Quantum coherence, evolution of the Wigner function, and transition from quantum to classical dynamics for a chaotic system*

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The paper deals with dynamics of a quantum chaotic system under influence of an environment. The effect of an environment is known to destroy the quantum coherence and can convert the quantum dynamics of a system to classical. We use a semiclassical technique for studying the process of decoherence. The condition for transition from quantum to classical dynamics is obtained in general form and checked numerically for a particular chaotic system, known as quantum the standard map on a torus. The relevance of the obtained results to the problem of correspondence between quantum and classical mechanics is briefly discussed. © *1996 American Institute of Physics.*
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Numerous connections between quantum and classical mechanics clearly indicate that classical behavior is described perfectly well by an extension of quantum physics. However, the details of the quantum-to-classical transition are still uncertain, as are the conditions for which this transition is applicable. For this present work, we have examined the transition from quantum to classical dynamics for a chaotic system.

I. INTRODUCTION

The problem of transition from quantum to classical mechanics has been a subject of permanent interest since the foundation of quantum mechanics. One of the motivations for this interest comes from the well known experimental fact that one and the same quantum system can behave in different laboratory conditions either as a quantum system or a classical one. It seems that common agreement on explanation of such a phenomenon has been achieved. The key idea is the following—there are no systems which are completely isolated from their environment (heat bath, cosmic microwave background, measurement device and so on) and this influence can be important. The exploration of this idea (so-called ''environmental approach'') makes the context of a large part of the recent papers devoted to the correspondence problem (see, for example, Refs. 1–4 and references therein).

Within the environmental approach, the ''whole'' problem of correspondence can be divided into two subproblems. The first one is the emergence of a ''definite'' classical trajectory of a particle from ''indefinite'' quantum dynamics. The second subproblem keeps us within the probabilistic approach. The main question here is how to obtain the classical distribution function from a quantum equation. The present paper deals entirely with the second subproblem. It is natural to take the Wigner function of a quantum system as initial point for approaching to classical mechanics. In fact, the Wigner function has a number of properties of the classical distribution function and seems to be its closest analog in the realm of quantum mechanics. Thus our question of interest can be reformulated in the following way—we want to obtain a condition for which the Wigner function converts into the classical distribution function.

A new impetus to increasing the interest in the correspondence problem has been recently given by study of the so-called quantum chaotic systems—those systems which have chaotic dynamics in the classical limit.[5–14] For an isolated (from its environment) chaotic system, the discrepancy between the Wigner function and the classical distribution function becomes apparent after an extremely short time $t_c \sim \lambda^{-1}\ln(I/\hbar)$, where I is some characteristic action of the system and λ is the Lyapunov exponent.[15] To give a feeling of how small the time t_c is, let us choose $I \sim 10^{-7}$ J·s (a particle with mass 1 g moving with speed 1 cm/s) and $\lambda \sim 1\ \mathrm{s}^{-1}$: then $t_c \approx 22$ s. This short time contradicts a common perception of classical mechanics as a limiting case from quantum mechanics. Thus it is no wonder that quantum chaotic systems have attracted much attention in connection with the correspondence problem in general, and in connection with the environmental approach in particular.

The beginning of study of an open (i.e., under influence of an environment) quantum chaotic system can be dated back to the paper of Ott *et al.*,[5] where dynamics of the quantum kicked rotor affected by external noise was analyzed. It was shown that as the strength of noise exceeds some critical value, the kicked rotor recovers classical diffusion for mean energy. The study of open quantum chaotic systems was continued in the papers of Graham and Dittrich.[6] The authors used an approach based on a master equation for the density matrix of a system (the kicked rotor and similar systems with discrete time) coupled to a reservoir of linear oscillators, or to a macroscopic system acting as a measurement device. Both effects of dissipation and noise caused by a bath or measurement device were considered. An approximation that noise is δ-correlated was used. Cohen studied dynamics of

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the quantum kicked rotor without use of the Markov approximation.[7] In particular it was shown that under an appropriate choice of the interaction term, the effect of dissipation can be neglected, and therefore, the problem reduces to that studied in Ref. 5. The stochastic equation for both high and low temperature limits were constructed and the effect of a noise correlation was studied in detail. The problem of the recovery of classical dynamics by external stimulus was considered by Toda, Adachi and Ikeda.[8] A striking result was obtained; that one can recover classical behavior by coupling the quantum kicked rotor to another quantum chaotic system (in fact another quantum kicked rotor was used). We also note a recent paper,[16] where the authors study the effect of the environment on dynamics of the quantum cat map—one of the models of quantum chaos, which we also use in our numerical simulations.

It should be noted that the analyses performed in all cited papers are more or less model dependent, where the singular form of the kicked rotor Hamiltonian is strongly in use. For a generic quantum chaotic system a condition for the recovery of classical dynamics was independently formulated in recent papers[12,13] by analyzing the Moyal equation for the Wigner function. In the present paper we use semiclassical techniques to obtain such a condition.[14] The semiclassical approach provides a deeper look inside the process which causes a transition from quantum to classical dynamics. Besides this, the method suggested is more accurate and, in principle, allows one to find the threshold exactly, provided the classical dynamic of a system is known in detail.

The structure of the paper is the following. In the next section we briefly discuss a common approach for treating an open system. We also show, using a qualitative argument, why influence of an environment ''prevents'' a quantum chaotic system from logarithmically small correspondence time. The condition of correspondence is formulated in Sec. III. The condition obtained is shown to be consistent with that in Refs. 12 and 13. Section V contains the numerical results for dynamics of the Wigner function for a particular chaotic system, known as the standard map on a torus. These numerical simulations are aimed to illustrate and check the analytical results of Sec. III. A short Sec. IV, devoted to the definition of the Wigner function for different topologies of the system phase space (plain, cylinder or torus), precedes the numerical simulation. Finally, the concluding Sec. VI is devoted to general discussion of classical mechanics as the ''$m\to\infty$'' limit from quantum mechanics.

