Decoherence with a Chaotic Environment: Quantum Walk with the Quantum Baker Map

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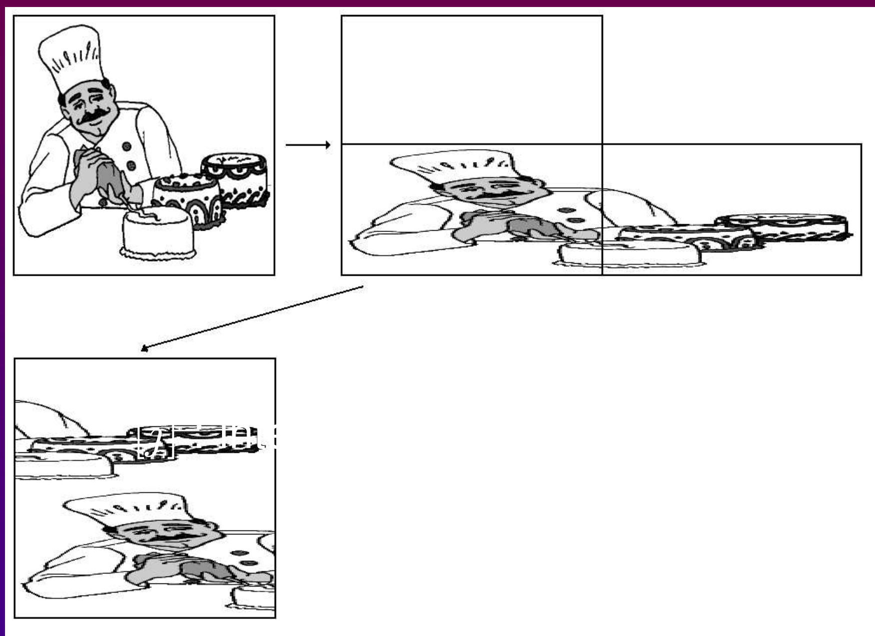
- Conclusions

Classical Baker Map



Transformation in $q, p \in [0,1]$

$$
\begin{aligned}
q_{i+1} &= 2q_i - [2q_i] \\
p_{i+1} &= (p_i + [2q_i])/2
\end{aligned}
$$

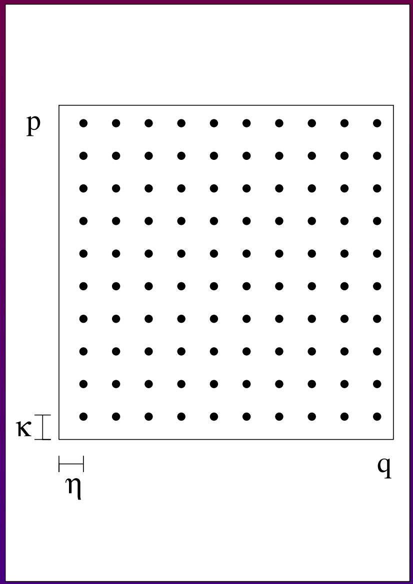
Symbolic Dynamics

$q = 0.\epsilon_0\epsilon_1\ldots,\ \ p = 0.\epsilon_{-1}\epsilon_{-2}\ldots$

$$
\begin{aligned}
(p,q) &= \ldots\epsilon_{-2}\epsilon_{-1} \bullet \epsilon_0\epsilon_1\epsilon_2\epsilon_3\ldots \\
&\downarrow\ \mathcal{B} \\
(p',q') &= \ldots\epsilon_{-2}\epsilon_{-1}\epsilon_0 \bullet \epsilon_1\epsilon_2\epsilon_3\ldots
\end{aligned}
$$

Quantum Baker Map

- Prequantization
  - D-dimensional Hilbert space
  - Imposing periodicities:
    $q_j = \frac{j+\eta}{D}; \quad p_k = \frac{k+\kappa}{D}$
    - $hD = 1$; $j, k = 0, \ldots, D-1$
    - $\eta, \kappa \in [0, 1) \rightarrow$ Floquet angles



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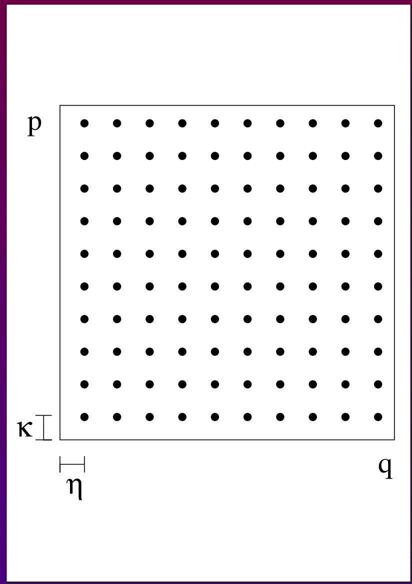

- Quantization

Converting $MSB(q) \rightarrow MSB(p)$ (Most Significative Bit)

$$B_{\text{BVS}} \equiv B^{\eta,\kappa}_{pos} = \left(F^{\eta,\kappa}_{D}\right)^{-1} \begin{pmatrix} F^{\eta,\kappa}_{\frac{D}{2}} & 0 \\ 0 & F^{\eta,\kappa}_{\frac{D}{2}} \end{pmatrix}$$

