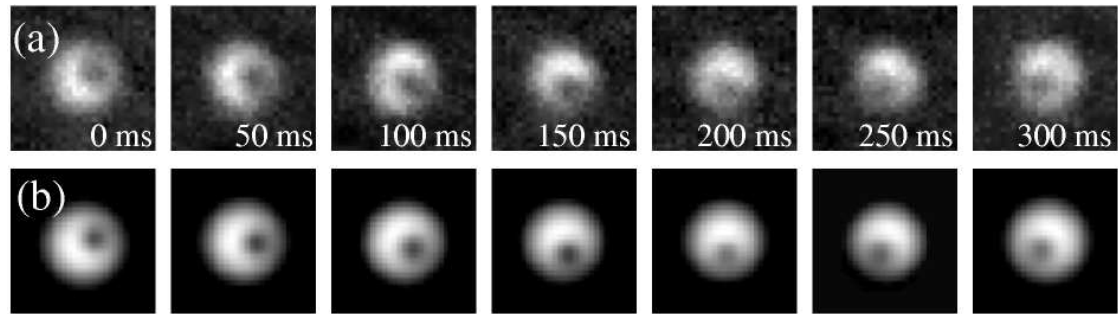
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Phys. Rev. A **72**, 053609 (2005)

Phys. Rev. A **72**, 053624 (2005)



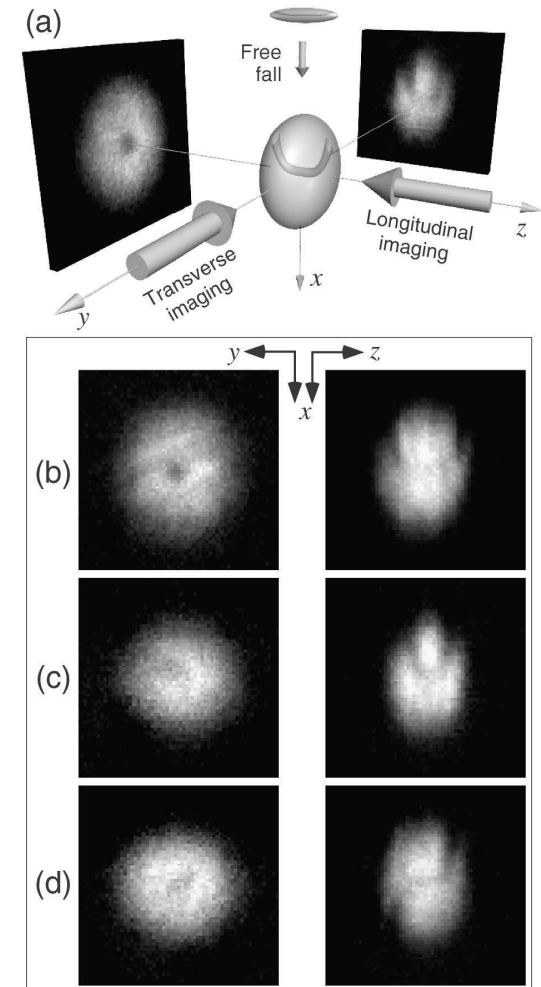
Observation of precessing vortices



[Anderson, Haljan, Weiman, Cornell, PRL, 2000]

[Rosenbusch, Bretin, Dalibard, PRL, 2002] →→

Vortex lines



Formulation

We suppose the condensate wave function $\Psi(\mathbf{r}, t)$ and its energy functional [E.P. Gross, 1961; L.P. Pitaevskii, 1961]:

$$E = \frac{\hbar^2}{2M} \int |\nabla \Psi|^2 d^3x + \int V |\Psi|^2 d^3x + 2\pi \frac{\hbar^2 a_s}{M} \int |\Psi|^4 d^3x.$$

Introduce the axisymmetric harmonic potential: $V(\rho, z) = \frac{1}{2} M(\omega_{\perp}^2 \rho^2 + \omega_{\parallel}^2 z^2)$, and appropriate units, to obtain the form:

$$E = \frac{1}{2} \int |\nabla \Psi|^2 d^3x + \int V |\Psi|^2 d^3x + 2\pi \frac{N a_s}{a_{\parallel}} \int |\Psi|^4 d^3x, \quad V(\rho, z) = \frac{1}{2} (\rho^2 + \beta^2 z^2).$$

$a_{\perp, \parallel} \equiv \sqrt{\hbar / (M \omega_{\perp, \parallel})}$: units of length, $\beta \equiv \omega_{\perp} / \omega_{\parallel}$,

N : the number of atoms, a_s : the scattering length.