II. MASTER EQUATION FOR AN OPEN SYSTEM

We remind the reader of a common approach to analysis of an open system. The starting point is the Hamiltonian for the composed system

$$H_{\rm tot}=H+H_{\rm en}+H_{\rm int}. \qquad (1)$$

In Eq. (1) $H=H(x,p,t)$ is the Hamiltonian of the system considered, $H_{\rm en}=H(\mathbf{q})$ is the Hamiltonian of its environment ($\mathbf{q}$ is the set of the environmental variables), and the interaction between the system and environment is assumed to be small $H_{\rm int}\sim\epsilon$. In what follows we shall restrict ourselves to the case $H_{\rm int}=VV_{\rm en}$, where V and $V_{\rm en}$ depend only on the system and environment variables respectively. The aim is to obtain an equation for the reduced distribution function

$$f(x,p,t)=\int d\mathbf{q}\; f_{\rm tot}(x,p,\mathbf{q},t) \qquad (2)$$

in the classical approach, or for the reduced density matrix $\hat\rho(t)=\mathrm{Tr}_{\mathbf{q}}[\hat\rho_{\rm tot}(t)]$ in the quantum approach. We shall display the equations for the classical case. The quantum equations have the same structure with the Poisson brackets substituted by the commutator: $\{\ldots\}\to(-i/\hbar)[\ldots]$.

To realize the aim one has to assume (to postulate) a number of properties for the environment or to choose a particular model for $H_{\rm en}$. Historically the second way was preferred, where the bath of linear oscillators was mainly considered as $H_{\rm en}$. The equation for $f(x,p,t)$ takes the simplest form in the case of the Ohmic environment ($H_{\rm en}$ corresponds to 1-dimensional scalar field, $V\sim x$)[1,2]

$$\frac{\partial f}{\partial t}=\{H,f\}+2\gamma\frac{\partial(pf)}{\partial p}+D\frac{\partial^2 f}{\partial p^2}. \qquad (3)$$

In Eq. (3) $D=2\gamma mk_BT$, T is the temperature of the field, and γ is the relaxation constant. The first term on the right hand side of Eq. (3) corresponds to unperturbed evolution of the system, the second describes the energy lost from the system, and the third is the influence of noise coming from the thermostat. For the problem considered in this paper the third term is of the most importance,[6–9,12,13] so we shall neglect dissipation in what follows. (This also can be considered as the limit $\gamma\to0$ and $T\to\infty$ while D is kept constant.) For a weak coupling the equation similar to Eq. (3) can be also obtained for arbitrary V.[17] Neglecting the dissipation term, it has the form

$$\frac{\partial f}{\partial t}=\{H,f\}+\tilde D\{V,\{V,f\}\}. \qquad (4)$$

It is easy to see that Eq. (4) is again a diffusion-like equation, but now diffusion is inhomogeneous in phase space, with local diffusion constant $D=\tilde D(\partial V/\partial x)^2$.

A new trend in treating an open system is connected with the study of a ''thermostat'' with a finite number of degrees of freedom (the term ''booster'' is often used).[18–20] It has been found that the assumption about an infinite number of degrees of freedom for $H_{\rm en}$ is not a necessary one and can be ''substituted'' by the assumption that the dynamic of the system $H_{\rm en}$ is chaotic. Then, irrespective of the particular form for $H_{\rm en}$, one can obtain a self-consistent equation for a system weakly coupled with the booster. This equation has the same form as Eq. (4),[20] where the constant $\tilde D$ is entirely defined by the correlation function $\langle V_{\rm en}(t)V_{\rm en}(t')\rangle \sim \exp[(t-t')/\tau]$ of the booster. We shall return to the case discussed in Sec. VI.

Now we come to the question of how the influence of an environment prevents a chaotic system from having a logarithmically small correspondence time. First, let us consider an isolated system. Following Refs. 12 and 13 we consider a 1-dimensional chaotic system with the Hamiltonian

$$H=\frac{p^2}{2m}+U(x,t). \tag{5}$$

Then the dynamics of its Wigner function $w=w(x,p,t)$ obeys the following equation:[21,22]

$$\frac{\partial w}{\partial t}=\{H,w\}_M \equiv\{H,w\}+\sum_{n\geqslant 1}\frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}U}{\partial x^{2n+1}}\frac{\partial^{2n+1}w}{\partial p^{2n+1}}. \tag{6}$$

In Eq. (6) we denote by $\{\ldots\}_M$ the Moyal brackets and we display their explicit form for the 1-dimensional Hamiltonian considered. For $\hbar=0$ this equation coincides with the Liouville equation for the distribution function, and therefore the terms containing Planck's constant can be regarded as ''quantum corrections'' to classical Liouville flow. The crucial point is that for the chaotic regime the partial derivatives of the classical distribution function $f(x,p,t)$ behave as $\partial^n f/\partial p^n\sim\exp(\lambda nt)$, with the increment λ given by the Lyapunov exponent. Thus the quantum corrections grow exponentially and after a characteristic time $t_c\sim\lambda^{-1}\ln(I/\hbar)$ (this time is often cited as Zaslavsky's time[15]) they become of the same order as the main term.

The situation will change if we take into account the influence of an environment. In fact, let us denote by $\delta p(t)$ the characteristic scale of the distribution function structure in Eq. (4). If $\tilde{D}=0$ in Eq. (4), then $\delta p(t)$ would decrease as $\delta p(t)\sim\delta p(0)\exp(-\lambda t)$. However, since $\tilde{D}\neq 0$ the diffusion term ''tries'' to smooth the distribution function as $\delta p(t)\sim[\delta p(0)+Dt]^{1/2}$ [here D is the local diffusion constant, $D=\tilde{D}(\partial V/\partial x)^2$]. Combination of these two processes will fix some minimal $\delta p_{\rm min}\sim(D/2\lambda)^{1/2}$. This means that the partial derivatives $\partial^n f/\partial p^n\sim(1/\delta p_{\rm min})^n$ are restricted for any time. Thus the quantum corrections do not grow and we have a possibility to reconcile quantum with classical evolution. This kind of argument was used to formulate a condition of quantum-classical correspondence in the papers Refs. 12 and 13. In the notation used it reads

$$\left(\frac{a\,\delta p_{\rm min}}{\hbar}\right)^2=\left(\frac{a}{\hbar}\sqrt{\frac{D}{2\lambda}}\right)^2\gg 1,\quad D=\tilde{D}(V')^2, \tag{7}$$

where V' stands for the characteristic value of the derivative $\partial V/\partial x$ and a denotes a characteristic length for variation of the potential $U(x,t)$. In the next section we shall obtain this condition in a different way.