$(\hat{F}^{\eta,\kappa}_{D})_{kj} \equiv \langle p_k | q_j \rangle = \frac{1}{\sqrt{D}} e^{-i \frac{2\pi}{D}(j+\eta)(k+\kappa)}$

Quantum Baker Map Families

- QBM families on N qubits, for $n = 1, \ldots, N$

$$\hat{B}_{N,n} \equiv \hat{G}_{n-1} \circ \hat{S}_n \circ \hat{G}_n^{-1} = \left(\hat{I}_{2^{n-1}} \otimes \hat{B}_{\,N-n+1,\,1} \right) \circ \hat{S}_n$$

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$\hat{S}_n$: Shift operator acting only on the first n qubits

$$\hat{S}_n |x_1\rangle|x_2\rangle \ldots |x_n\rangle \ldots |x_N\rangle = |x_2\rangle \ldots |x_n\rangle|x_1\rangle \ldots |x_N\rangle$$

$\hat{G}_{n-1}$: Partial Fourier Transform

$$\hat{G}_n \equiv \hat{I}_{2^n} \otimes \hat{F}^{\eta,\kappa}_{2^{N-n}}, \qquad n = 0, \ldots, l$$

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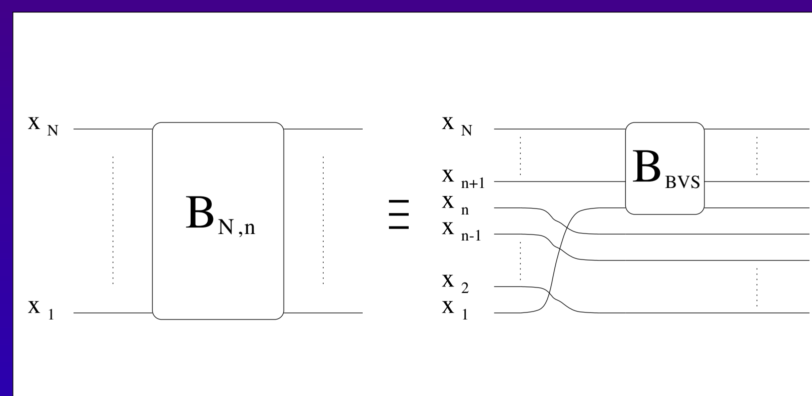
✐ Circuit Representation $\longrightarrow$



Quantum Walk

- Hilbert space: $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_C$
  - Particle: $\mathcal{H}_P$, $\{|j\rangle; j \in \mathbb{Z}\}$ (line)
  - Coin: $\mathcal{H}_C$, $\{|0\rangle, |1\rangle\}$

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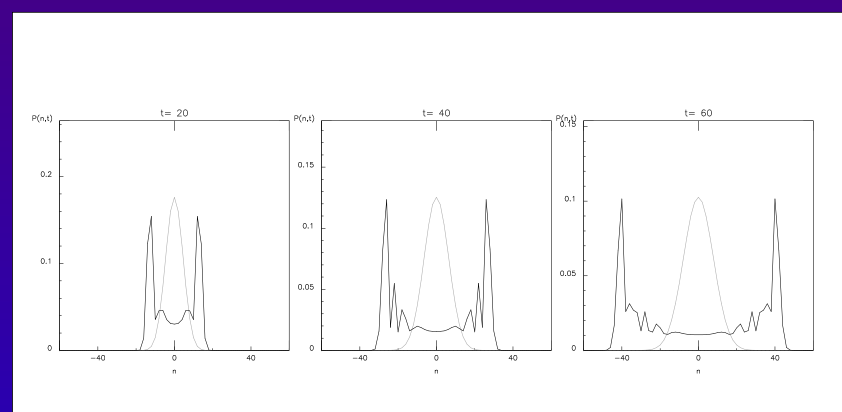
Hadamard Walk

$$\hat{C} = \hat{H} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



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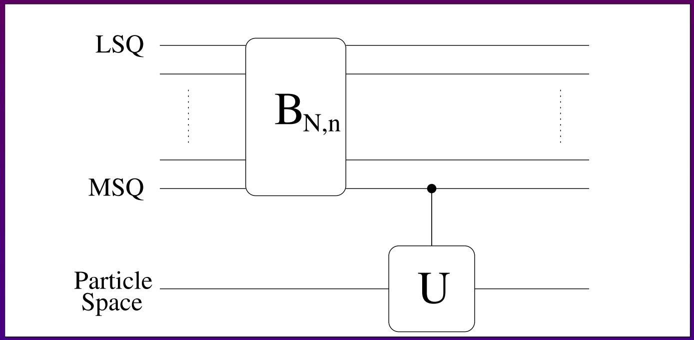

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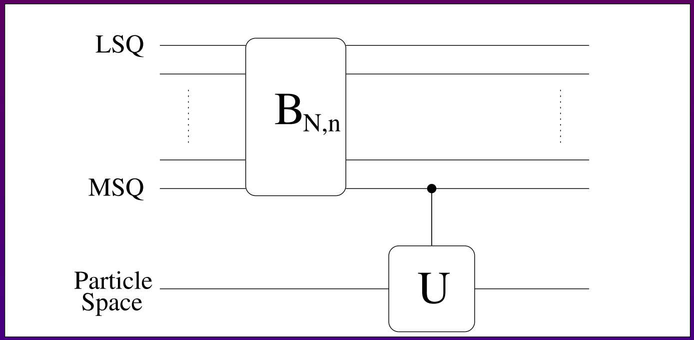

