Classification of Tripartite Entanglement with one Qubit

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Introduction

Local operation on a shared entangled states*

$$
\begin{array}{ccccccccc}
\text{CC} & \Longleftrightarrow & \Longleftrightarrow & \Longleftrightarrow & \Longleftrightarrow & \Longleftrightarrow & \Longleftrightarrow & \Longleftrightarrow & \text{CC} \\
\Updownarrow & & & & & & & & \Updownarrow \\
\Updownarrow & & & & \text{Source} & & & & \Updownarrow \\
\text{Alice} & \leftarrow & n-\text{level} & \leftarrow & |\psi\rangle & \rightarrow & \text{qubit} & \rightarrow & \text{Carol} \\
\Updownarrow & & & & \downarrow & & & & \Updownarrow \\
\Updownarrow & & & & n-\text{level} & & & & \Updownarrow \\
\Updownarrow & & & & \downarrow & & & & \Updownarrow \\
\text{CC} & \Longleftrightarrow & \Longleftrightarrow & \Longleftrightarrow & \text{Bob} & \Longleftrightarrow & \Longleftrightarrow & \Longleftrightarrow & \text{CC}
\end{array}
$$

Stochastical local operations and classical communication (SLOCC).

$$\text{Bipartite case} \Rightarrow |\psi\rangle = \sum_i^n \sqrt{\lambda_i}\, |\lambda_i\rangle \otimes |\lambda_i'\rangle \Rightarrow \text{Schmidt Rank}$$

*C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, arXiv:quant-ph/9908073 (2000).

- Three qubits case $\Rightarrow$ two classes: W and GHZ*

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

*W Dür, G. Vidal and J. I. Cirac, Phys. Rev. A63, 062314 (2000).

- Other works concerning SLOCC classification:

1. Four qubits*
2. 2 qubits and one n-level system†
3. Aspects of SLOCC classification‡

*F. Verstraete, J. Dehaene, B. De Moor and H. Verschelde, Phys. Rev. A65, 052112 (2002).

†A. Miyake and F. Verstraete, Phys. Rev. A69, 012101 (2004).

‡A. Miyake, Phys. Rev. A67, 012108 (2003).

- Our work

1. the number of products in the smallest decomposition is a SLOCC invariant.*

2. Describe how to find these decompositions for entangled states with local supports $(n, n, 2)$.

3. Use these decompositions to get the SLOCC classification.

*W Dür, G. Vidal and J. I. Cirac, Phys. Rev. A63, 062314 (2000).

Tripartite states with one qubit

1. Let $|\psi\rangle$ an entangled state with local supports $(n, n, 2)$.

2. The local support of $|\psi\rangle$ on $s_{ab} = s_a + s_b$ is a 2D plane $\mathcal{P} \subset C_a^n \otimes C_b^n$.

3. $\mathcal{P} \subset C_a^2 \otimes C_b^2$ generated by entangled states has either:*

one product state $\Rightarrow W$ class

two product states$\Rightarrow GHZ$ class†

*A. Sanpera, R. Tarrach and G. Vidal, Phys. Rev. A58, 826 (1998).
†W Dür, G. Vidal and J. I. Cirac, Phys. Rev. A63, 062314 (2000).

4.

$$|\psi\rangle = \sum_{k=0,1} c_k|r_k\rangle|k\rangle \Rightarrow |\phi\rangle = \alpha_0\,|r_0\rangle + \alpha_1\,|r_1\rangle$$

where $|r_k\rangle \in \mathcal{P} \subset C_a^n \otimes C_b^n \Rightarrow |r_k\rangle$ span $\mathcal{P}$.

5. $|\phi\rangle$ can be seen as the linear mapping

$$\begin{array}{rcl} |\phi\rangle : C_a^{n*} & \longrightarrow & C_b^n \\ \langle u_a| & \longrightarrow & \langle u_a|\phi\rangle \end{array}$$

The rank of this linear mapping is the Schmidt rank of $|\phi\rangle$.