Numerics for a vortex in a 3D trap

We introduce the **chemical potential** μ and substitute

$$E \rightarrow E - \mu N \quad N \equiv \int |\Psi|^2 dV.$$

A vortex **precessing** with **angular frequency** ω is an extremum of

$$E_{\text{rot}} = E - \mu N - \omega \ell.$$

$$\ell = \frac{1}{i} \int \Psi^* \frac{\partial \Psi}{\partial \phi} d^3x \quad \text{:angular momentum.}$$

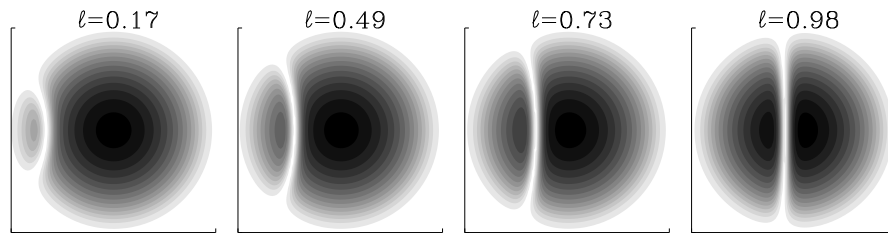
But: we use a **norm-preserving relaxation algorithm** to find **extrema** of the extended energy functional:

$$E_{\text{rot}} = E - \mu N + \frac{a}{2} (\ell - b)^2, \quad a, b : \text{const.}$$

The solutions are precessing at an angular frequency $\omega = -a(\ell - b)$.

A spherical trap

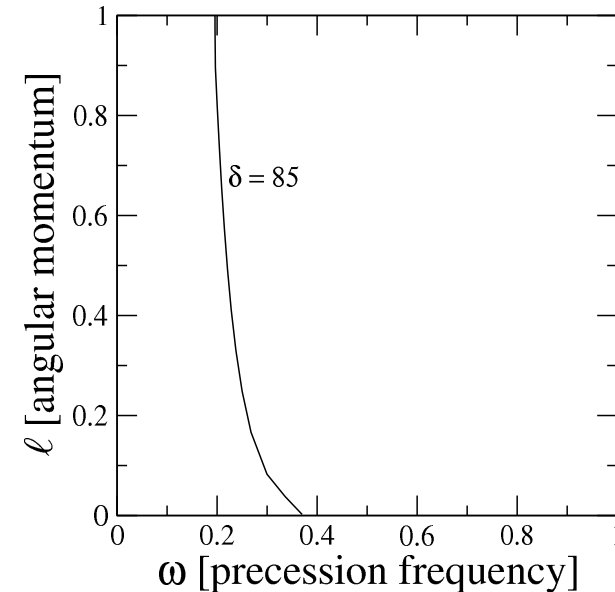
Experiment by [B.P. Anderson, *et al*, PRL **85**, 2857 (2000)].



The particle density $|\Psi|^2$ for vortex solutions corresponding to experiment.

D.L. Feder *et al*, PRL **86**, 564 (2001)

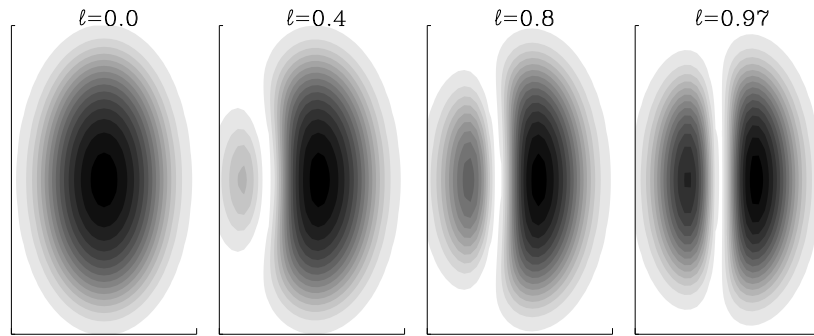
would give $\omega(\ell \rightarrow 0) = 2\pi \times 1.58$ Hz.