To conclude this section we would like to make a remark concerning the numerical method used to solve the master equation for an open system. The quantum counterpart of Eq. (4) has the form

$$\frac{\partial\hat{\rho}(t)}{\partial t}=-\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)]-\frac{\tilde{D}}{\hbar^2}[\hat{V},[\hat{V},\hat{\rho}(t)]]. \tag{8}$$

The numerical method used is based on evidence that this equation for the density matrix is equivalent to the following stochastic Schrödinger equation:

$$i\hbar\frac{\partial\psi(t)}{\partial t}=\hat{H}+\epsilon\xi(t)\hat{V}\psi(t), \tag{9}$$

where $\xi(t)$ is a random process with correlation function $\overline{\xi(t)\xi(t')}=\delta[(t-t')/\tau]$, and $\tilde{D}=\epsilon^2\tau)$. (We note that using a technique suggested in Ref. 23 it is possible to construct a stochastic Schrödinger equation for a master equation of a rather arbitrary form.) Having obtained the solution of Eq. (9), one obtains the solution of the problem (8) by averaging the pure density matrix over the different realizations of the stochastic process $\xi(t)$

$$\hat{\rho}(t)=\{|\psi(t,\xi)\rangle\langle\psi(t,\xi)|\}_\xi. \tag{10}$$

This method is essentially more effective than the direct solving of Eq. (8). We have found that one hundred runs are already enough to get a convergence. We also use this analogy between the stochastic equation for the wave function and the master equation for the system density matrix in our analytical calculation.

III. CONDITION OF QUANTUM CLASSICAL CORRESPONDENCE

We shall analyze the problem in the Wigner representation which, as has already been mentioned, is the most appropriate representation for the purpose of comparison between quantum and classical mechanics. Besides this, we are interested in the case $I/\hbar\gg 1$ (I is some characteristic action of the system), when such a comparison is possible in principle. Thus, our nearest aim is to obtain the semiclassical expression for the evolution of the Wigner function.

The Wigner function is defined according to the following formula:[21,22]

$$w(X,P,t)=\frac{1}{2\pi\hbar}\int dx\,\exp\left(\frac{iPx}{\hbar}\right)\rho\left(X-\frac{x}{2},X+\frac{x}{2},t\right), \tag{11}$$

where $\rho(x',x'',t)$ in the right hand side of the equation is the density matrix of the system in the coordinate representation. Due to the equivalence between Eq. (8) and Eq. (9) we can present $\rho(x',x'',t)$ as follows:

$$\rho(x',x'',t)=\{\psi(x',t)\psi^*(x'',t)\}_\xi. \tag{12}$$

Let us denote by $G(X,Y,t)$ the Green's function for the stochastic Schrödinger equation (9)

$$\psi(X,t)=\int dY\,G(X,Y,t)\psi(Y,0). \tag{13}$$

Substituting Eq. (13) and Eq. (12) into Eq. (11), we obtain after simple transformations

$$w(X,P,t)=\int\int F(X,P;Y,Q;t)w(Y,Q,0)dY\,dQ, \tag{14}$$

where

$$F(X,P;Y,Q;t) = \left\{ \frac{1}{2\pi\hbar} \int \int \exp\left(i \frac{Px}{\hbar} \right) G\left(X - \frac{x}{2}, Y - \frac{y}{2}, t \right) \times G^*\left(X + \frac{x}{2}, Y + \frac{y}{2}, t \right) \exp\left(-i \frac{Qy}{\hbar} \right) dx\, dy \right\}_\xi \quad (15)$$

is the evolution operator. The ''semiclassical evolution'' corresponds to the choice $G(X,Y,t)$ in the common form[24]

$$G(X,Y,t) = \sum_\alpha g_\alpha(X,Y,t), \quad (16)$$

$$g_\alpha(X,Y,t) = \left(\frac{1}{2\pi i\hbar} \right)^{1/2} \left(-\frac{\partial^2 S^\alpha(X,Y,t)}{\partial X \partial Y} \right)^{1/2} \times \exp\left[\frac{i}{\hbar} S^\alpha(X,Y,t) + \frac{i\pi}{2} \nu_\alpha \right], \quad (17)$$

where the index α labels the different classical trajectories connecting the points X and Y, $S^\alpha(X,Y,t)$ is the principal Hamilton's function, and ν_α is the Maslov index. Substituting Eq. (16) in Eq. (15) we finally obtain

$$F(X,P;Y,Q;t) = \sum_{\alpha,\beta} f_{\alpha,\beta}(X,P;Y,Q;t), \quad (18)$$

where

$$f_{\alpha,\beta}(X,P;Y,Q;t) = \left\{ \frac{1}{2\pi\hbar} \int \int \exp\left(i \frac{Px}{\hbar} \right) g_\alpha\left(X - \frac{x}{2}, Y - \frac{y}{2}, t \right) \times g_\beta^*\left(X + \frac{x}{2}, Y + \frac{y}{2}, t \right) \exp\left(-i \frac{Qy}{\hbar} \right) dx\, dy \right\}_\xi \quad (19)$$

and $g_\alpha(X,Y,t)$ is given by Eq. (17).

One might doubt if the semiclassical approximation can be a basis for studying of a chaotic system. In fact, as we have already mentioned, the time of correspondence between pure classical and quantum evolutions is extremely small and scales as $t_c \sim \ln(1/\hbar)$. Fortunately, this does not imply that the semiclassical approximation fails after t_c. It was shown in the papers of Heller and Tomsovic[25] that Eqs. (16)–(17) can be used to calculate the system wave function for time much larger than t_c.