6. We are looking for α_0 and α_1 such that the equation

$$\langle u_a|(\alpha_0|r_0\rangle + \alpha_1|r_1\rangle) = 0 \tag{1}$$

has a at least one non-trivial solution $\langle u_a| \in C_a^{n*}$.

7. Interpretation of $|u_a\rangle$:

If we found s_a in state $|u_a\rangle \Rightarrow |\psi\rangle$ reduces to a product state.*

*Three qubits: A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre and R. Tarrach, Phys. Rev. Lett. 85, 1560 (2000).

8. Let $\{|i\rangle\}$ and $\{|j\rangle\}$ being basis in C_a^n and C_b^n

$$(\alpha_0 R_0 + \alpha_1 R_1)\, u_a^* = 0 \Rightarrow (R_1^{-1} R_0 - \lambda) u_a^* = 0$$

where $[R_k]_{ij} = \langle ji | r_k \rangle$, $u_{a_i}^* = \langle u_a | i \rangle$ and $\lambda = -\alpha_1/\alpha_0$.

9. Choosing another base $\{|\phi_k\rangle\}$ for $\mathcal{P}$

$$|\phi\rangle = \beta_0 |\phi_0\rangle + \beta_1 |\phi_1\rangle \Rightarrow (\Phi_1^{-1}\Phi_0 - \mu) u_a^* = 0$$

where $\mu = -\beta_1/\beta_0$ and $[\Phi_k]_{ij} = \langle ji | \phi_k \rangle$.

10. What aspects are common to matrices $R_1^{-1} R_0$ and $\Phi_1^{-1}\Phi_0$ and how are their respective eigenvalues λ_l and μ_l related?

11. Definition 2: Jordan family

two matrices, A and B, are at the same Jordan family iff

$$\lambda_l \text{ of } A \Leftrightarrow \mu_l \text{ of } B$$

$$\text{rank}(A - \lambda_l)^k = \text{rank}(B - \mu_l)^k$$

12. Theorem 1:

Let R_0 and R_1 be two n by n matrices, R_1 invertible

$$\begin{array}{l} \Phi_0 = aR_0 + bR_1 \\ \Phi_1 = cR_0 + dR_1 \end{array} \quad \text{with} \quad (ad - bc) = 1 \text{ and } \Phi_1 \text{ invertible}$$

$\Rightarrow R_1^{-1}R_0$ and $\Phi_1^{-1}\Phi_0$ are at the same Jordan family and

$$\mu_l = \frac{a\lambda_l + b}{c\lambda_l + d}.$$

13. Interchanging the subsystems s_a and s_b, the result is equivalent.

R_k goes to R_k^T and $R_1^{-1}R_0$ goes to $(R_0R_1^{-1})^T$ which is similar to $R_1^{-1}R_0$.

14. Let $|\phi_1\rangle$ and $|\phi_2\rangle$ to states in $\mathcal{P}$ with Schmidt rank smaller than n

$$|\psi\rangle = |\phi_1\rangle|c_1\rangle + |\phi_2\rangle|c_2\rangle, \tag{2}$$

where $|c_1\rangle$ and $|c_2\rangle$ are appropriate non-normalized states in C_c^2.

15. $R_1^{-1}R_0$ may have only one eigenvalue.

16. In general, $R_1^{-1}R_0$ has m solutions, there are $\begin{pmatrix} m \\ 2 \end{pmatrix}$ combinations.

17. For each Jordan family of $R_1^{-1}R_0$ we can associate a family of entangled states $|\psi\rangle$. States which belong to distinct families belong also to distinct SLOCC classes.

Example 1: Three qubits.

two Jordan families

$$\text{(a):}\quad \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \Rightarrow |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

$$\text{(b):}\quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow |GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

where $\lambda_1 \neq \lambda_2$.