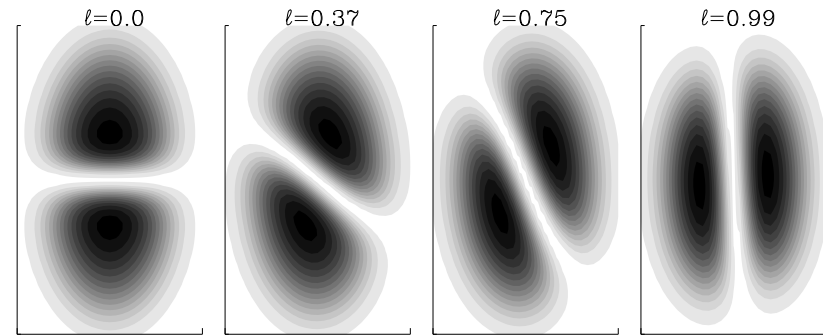
Angular momentum per particle ℓ (in units of \hbar) versus precession frequency ω (in units of trap frequency).

We have $1.52\text{Hz} < \omega/(2\pi) < 1.95\text{Hz}$ (e.g., 1.72 Hz for $\ell = 0.5$).

An elongated trap



U-vortex.

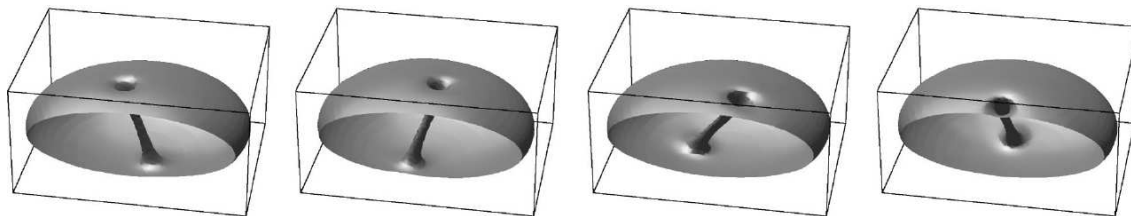


S-vortex.

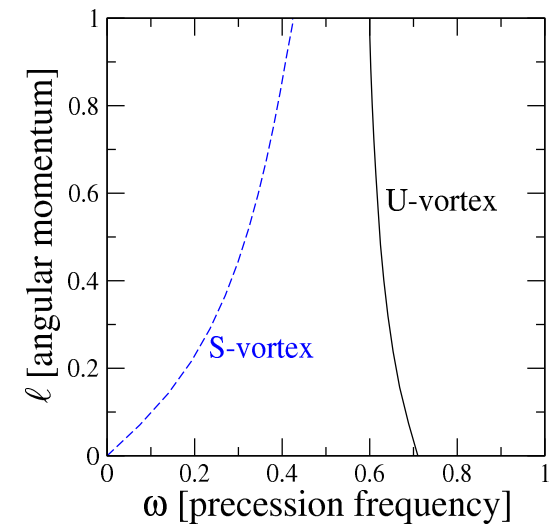
J.J. Garcia-Ripoll and V.M. Perez-Garcia, PRA **64**, 53611 (2001).

M. Modugno, L. Pricoupenko, Y. Castin, EPJ D **22**, 235 (2003).

A. Aftalion and I. Danaila, PRA **68**, 23603 (2003).



Smith, Heathcote, Krueger, Foot, PRL (2004)



Lowest Landau level

The single (off-center) U-vortex mode in the LLL derived by O.K. Vorov *et al*, Phys. Rev. Lett. (2005):

$$\Psi = \frac{\ell^{\frac{1}{4}}}{\sqrt{\pi}} [(x-b) + iy] e^{-\frac{1}{2}[(x-a)^2 - 2ia y + y^2]}$$

$$a = (\sqrt{\ell} - \ell)^{1/2}, \quad b = \frac{1 - \ell}{a},$$

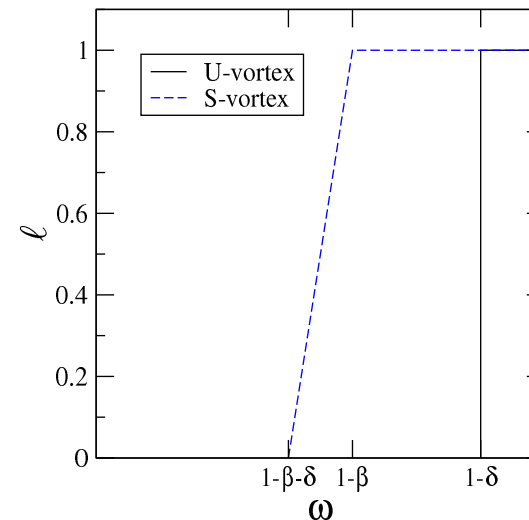
$0 \leq \ell \leq 1$, :angular momentum,

had been found numerically by [Butts and Rokhsar, Nature (1998)].