Now we shall show that the terms with $\alpha = \beta$ in Eq. (18) define the classical evolution of the system. In fact, substituting $g_\alpha(X \mp x/2, Y \mp y/2, t)$ into Eq. (19) in the following form:

$$g_\alpha\left(X \mp \frac{x}{2}, Y \mp \frac{y}{2}, t \right) \approx \left(\frac{-1}{2\pi i\hbar} \frac{\partial^2 S^\alpha}{\partial X \partial Y} \right)^{1/2} \exp\left(\frac{i}{\hbar} \left[S^\alpha \mp \frac{\partial S^\alpha}{\partial X} \frac{x}{2} \mp \frac{\partial S^\alpha}{\partial Y} \frac{y}{2} + \frac{\partial^2 S^\alpha}{\partial X^2} \frac{x^2}{8} + \frac{\partial^2 S^\alpha}{\partial Y^2} \frac{y^2}{8} + \frac{\partial^2 S^\alpha}{\partial X \partial Y} \frac{xy}{4} \right] + \frac{i\pi}{2} \nu_\alpha \right) \quad (20)$$

[here $S^\alpha \equiv S^\alpha(X,Y,t)$] we obtain:

$$\sum_\alpha f_{\alpha,\alpha}(X,P;Y,Q;t) = \sum_\alpha \{ -(\partial P_t^\alpha(X,Y)/\partial Y) \delta[P - P_t^\alpha(X,Y)] \times \delta[Q - P_0^\alpha(X,Y)] \}_\xi, \quad (21)$$

where $P_t^\alpha(X,Y) = \partial S^\alpha(X,Y,t)/\partial X$ and $P_0^\alpha(X,Y) = -\partial S^\alpha(X,Y,t)/\partial Y$ are the initial and final momentum of a classical particle moving from Y to X. Using the identity $\delta[\chi(\phi) - \chi_0](d\chi/d\phi) = \delta[\phi - \chi^{-1}(\chi_0)]$, we come from Eq. (21) to the final expression

$$\sum_\alpha f_{\alpha,\alpha}(X,P;Y,Q;t) = \{ \delta[Y - X_0(X,P,t)] \times \delta[Q - P_0(X,P,t)] \}_\xi. \quad (22)$$

In Eq. (22) $X_0 = X_0(X,P,t)$, $P_0 = P_0(X,P,t)$ is the solution of the classical equation of the motion backwards in time and we omit the sum over α since for given X and P there is only one classical trajectory with the specified initial coordinate Y and momentum Q.

We note that Eq. (22) can also be derived in a more accurate way by using the stationary phase method. In fact, let us continue expansion in Eq. (20) to higher power of x and y. Then the equation for the saddle point has the form $\boldsymbol{\nabla}[(P - P_t^\alpha)x + (P_0^\alpha - Q)y - \frac{1}{24}(S_{XXX}^\alpha x^3 + 3S_{XXY}^\alpha x^2 y + 3S_{XYY}^\alpha xy^2 + S_{YYY}^\alpha y^3) + \ldots] = 0$. For given P, Q this equation has a solution $x = x^*$, $y = y^*$ and one should estimate the second derivative at this point. It is easy to see that second derivatives from the function in square brackets tend to zero when x^* and y^* approach zero. Thus only the point $x^* = 0$, $y^* = 0$ is important in the semiclassical limit and we can approximate the integral (19) by a 2-dimensional integral of Airy type

$$f_{\alpha,\alpha}(X,P;Y,Q;t) = \left\{ -\frac{\partial P_t^\alpha}{\partial Y} \frac{1}{(2\pi\hbar)^2} \int \int dx\, dy \times \exp\left(\frac{i}{\hbar} \left[(P - P_t^\alpha)x + (P_0^\alpha - Q)y - \frac{1}{24}(S_{XXX}^\alpha x^3 + 3S_{XXY}^\alpha x^2 y + 3S_{XYY}^\alpha xy^2 + S_{YYY}^\alpha y^3) \right] \right) \right\}_\xi. \quad (23)$$

To proceed further let us assume for simplicity that $S_{XXY}^\alpha = S_{XYY}^\alpha = 0$ and $S_{XXX}^\alpha = S_{YYY}^\alpha = 8s$. Then Eq. (23) takes the form

$$f_{\alpha,\alpha}(X,P;Y,Q;t) = \left\{ -\frac{\partial P_t^\alpha}{\partial Y} \left(\frac{1}{s^{1/3}\hbar^{2/3}} \right)^2 \mathrm{Ai}\left(\frac{P_t^\alpha - P}{s^{1/3}\hbar^{2/3}} \right) \mathrm{Ai}\left(\frac{Q - P_0^\alpha}{s^{1/3}\hbar^{2/3}} \right) \right\}_\xi. \quad (24)$$

We remind the reader that the Airy function is normalized by the unit $\int \mathrm{Ai}(\nu) d\nu = 1$ and has the asymptotes $\mathrm{Ai}(\nu) \approx (4\pi)^{-1/2} \nu^{-1/4} \exp(-\frac{2}{3}\nu^{3/2})$ for $\nu \gg 0$ and $\mathrm{Ai}(\nu)$

$\approx \pi^{-1/2}\nu^{-1/4}\sin(-\frac{2}{3}|\nu|^{3/2}+(\pi/4))$ for $\nu \ll 0$. When we average the right hand side of Eq. (24) over ξ, the oscillating tails of the Airy function disappear and with good accuracy one can approximate Eq. (24) by Eq. (21). We also note that this approximation is consistent with the limit $t\to\infty$ and does not imply any condition on the intensity of the stochastic process $\xi(t)$. In fact, the variations of the momentum $P_0^\alpha(X,Y)$ and $P_t^\alpha(X,Y)$ due to the stochastic term is proportional to $\int_0^t\xi(t)dt=(t/\tau)^{1/2}$ [see Eq. (26) below], while the ''characteristic width'' of the Airy function, given by $s^{1/3}\hbar^{2/3}\sim[S(X,Y,t)]^{1/3}$, grows only as $t^{1/3}$.