  - Momentum Representation: $\langle k|\hat{M}|k'\rangle = \delta_{k,k'}\hat{M}_k$

$$\hat{M}_k = \begin{pmatrix} e^{-\varphi_k} & 0 \\ 0 & e^{\varphi_k} \end{pmatrix} \hat{B}_{N,n}; \qquad \varphi_k = \begin{cases} k & \text{in a line} \\ \frac{2\pi k}{M} & \text{in a cycle} \end{cases}$$

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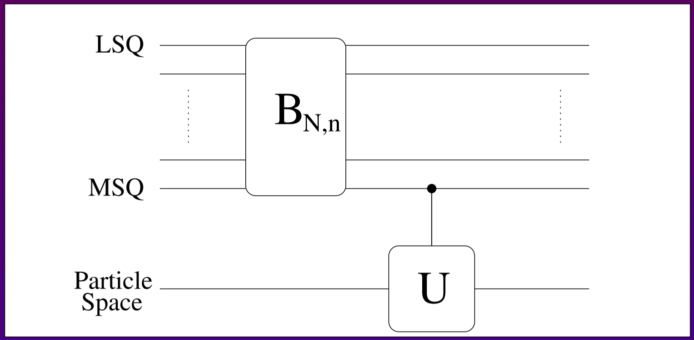

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- Probabilty distribution for $|\Psi_0\rangle = |0\rangle \otimes |\Phi_0\rangle$

$$p(x,t) = \int \frac{dk}{4\pi^2} \int dk' e^{-ix(k-k')} \langle\Phi_0|(\hat{M}_k^\dagger)^t(\hat{M}_{k'})^t|\Phi_0\rangle$$

Standard Deviation

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \sim \begin{cases} \text{CRW} & \rightarrow & O(\sqrt{t}) \\ \text{QW} & \rightarrow & O(t) \end{cases}$$

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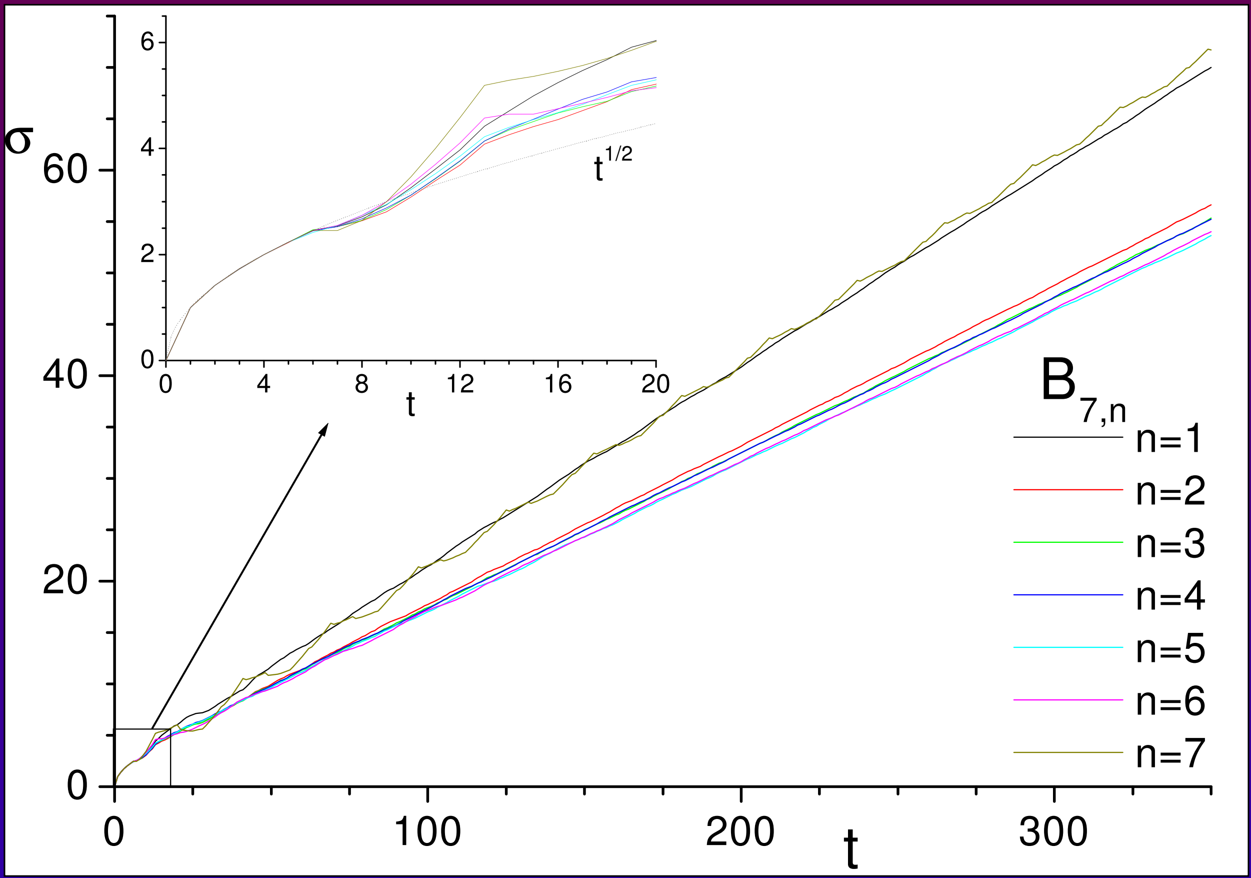