Example 2: $|\psi\rangle$ has local supports 3, 3 and 2 - Five Jordan families:

$$\text{(a):}\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \Rightarrow |\psi_a\rangle = \frac{1}{\sqrt{5}}[(|10\rangle + |21\rangle)|0\rangle + (|00\rangle + |11\rangle + |22\rangle)|1\rangle].$$

$$\text{(b):}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \Rightarrow |\psi_b\rangle = \frac{1}{2}[|21\rangle\,|0\rangle + (|00\rangle + |11\rangle + |22\rangle)|1\rangle].$$

$$\text{(c):}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix} \Rightarrow |\psi_c\rangle = \frac{1}{2}[(|00\rangle + |21\rangle)|0\rangle + (|11\rangle + |22\rangle)|1\rangle].$$

$$\text{(d):}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \Rightarrow |\psi_d\rangle = \frac{1}{\sqrt{3}}[|00\rangle\,|0\rangle + (|11\rangle + |22\rangle)|1\rangle].$$

$$\text{(e):}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \Rightarrow |\psi_e\rangle = \frac{1}{2}[(|00\rangle + |11\rangle)|0\rangle + (|11\rangle + |22\rangle)|1\rangle].$$

where $\lambda_l \neq \lambda_{l'}$ for $l \neq l'$.

Example 3: $|\psi\rangle$ has local supports 4, 4 and 2 - 13 Jordan families

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \Rightarrow \begin{aligned} &|GHZ\rangle \otimes |\phi^+\rangle = \\ &\quad \tfrac{1}{2}\big[(|00,00\rangle + |01,01\rangle)|0\rangle \\ &\qquad +(|10,10\rangle + |11,11\rangle)|1\rangle\big] \end{aligned}$$

$$B = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \Rightarrow \begin{aligned} &|W\rangle \otimes |\phi^+\rangle = \\ &\quad \tfrac{1}{\sqrt{6}}\big[\ (|00,10\rangle + |01,11\rangle + |10,00\rangle + |11,01\rangle)|0\rangle \\ &\qquad +(|00,00\rangle + |01,01\rangle)|1\rangle\big] \end{aligned}$$

Note that the Jordan family corresponding to B differs from that corresponding to

$$C = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \Rightarrow |\psi_c\rangle = \tfrac{1}{\sqrt{6}}\big[(|00,00\rangle + |01,01\rangle + |10,10\rangle + |11,11\rangle)|0\rangle + (|10,01\rangle + |11,10\rangle)|1\rangle\big]$$

only in that the ranks of $(B-\lambda_1)^k$ and $(C-\lambda_1)^k$ differ for $k=2$.

$$|W\rangle \otimes |\phi^+\rangle = \frac{1}{\sqrt{6}}\Big[\ (|00,10\rangle + |01,11\rangle + |10,00\rangle + |11,01\rangle)|0\rangle\ + (|00,00\rangle + |01,01\rangle)|1\rangle\Big]$$

$$\begin{array}{ccccccccccc}
|01\rangle & & & & |11\rangle & \vdots & |00\rangle & & & & |10\rangle \\
\uparrow & \nwarrow & & & \uparrow & \vdots & \uparrow & \nwarrow & & & \uparrow \\
\phi_0 & & \phi_1 & & \phi_0 & \vdots & \phi_0 & & \phi_1 & & \phi_0 \\
\uparrow & & & \nwarrow & \uparrow & \vdots & \uparrow & & & \nwarrow & \uparrow \\
|11\rangle & & & & |01\rangle & \vdots & |10\rangle & & & & |00\rangle
\end{array}$$

$$|\psi_c\rangle = \frac{1}{\sqrt{6}}\Big[(|00,00\rangle + |01,01\rangle + |10,10\rangle + |11,11\rangle)|0\rangle + (|10,01\rangle + |11,10\rangle)|1\rangle\Big]$$

$$\begin{array}{ccccccccccc}
|00\rangle & \vdots & |01\rangle & & & & |10\rangle & & & & |11\rangle \\
\uparrow & \vdots & \uparrow & \nwarrow & & & \uparrow & \nwarrow & & & \uparrow \\
\phi_0 & \vdots & \phi_0 & & \phi_1 & & \phi_0 & & \phi_1 & & \phi_0 \\
\uparrow & \vdots & \uparrow & & & \nwarrow & \uparrow & & & \nwarrow & \uparrow \\
|00\rangle & \vdots & |01\rangle & & & & |10\rangle & & & & |11\rangle
\end{array}$$