We have shown that the ''diagonal'' terms in Eq. (18) correspond to classical evolution of the system. Having this result in mind we conclude that the transition to classical dynamics takes place under the condition of the terms with $\alpha\neq\beta$ in Eq. (18) vanishing. Let us obtain this condition. The use of the stationary phase method for $\alpha\neq\beta$ brings the complex prefactor $\exp\{i[S^\alpha(X,Y,t)-S^\beta(X,Y,t)]/\hbar\}$. Therefore, we can estimate $f_{\alpha,\beta}(X,P;Y,Q;t)$ to order of magnitude precision as

$$f_{\alpha,\beta}(X,P;Y,Q;t)\sim\left\{\exp\left(\frac{i}{\hbar}[S^\alpha(X,Y,t)-S^\beta(X,Y,t)]\right)\right\}_\xi. \tag{25}$$

Let us denote by $S_0(X,Y,t)$ the value of the principal Hamilton's function for $\epsilon=0$, and by $\delta S(X,Y,t)$ the variation of $S(X,Y,t)$ due to the stochastic term $[S(X,Y,t) = S_0(X,Y,t)+\delta S(X,Y,t)]$, and let $x_0(t)$ be the trajectory connecting the points X, Y in the case $\epsilon=0$ $[x(t)=x_0(t)+\delta x(t)]$. We restrict ourselves to the case where the operator $\hat{V}$ of the system interaction with the environment is the function of the coordinate x, i.e., $\hat{V}=V(x)$. To first order over parameter ϵ we have

$$\begin{aligned}S(X,Y,t)&=\int_0^t\left[\frac{m\dot{x}^2}{2}-U(x,t)-\epsilon\xi(t)V(x)\right]dt\\&\approx S_0(X,Y,t)+\epsilon\int_0^t[m\dot{x}_0\delta\dot{x}-U'(x_0,t)\delta x]dt\\&\quad-\epsilon\int_0^t\xi(t)V(x_0)dt\\&=S_0(X,Y,t)-\epsilon\int_0^t\xi(t)V(x_0)dt\end{aligned} \tag{26}$$

[we have used that $\delta x(0)=\delta x(t)=0$ and $m\ddot{x}_0=-U'(x_0,t)$]. Substituting Eq. (26) in Eq. (25) we obtain

$$f_{\alpha,\beta}(X,P;Y,Q;t)\sim\left\{\exp\left(\frac{i\epsilon}{\hbar}\int_0^t\xi(t)[V(x_0^\alpha)-V(x_0^\beta)]dt\right)\right\}_\xi. \tag{27}$$

Equation (27) can be simplified if we use the fact that the typical distance between two different chaotic trajectories x_0^α and x_0^β coincides with the characteristic length a for variation of the potential $U(x,t)$. Approximating $[V(x_0^\alpha)-V(x_0^\beta)]\sim V'a$, where V' is the characteristic value for $\partial V/\partial x$, we obtain

$$\begin{aligned}&f_{\alpha,\beta}(X,P;Y,Q;t)\\&\quad\sim\left\{\exp\left[\frac{i\epsilon V'a}{\hbar}\int_0^t\xi(t)dt\right]\right\}_\xi\\&\quad=\int_{-\infty}^{\infty}d\nu\,\exp\left[\frac{i\epsilon V'a}{\hbar}\nu\right](2\pi t/\tau)^{-1/2}\exp\left[-\frac{\nu^2}{2(t/\tau)}\right]\\&\quad=\exp(-\kappa t),\quad \kappa=\frac{(\epsilon V'a)^2\tau}{2\hbar^2}.\end{aligned} \tag{28}$$

Now we are able to formulate the condition of correspondence. We consider the case of chaotic dynamics. In this case the number of the classical trajectories contributing to Eq. (16) grows exponentially with increment η defined by the Lyapunov exponent λ: $\eta=\eta(\lambda)\sim\lambda$.[25] Thus one has to estimate by absolute value a series of the following form:

$$A(t)=\sum_{\alpha,\beta=1}^{N=\exp(\eta t)} f_{\alpha,\beta}\exp(-\kappa t),\quad |f_{\alpha,\beta}|\sim 1, \tag{29}$$

where $A(t)$ stands for the overall contribution of the interference terms $f_{\alpha,\beta}(X,P;Y,Q;t)$ with $\alpha\neq\beta$. The function $A(t)$ decays if $\kappa>2\eta$. Therefore, a required condition has the form

$$\frac{\epsilon^2V'^2a^2\tau}{\hbar^2}>\eta(\lambda). \tag{30}$$

In this formula the quantity $\epsilon V'a$ has units of energy and can be considered as a characteristic energy of the system interaction with the environment, τ is the ''memory time'' of the environment, and η is proportional to the Lyapunov exponent of the system considered. It is also easy to see that condition (30) is completely equivalent to that displayed in Sec. II [see Eq. (7)].

IV. WIGNER FUNCTION FOR DIFFERENT TOPOLOGIES OF PHASE SPACE

In the next section we shall check the condition obtained for a particular chaotic system known as the quantum standard map on the torus (QSMT). The Hamiltonian of this system has the form

$$H=\frac{p^2}{2}+u(x)\sum_n\delta(t-n),\quad u(x+2\pi)=u(x), \tag{31}$$

where periodic boundary conditions on both x and p are imposed. (In the numerical simulation we use scaled variables, the semiclassical parameter of the system is defined by the scaled Planck constant entering the momentum operator.) Because of the specific form of the potential $U(x,t)=u(x)\Sigma\,\delta(t-n)$, dynamics of QSMT can be explicitly described by the unitary map, which greatly simplifies the numerical simulation. This simplicity makes the system (31) one of the most popular models for studying the various aspects of quantum chaos.[26–32,16] Our aim is the system dynamics in terms of the Wigner function. Since there is a difference in the definition of the Wigner function for different topologies of the system phase space (plain, cylinder or torus), we consider a general discussion of the Wigner function structure to be useful.

In the case of plain phase space the Wigner function $w(X,P,t)$ is a continuous function of the coordinate X and the momentum P and is uniquely defined by the system density matrix $\hat{\rho}$. If we use the coordinate representation for the density matrix $\hat{\rho}(t)=\rho(x,x',t)$, then the Wigner function is defined according to Eq. (11). In the momentum representation the formula looks similar

$$w(X,P,t)=\frac{1}{2\pi\hbar}\int dp\ \exp\left(\frac{iXp}{\hbar}\right)\rho\left(P+\frac{p}{2},P-\frac{p}{2},t\right). \tag{32}$$

The properties of the Wigner function (32) are well known[21,22] and do not need to be cited here.