Classical: $(0 \leq t \leq N)$, Transition: $(N \leq t \leq D)$, Quantum: $(t \geq D = 2^N)$

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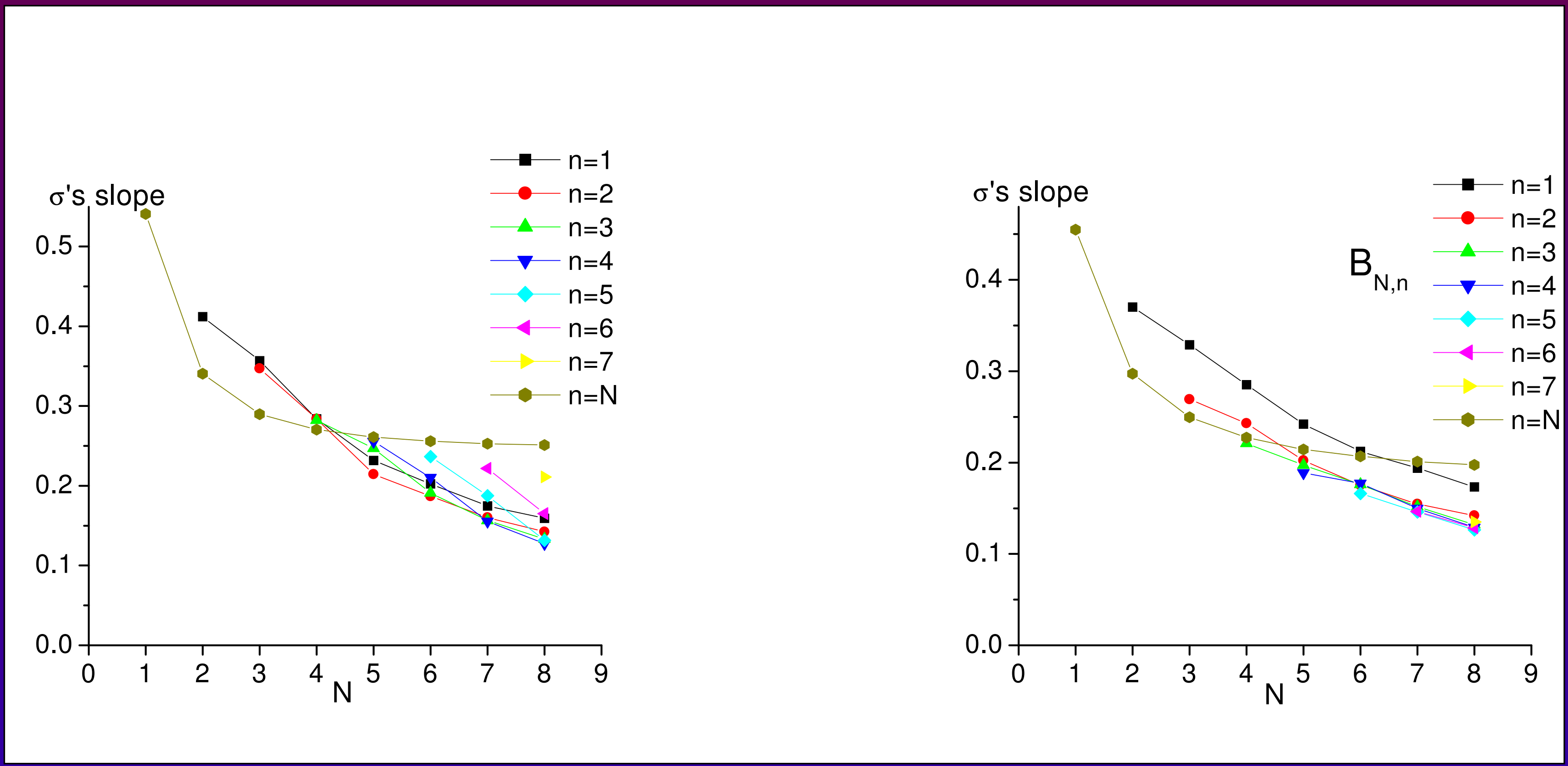
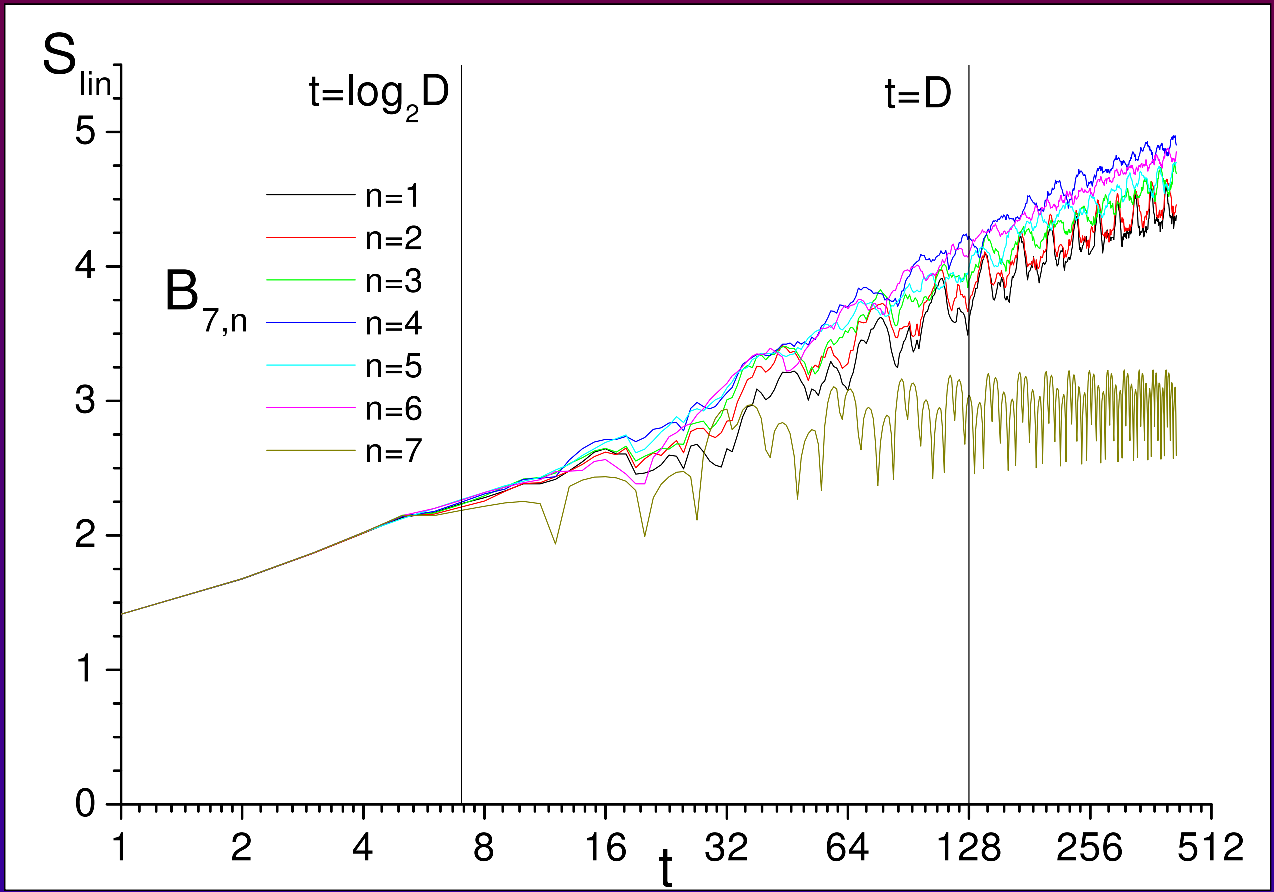