Another interesting family is

$$(d) \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \Rightarrow \quad |\psi_d\rangle = \frac{1}{\sqrt{4+2|a|^2}}\Big[(|11\rangle + a|22\rangle + |33\rangle)|0\rangle + (|00\rangle + a|11\rangle + |22\rangle)|1\rangle\Big]$$

which is the only one at this entanglement dimensionality that needs to be subdivided into an infinity of SLOCC classes and where $0 \neq a \neq 1$.

SLOCC Classification

$$|\psi\rangle \xrightarrow{\text{SLOCC}} |\psi'\rangle \text{ iff } |\psi'\rangle = A \otimes B \otimes C|\psi\rangle$$

where A, B and C are linear operators in C_a^n, C_b^n and C_c^2 respectively*.

$$|\psi'\rangle = \sum_{k=0,1} A \otimes B|r_k\rangle C(c_k|k\rangle) = \sum_{k=0,1} |\phi'_k\rangle|c'_k\rangle$$

where $|c'_k\rangle = C(c_k|k\rangle)$ and $|\phi'_k\rangle = A \otimes B|r_k\rangle$.

There exist C which maps $c_k|k\rangle$ into any two distinct $|c'_k\rangle$.

$$\Phi'_k = BR_kA^T \Rightarrow \Phi_1'^{-1}\Phi'_0 = A^{T^{-1}}R_1^{-1}R_0A^T$$

interchanging the subsystems s_a and s_b,

$$(\Phi'_0\Phi_1'^{-1})^T = B^{T^{-1}}(R_0R_1^{-1})^TB^T.$$

+ Theorem 1$\Rightarrow |\psi\rangle \xleftrightarrow{\text{SLOCC}} |\psi'\rangle$ only if they are in the same Jordan family.

*W Dür, G. Vidal and J. I. Cirac, Phys. Rev. A63, 062314 (2000).

$$
\begin{array}{ccc}
|\psi\rangle & & |\psi'\rangle \\
\Downarrow & & \Downarrow \\
R_1^{-1}R_0 & \text{in the same Jordan family of} & R_1'^{-1}R_0' \\
\Downarrow & & \Downarrow \\
\{\lambda_{l,r}\} & & \{\lambda_{l,r'}\} \\
\searrow & & \swarrow
\end{array}
$$

$$\text{rank}(R_1^{-1}R_0 - \lambda_{l,r})^k = \text{rank}(R_1'^{-1}R_0' - \lambda_{l,r'})^k$$

look for two matrices, $\Phi_1'$and Φ_2' that are superpositions of R_1' and R_0' and such that $\Phi_1'^{-1}\Phi_0'$ are similar to $R_1^{-1}R_0$.

$$\lambda_{l,r} = \mu_{l,\phi'} = \frac{a\lambda_{l,r'} + b}{c\lambda_{l,r'} + d} \Rightarrow \lambda_{l,r}\lambda_{l,r'}c + \lambda_{l,r}d - \lambda_{l,r'}a - b = 0$$

with the additional condition that $(ad - bc) = 1$.

Any non-trivial solution of linear system intersects $(ad - bc) = 1$.

There always exist at least one solution if that $L \leq 3$.

Discussion: More General Tripartite Entangled States

- When neither one of the subsystems is a qubit, we get the equation

$$(\sum_k \alpha_k R_k) u_a^* = 0$$

- When the entanglement has local supports n,m and 2, with $m \neq n$, there is no invertible matrix.

Conclusion

- We have described a constructive method to find decompositions of tripartite entangled pure states which involve a number of terms smaller than one obtains using two successive Schmidt decompositions for entangled states with local supports on each part n, n and 2.

- We use these decompositions to classify these states according their inter-convertibility through SLOCC.

- We show how to find the SLOCC operation which transform one state in another when they are in the same SLOCC class.

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