The properties of the Wigner function in the case of cylinder phase space ($-\infty<P<\infty$, $0\leqslant X<2\pi$) are less well known. The Wigner function in a cylinder was introduced for the first time in Ref. 33 and was then used in analysis of the quantum kicked rotor dynamics in Refs. 34, 6, and 7. If we denote by $\langle n|\hat{\rho}(t)|m\rangle$ the elements of the system density matrix in the momentum representation [momentum eigenfunctions are $|n\rangle=(2\pi)^{-1/2}\exp(inx)$], then the Wigner function can be defined as follows (the formula is adopted from Ref. 6):

$$w(X,P_l,t) =\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\exp(iXj)\,\frac{1+(-1)^{l+j}}{2}\left\langle\frac{l+j}{2}\right|\hat{\rho}(t)\left|\frac{l-j}{2}\right\rangle, \tag{33}$$

where l labels the quantized values of the momentum $P_l=\hbar l/2$. We note that P_l is multiple of a $\hbar/2$ but not $\hbar$ as is intuitively expected. This ''$\hbar/2$''-quantization rule leads to the appearance of a virtual interference pattern, which we shall discuss below for the case of torus phase space.

Our prime interest is in the case of torus phase space. The periodic boundary condition along the momentum axis is satisfied by a special choice of the Planck constant $\hbar=2\pi/N$. Then the quantum Hamiltonian (31) defines an N-level system with basis functions $|n\rangle=(2\pi)^{-1/2}\times\exp(inx_k)$, where $x_k=2\pi k/N$, $k=0,\ldots,N-1$. Therefore, the Wigner function is now defined on grid of the size $2N\times 2N$ and the formula (33) takes the form

$$w(X_k,P_l,t) =\sum_{j=0}^{2N-1}\exp\left(i\frac{\pi kj}{N}\right)\frac{1+(-1)^{l+j}}{2}\left\langle\frac{l+j}{2}\right|\hat{\rho}(t)\left|\frac{l-j}{2}\right\rangle, \tag{34}$$

where $P_l=(\hbar/2)l=\pi l/N$ and $X_k=\pi k/N$ $(0\geqslant X,P>2\pi)$. Let us discuss this particular case in more detail.

The first row in Fig. 1 shows evolution of the Wigner function $w(X_k,P_l,t)$ for the case of the initial state chosen in the form of the wave packet with minimal uncertainty. The packet was centered at $X=0$, $P=0$, and the function $u(x)=K\cos(x)$ in Eq. (31). Here, in Fig. 1, and below we use a three-color representation for the Wigner function. Red corresponds to a positive value of $w(X_k,P_l,t)$, blue denotes negative value and white corresponds to the value of the Wigner function being near zero $|w(X_k,P_l,t)|<\varepsilon$ (ε is a small number of the order of N^{-2}). It is seen that the initial wave packet has three images at $P=0$, $X=\pi$, and $P=\pi$, $X=0,\pi$. We shall call them ''virtual wave packets'' because they disappear after partial integration of the Wigner function. In the other words, the functions $w_x(X_k,t)=\sum_{l=0}^{2N-1}w(X_k,P_l,t)$ and the functions $w_p(P_l,t)=\sum_{k=0}^{2N-1}w(X_k,P_l,t)$ have only one peak. We also note that the partial integration recovers the usual quantization rule, i.e., $w_p(P_l,t)\equiv 0$ for odd l and $w_x(X_k,t)\equiv 0$ for odd k.

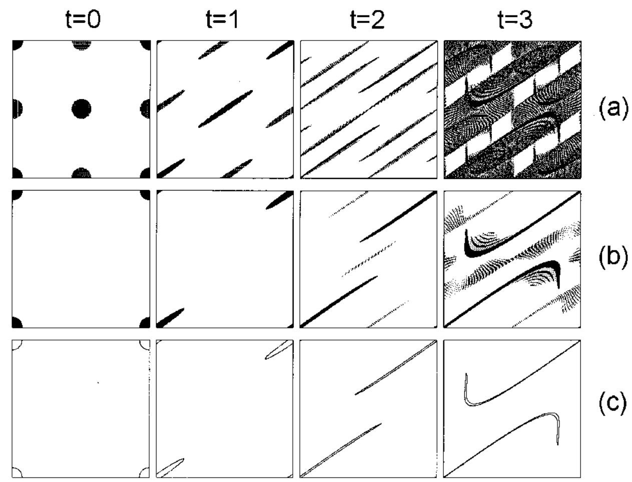

FIG. 1. (a) Dynamics of the Wigner function for QSMT in the case $u(x)=K\cos x$, $K=1.71$. The initial state is chosen in the form of the wave packet with minimal uncertainty ($\hbar=2\pi/N$, $N=128$). The grid of the size $2N\times 2N$ is shown. Red corresponds to positive value of $w(X_k,P_l,t)$, blue denotes negative value and white corresponds to value of the Wigner function near zero ($|w(X_k,P_l,t)|<0.000015\sim N^{-2}$). (b) Dynamics of the coarse-grained Wigner function. (c) Contour line for the classical distribution function $f(X,P,t)$ for the same initial state.

Since $w(X_k,P_l,t)$ is a rapidly oscillating function in the vicinity of the virtual wave packets, one can try to remove this specific interference pattern by coarse-graining of the Wigner function $\widetilde{w}(X,P,t)=\int dx\ dp\ B(x,p)w(X+x,P+p,t)$, using, for example, a Gaussian with variance σ_x, σ_p as $B(x,p)$. (The variance should be taken less than $\hbar/2$, otherwise one removes any interference pattern.) However, this procedure does not lead us to the desired results. With time the virtual packets give rise to the interference pattern in a larger scale (see Fig. 1, second row) and after a rather short time it becomes impossible to distinguish between the ''imaginary'' and ''real'' interference patterns.

In what follows we use a different approach in dealing with the virtual interference. Without being universal it nevertheless completely suits our main purpose—the study of the transition between quantum and classical dynamics. Instead of the function (34) we shall consider the modified Wigner function

$$\widetilde{w}(X_k,P_l,t)=\tfrac{1}{4}[w(X_k,P_l,t)+w(X_{k+N},P_l,t)+w(X_k,P_{l+N})+w(X_{k+N},P_{l+N},t)]. \quad (35)$$

It is easy to show that the function $\widetilde{w}(X_k,P_l,t)$ equals zero for odd l, k, and therefore the modified Wigner function is defined on the grid $N\times N$. In the classical case the procedure (35) means that we consider the system dynamics on modulo π instead of modulo 2π.