$|\psi_0 = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $\eta = \kappa = 0.5$, (left), $\eta = \kappa = 0$, (right).

Entanglement

$S_L \equiv -\ln\left(Tr[\rho_P^2]\right)$: entanglement between *Particle* and *Environment*.



Sat. time $\begin{cases} n = N & \rightarrow & t \sim O(N) \\ n \neq N & \rightarrow & t \sim O(D) \end{cases}$ Sat. value $\begin{cases} n = N & \rightarrow & S_0 \sim O(\log N) \\ n \neq N & \rightarrow & S_0 \sim O(N) \end{cases}$

Distance in phase-space

Wigner function $\longrightarrow W(q,p) = \frac{1}{M} Tr[\rho A(q,p)]$

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$$A(q,p) = U^q R V^{-p} \exp\left(i\pi pq/M\right);$$

$$U|n\rangle = |n+1\rangle,\ \ R|n\rangle = |-n\rangle,\ \ V|k\rangle = |k+1\rangle$$

- $W \in \Re$
- W is complete $\rho = \sum_{q,p} W(q,p) A(q,p)$
- Marginal probabilities $= \sum_{\mathrm{line}} W$

Distance to QRW (Δ):

$$\Delta = \sum_{q,p} [W_{crw}(q,p) - W(q,p)]^2 \propto Tr[(\rho_{crw} - \rho)^2]$$