The S-vortex

corresponds to the linear combination:

$$\begin{aligned} \Psi_1 &\sim z e^{-(\rho^2 + \beta z^2)/2}, \\ \Psi_2 &\sim \rho e^{i\phi} e^{-(\rho^2 + \beta z^2)/2}, \\ \Psi &= c_1 \Psi_1 + c_2 \Psi_2. \end{aligned}$$



δ : interaction strength

$\beta = \omega_{\perp} / \omega_{\parallel}$: trap aspect ratio

Formulation II

We suppose the condensate wave function $\Psi(\mathbf{r}, t)$ and its equation of motion:
[E.P. Gross, 1961; L.P. Pitaevskii, 1961]

$$i \frac{\partial \Psi}{\partial t} = -\frac{1}{2} \Delta \Psi + V(\rho) \Psi + \frac{g}{2} (\Psi^* \Psi) \Psi - \mu \Psi.$$

We have introduced the chemical potential μ and have substituted
 $\Psi(\mathbf{r}, t) \rightarrow \Psi(\mathbf{r}, t) e^{-i\mu t}$. μ should be chosen so that $\int (\Psi^* \Psi) d^3x = 1$.

g : the interaction strength.

[$1/\omega_{\perp}$: unit of time]

Steady state precession: Equations of motion

Suppose a vortex in an axisymmetric trap, as a steady state precessing with frequency ω . Its wavefunction is:

$$\bar{\Psi}(x, y, t) = \Psi(x', y' | \mu, \omega) e^{-i\mu t},$$
$$x' = x \cos \omega t + y \sin \omega t, \quad y' = -x \sin \omega t + y \cos \omega t,$$

This satisfies the Gross-Pitaevskii equation ($\varepsilon_{\alpha\beta}$: antisymmetric tensor):

$$\mu\Psi - i\omega\varepsilon_{\alpha\beta}x_{\alpha}\partial_{\beta}\Psi = -\frac{1}{2}\Delta\Psi + V(\rho)\Psi + g(\Psi^*\Psi)\Psi. \quad (1)$$

where, e.g., $V(\rho) = \frac{1}{2}\rho^2$ is the trapping potential, g the interaction strength.

Virial relations

Suppose a vortex located at $\mathbf{R} = (R, 0)$, with linear momentum $\mathbf{P} = (0, P)$.

Apply field-theoretical methods to the equations of motion to derive:

$$P - \omega R = 0, \quad (I)$$

$$\omega P - \int \frac{x}{\rho} \frac{dV}{d\rho} n \, dx dy = 0, \quad (II)$$

where ($\alpha = 1, 2$)

$$P_\alpha \equiv \int J_\alpha \, dx dy, \quad \text{the linear momentum,}$$

$$R_\alpha \equiv \int x_\alpha n \, dx dy, \quad \text{the mean position of the configuration,}$$

$$n = \Psi^* \Psi \text{ (particle density),} \quad J_\alpha = \frac{1}{2i} (\Psi^* \partial_\alpha \Psi - \Psi \partial_\alpha \Psi^*) \text{ (current).}$$

Harmonic trap: $V = \frac{1}{2} \rho^2$

Virial relation I: $P - \omega R = 0$

Virial relation II: $\omega P - R = 0$

$$\Rightarrow P = 0 = R \quad (\text{for } \omega \neq 1).$$

The result is satisfied by: (i) the calculated U- and S-vortices, (ii) the LLL closed form solution (by Vorov *et al*).

Anharmonic trap: $V = \frac{1}{2} \rho^2 (1 + \lambda \rho^2)$

Virial relation I: $P - \omega R = 0$

Virial relation II: $\omega P - R = 2\lambda Q, \quad Q \equiv \int x \rho^2 n \, dx dy.$

$$\Rightarrow P = -\frac{2\lambda\omega}{1-\omega^2} Q, \quad R = -\frac{2\lambda}{1-\omega^2} Q,$$