V. TRANSITION TO THE CLASSICAL DYNAMICS FOR QSMT

We proceed to the study of the transition from quantum to classical dynamics for QSMT. In the numerical simulation we choose the momentum eigenfunction $|0\rangle$ as the initial state of the system. In the classical approach this initial state corresponds to an ensemble of the classical particles with $P=0$ and uniform distribution of X over the interval $[0,2\pi)$. The function $u(x)$ in the Hamiltonian (31) is chosen in the form

$$u(x)=\left[K\left(\pi x-\frac{x^2}{2}\right)\right]. \quad (36)$$

In this special case the classical evolution of the system can be described by the linear map

$$P'=[P+K(x-\pi)]\text{mod:}\pi,$$
$$X'=[X+P']\text{mod:}\pi. \quad (37)$$

(For the purpose of the comparison with the quantum case we consider the classical dynamics on modulo π instead of modulo 2π.) For $K>0$ this map is homogeneously unstable with the Lyapunov exponent $\lambda=\ln\{(2+K)/2+[(2+K)^2/4-1]^{1/2}\}$ given by the eigenvalue of the stability matrix of Eq. (37). For the first six time steps the evolution of the classical ensemble is illustrated by the third row in Fig. 2 for $K=1.71$, and Fig. 3 for $K=2$. Here the dynamics of the particles were additionally influenced by noise. The unperturbed dynamics would correspond to zero width of the branches.

The first row in Fig. 2 shows a typical evolution of the modified Wigner function (35) for $\hbar=2\pi/N$, $N=512$. (Quarter of the phase space $0\leq X,P<\pi$ is shown.) It is seen that the Wigner function exhibits a rich interference pattern already after the first time step. We especially note that the variation of $\widetilde{w}(X_k,P_l,t)$ is rather large and exceeds the mean value (equal to N^{-2}) by a factor of between 10 and 20.

A remarkable feature of QSMT with $u(x)$ in the form (36) is that along with the typical interference pattern shown in Fig. 2 it can display a very specific pattern for integer values of K. [For integer K the map (37) defines the so-called Arnold's cat map.] This interference pattern is shown in Fig. 3 for $K=2$. Now the function $\widetilde{w}(X_k,P_l,t)$ equals zero everywhere except at N points where it takes constant value $\widetilde{w}(X_k,P_l,t)=1/N$. Thus evolution of the Wigner function coincides with evolution of an ensemble of $M=N$ classical particles, which move also on the grid for integer K. The connection of the quantum dynamics with classical dynamics in discrete phase space was mentioned for the first time in Ref. 35, where the authors studied quantum evolution of the kicked rotor. Then this idea was explored and partially justified in papers[34] devoted to the kicked rotor dynamics in the Wigner representation. The quantum Arnold's cat map considered here seems to be a unique system where (as it was first shown in Refs. 28 and 29) the correspondence between quantum dynamics and ''discrete classical dynamics'' is exact. Of course, this case is an exception to the rule. However, because of the transparent character of the inter-

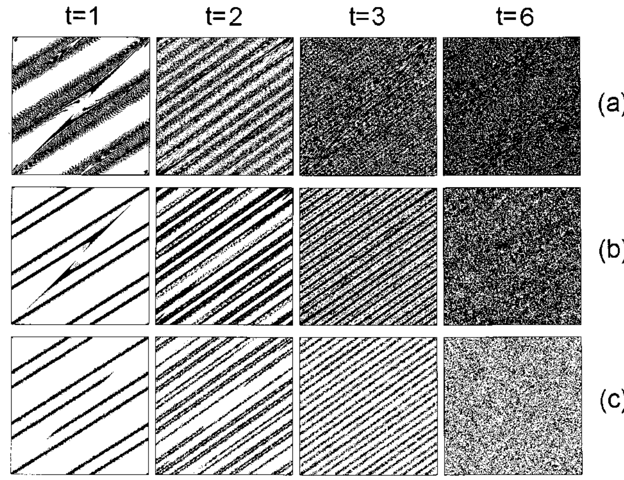

FIG. 2. Comparison between the quantum ($\hbar=2\pi/N$, $N=512$) and classical dynamics for $u(x)$ in the form (36) with $K=1.71$. One quarter of the phase space ($0\leq X,P<\pi$) is shown. Initial state corresponds to $P=0$ and uniform distribution over X. (a) Unperturbed dynamics of the modified Wigner function $\widetilde{w}(X_k,P_l,t)$. (b) The modified Wigner function averaged over 128 realizations of the stochastic process $\xi(t)$ with intensity $\epsilon=0.05$. (c) Dynamics of an ensemble of $M=24064$ classical particles for the same value of ϵ.

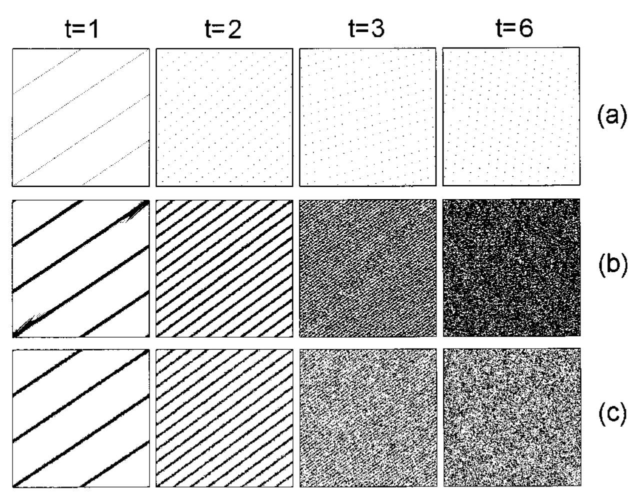

FIG. 3. The same as in Fig. 2, but $K=2$.