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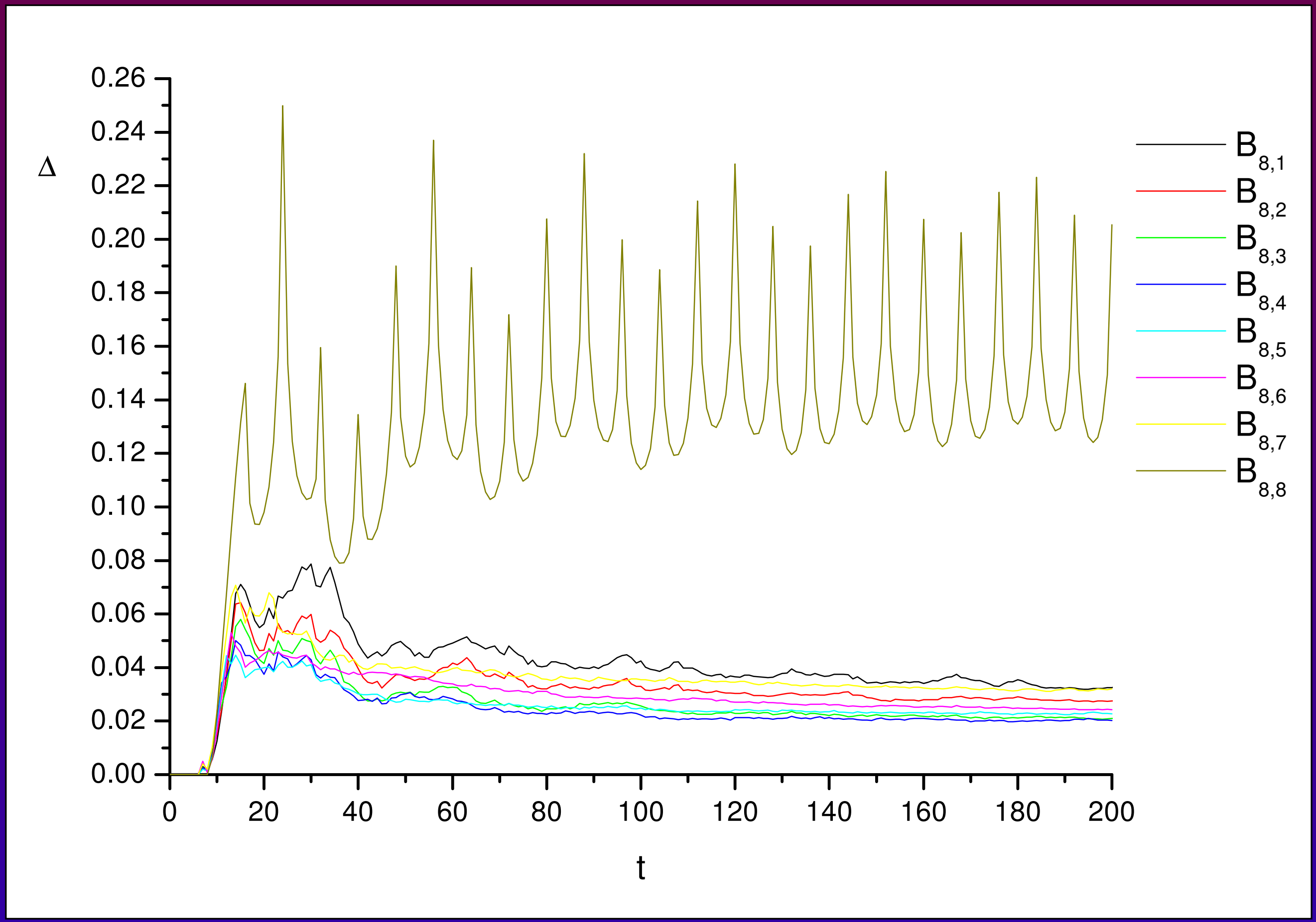


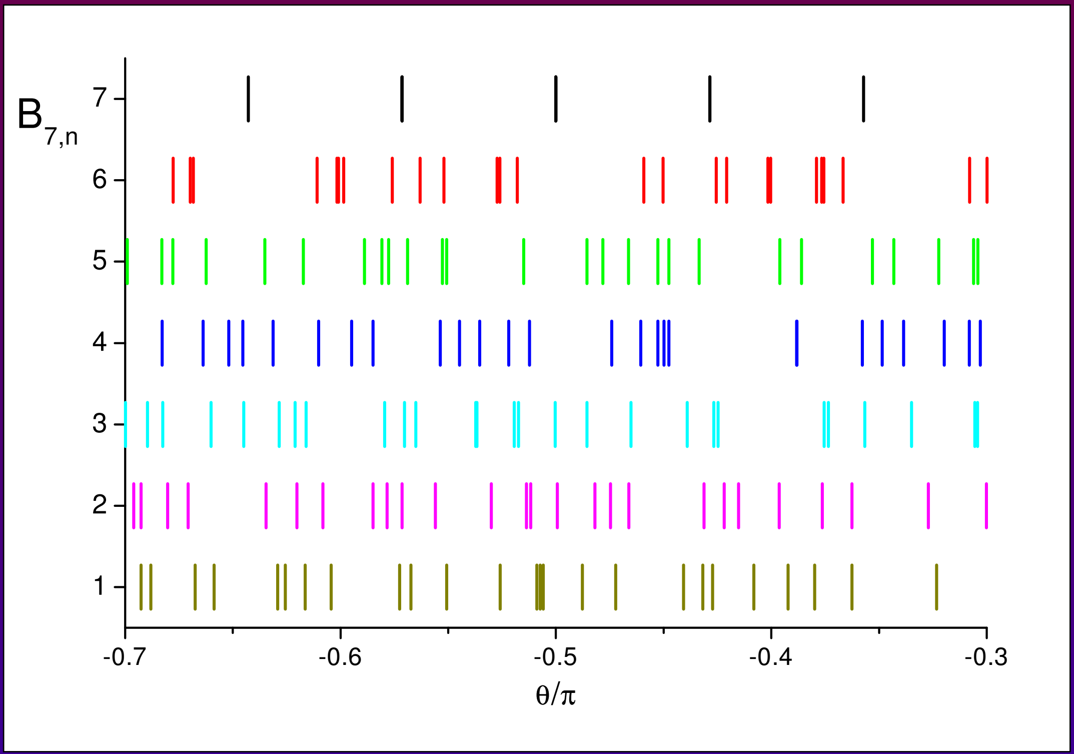
$$\begin{cases} n = N & \rightarrow \text{(significant) least decoherent} \\ n \sim \frac{N}{2} & \rightarrow \text{(fine) most decoherent} \end{cases}$$

(in coincidence with S_L and σ).

Spectrum

Consecutive level spacing

$$\hat{B}_{N,n}|\Phi_j\rangle = e^{i\theta_j}|\Phi_j\rangle$$



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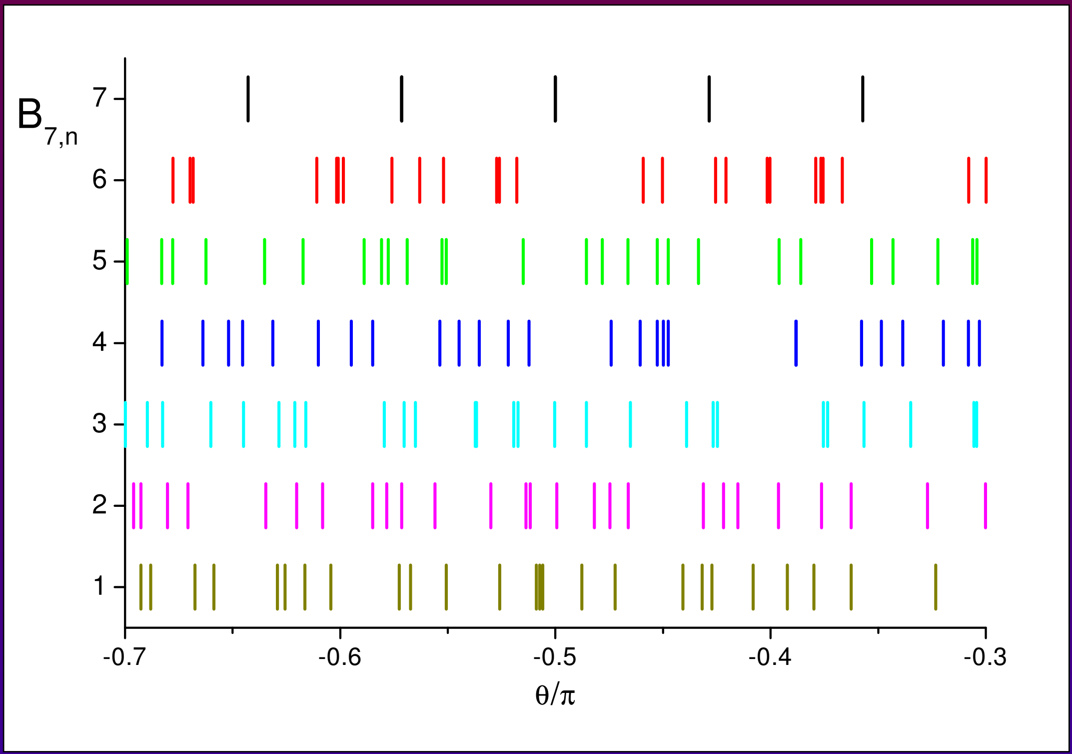