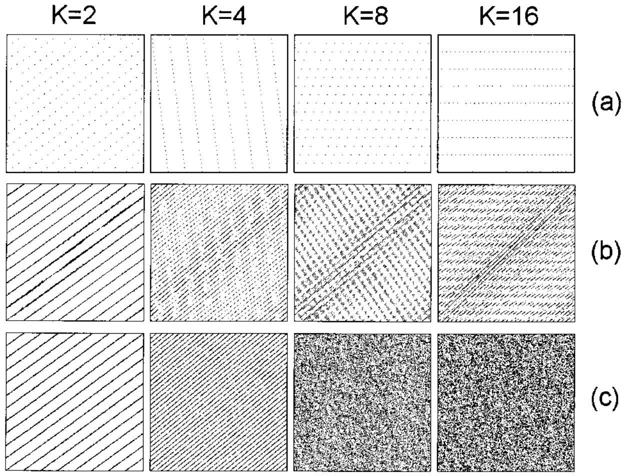

FIG. 4. The Wigner function for $t=2$ and different values of K: (a) $\epsilon=0$, (b) $\epsilon=0.01$, (c) classical distribution function for $\epsilon=0.01$.

ference pattern exhibited, it is ideally suited for the numerical study of the transition between quantum and classical dynamics.

Let us turn to our main aim—a conversion of quantum dynamics affected by a stochastic process (which mimics the influence of the environment) to classical dynamics. We choose the interaction operator in Eq. (9) in the form of the first derivative from $U(x,t)$, i.e., $\hat{V}=\partial U/\partial x$. (Such a choice for the interaction operator will be explained in Sec. VI.) The middle row in Fig. 2 and Fig. 3 shows the Wigner function averaged over 128 runs of the stochastic process $\xi(t)$. It is seen that interference pattern disappears and we have a good correspondence with classical dynamics influenced by noise of the same intensity.

Varying the intensity of the noise one can find a critical value of ϵ necessary to destroy the quantum coherence and to convert the Wigner function into the classical distribution function. The numerical simulation performed confirms that the critical value ϵ_{cr} is proportional to $\hbar$ (i.e., to the semiclassical parameter of the system, if one uses unscaled variables). This is actually not surprising, since ϵ and $\hbar$ enter into all formulas only through the combination $\epsilon/\hbar$. The dependence of ϵ_{cr} on the Lyapunov exponent λ is less obvious. To check the dependence predicted we performed the numerical simulation for different values of λ [different K in Eq. (36)] at fixed value of ϵ. The results are presented in Fig. 4. It is seen that as K increases the Wigner function exhibits signs of an interference pattern.

VI. CLASSICAL MECHANICS AS THE "$m\to\infty$"-LIMIT OF QUANTUM MECHANICS

In the previous sections we have obtained the condition of correspondence (30) between the classical and quantum dynamics and checked it numerically for a particular chaotic system. In this section we discuss an issue of the result (30) for the problem of crossover from quantum to classical mechanics when we pass the border between micro- and macroworlds.

Usually, classical mechanics is considered as a limiting case of quantum mechanics when the mass of a particle tends to infinity. In fact, it is easy to show that for a generic system the limit $m\to\infty$ is equivalent to the limit $\hbar\to 0$ and, therefore, one has the formal transition from quantum to classical mechanics. However, there is a well known shortcoming in this simple scheme—the limits $m\to\infty$ and $t\to\infty$ do not commute. In other words, for any finite m we have a finite time of correspondence between quantum and classical dynamics. For a chaotic system, as it was shown in Sec. I, this time is too short for this naive approach to be realized in nature. In this section we discuss a different way for the transition to classical mechanics which is free of the above mentioned shortcoming.

First of all we note that the limit $m\to\infty$ necessarily implies a complex particle consisting of many "elementary" particles. Let

$$\hat{H}_{\rm tot}=\sum_{i=1}^{N} -\frac{\hbar^2}{2m_i}\frac{\partial^2}{\partial x_i^2}+\sum_{i\neq j} W(x_i-x_j)+\sum_i q_iU(x_i,t) \tag{38}$$

be the total Hamiltonian of such a particle. Here x_i denotes the coordinates of the "elementary" particles (atoms), $W(x_i-x_j)$ is the interaction between them, and $U(x,t)$ is some external potential field. We are interested in the dynamics of the particle as a whole. Denoting by x the coordinate of the gravity center we reduce Hamiltonian (38) to the form

$$\hat{H}_{\rm tot}\approx\hat{H}+\hat{H}_R+d\frac{\partial U}{\partial x}, \tag{39}$$

$$\hat{H}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+qU(x,t), \tag{40}$$

$$\hat{H}_R=\sum_{i=1}^{N} -\frac{\hbar^2}{2m_i}\frac{\partial^2}{\partial \zeta_i^2}+\sum_{i\neq j} W(\zeta_i-\zeta_j), \tag{41}$$

where $m=\Sigma m_i$, $q=\Sigma q_i$, $d=\Sigma q_i\zeta_i$, $\zeta_i=x-x_i$. Thus any macroscopic particle is an open system, even if we neglect its interaction with the cosmic background radiation. Its own internal degrees of freedom can play the role of an environment.

A self-closed equation of motion for the system gravity center can be obtained under the assumption of chaotic dynamics for the internal degrees of freedom.[20,36] This equation coincides with Eq. (8) where we should substitute ϵ by mean squared fluctuation of the particle dipole moment d and operator $\hat{V}$ by $\hat{V}=\partial U/\partial x$. As was shown in Sec. III, Eq. (8) provides a transition to classical dynamics if the condition (30) is fulfilled. For a generic chaotic system the Lyapunov exponent λ is of the same order of magnitude as the characteristic frequency of macro-motion and scales as $1/\sqrt{m}$. The fluctuation of the dipole momentum grows with the size (and, hence, the mass) of a particle. The correlation time τ is fixed at some value which is defined by the characteristic frequency of the internal motion. Thus the condition (30) will be necessarily satisfied when we consider the limit $m \to \infty$.

We especially note that the condition (30) can already be met for a particle with mass of a few atoms. In fact, let us consider a charged ($q = 1.6 \times 10^{-19}$ C) fragment of a multi-atom molecule ($m \sim 10^{-24}$ kg) moving in typical laboratory field $U \sim 10^5$ V, $\partial U/\partial x \sim 10^7$ V/m. Then, the characteristic frequency of macro-motion and the Lyapunov exponent $\lambda \sim (qU''/m)^{1/2} \sim 10^7$ s^{-1}. The inverse correlation time τ^{-1} is of the order of vibration frequency, i.e., $\tau \sim 10^{-14}$ s. Taking the mean squared fluctuation of the order of molecule dipole momentum $d \sim 10^{-29}$ we satisfy the condition with a factor of more than 10.

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