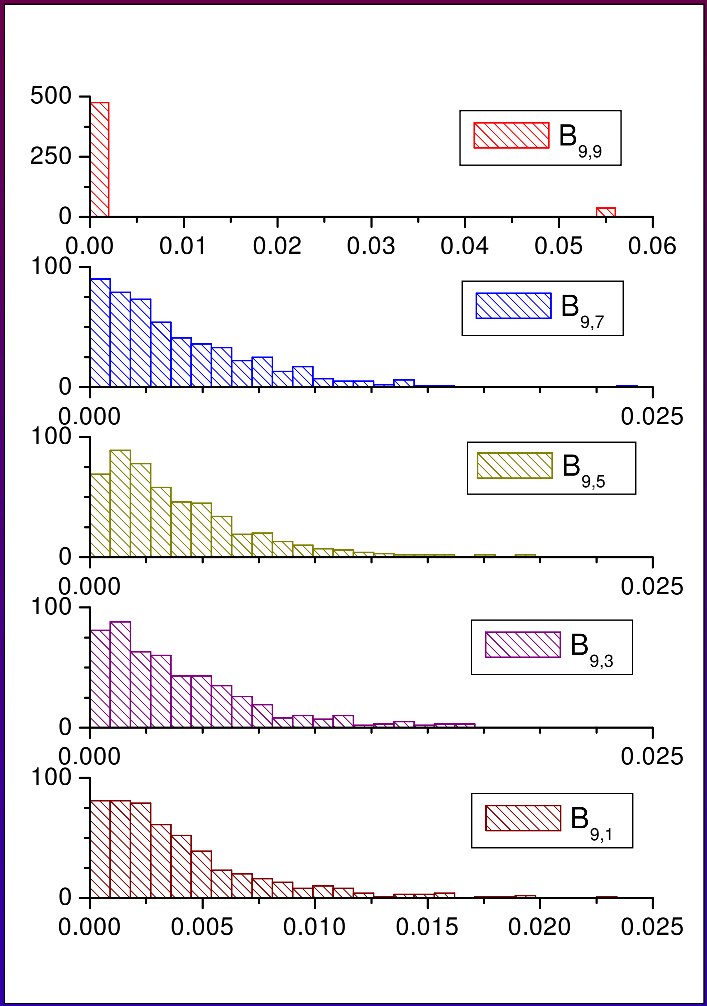

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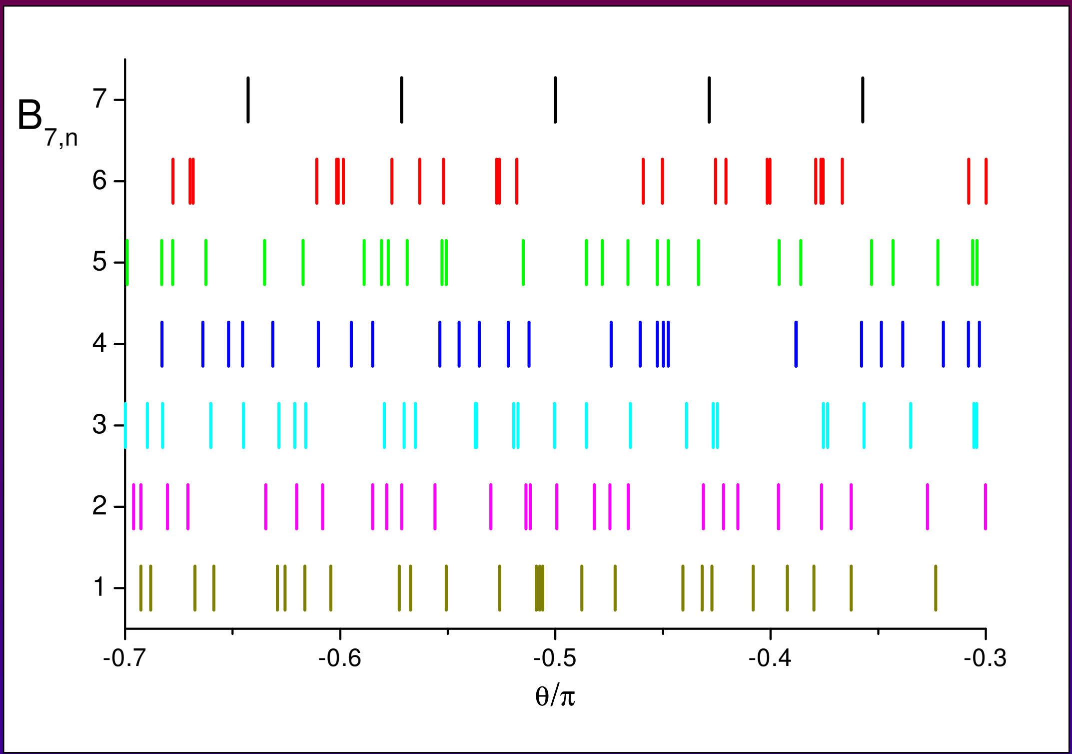

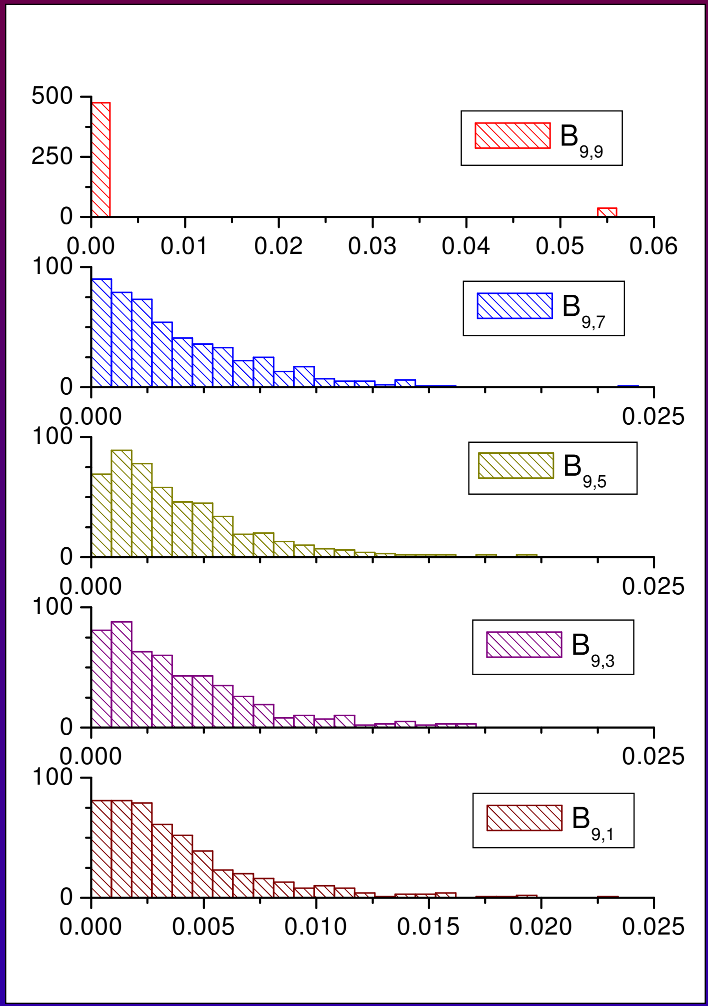

$B_{N,n}$ differs with RMT prediction because of the dimension ($D = 2^N$) and the simmetries

"most chaotic maps" (middle members) $\longrightarrow$ *"best environments"*

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- Middle members of QBM are the *best environments* and the *most chaotic maps*.

References

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