Optimal estimation of group transformations using entanglement

Giulio Chiribella (Università di Pavia, Italia)

ISSQUI05, MPIPKS Dresden, 01/09/2005.

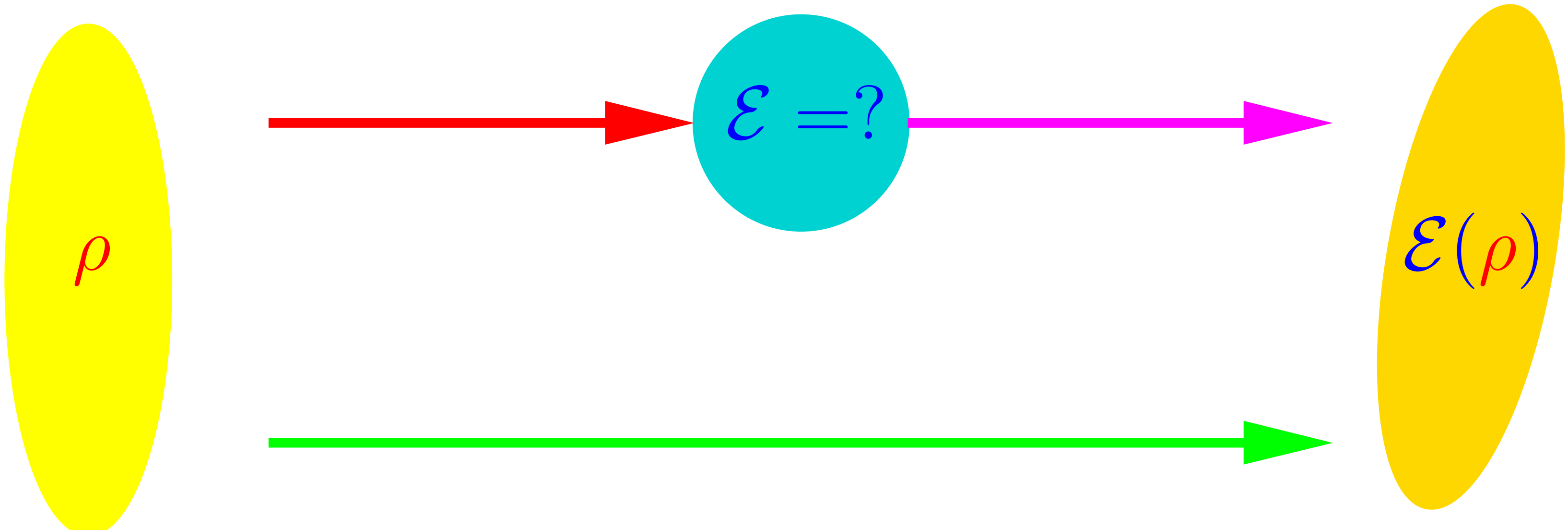
In collaboration with

G.M. D'Ariano and M. F. Sacchi.

Outline

- Introduction: estimating an unknown black box
- Estimating group transformations: the Bayesian approach
- A generalized Holevo class of cost functions
- Optimal input states
- Optimal estimation strategies

Estimating an unknown black box



What is the best input state?

What is the best estimation strategy?

Does entanglement improve the estimation?

The case of unitary channels

- **Dense coding:** **dramatic effect of entanglement**

Bennett and Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).

- **Discriminating two unitaries:** **entanglement is not useful...**

D'Ariano at al, Phys. Rev. Lett. **87**, 270404 (2001)

Acin, Phys. Rev. Lett **87**, 177901 (2001)

- **Estimating an $\mathbb{SU}(d)$ channel:** **entanglement is very useful...**

Fujiwara, Phys. Rev. A **65**, 012316 (2001)

Acin et al, Phys. Rev. A **64**, 050302 (2002)

and many others...

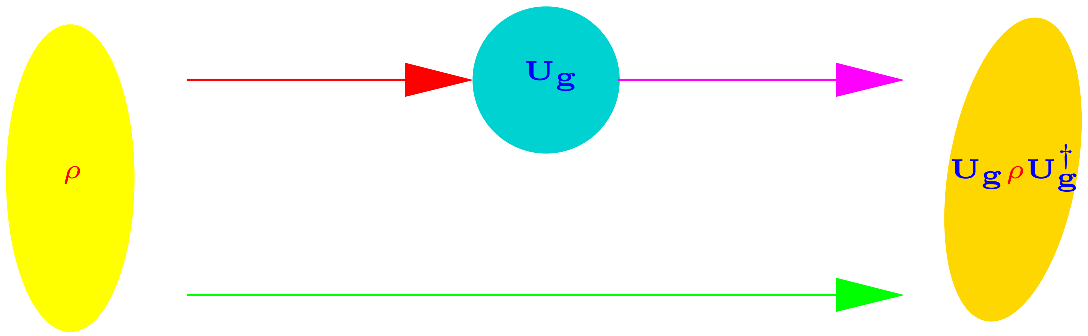
A natural question

Is it possible to understand the role of entanglement

once for all

in a general way?

Estimating group transformations



The rules of the game:

- we are asked to **give an estimate** $\hat{g}$ of the transformation g
- we know **only** that g is an element of the group $\mathbf{G}$
- we are allowed to **prepare any input state** and to **perform any quantum measurement** on the output state.

What is the "best measurement"?

- Fix the **a priori distribution** $\mathrm{d}\, g$

  complete ignorance $\longrightarrow$ uniform a priori distribution

- Fix a **cost function** $c(\hat{g}, g)$

  the "cost" is as smaller as the estimate $\hat{g}$ is nearer to the true value g.

- Minimize the average cost $\langle c \rangle$

$$\langle \mathbf{c} \rangle = \int \mathrm{d}\,\mathbf{g} \int \mathrm{d}\,\hat{\mathbf{g}} \;\; \mathbf{c}(\hat{\mathbf{g}}, \mathbf{g}) \; \mathbf{p}(\hat{\mathbf{g}}|\mathbf{g})$$

($=$ how much we need to pay if we play the game many times)

The "best measurement" **depends** on the choice of the cost function.

Cost functions in the case of phase estimation

For phase estimation one has the **Holevo class:**

$$c(\hat{\phi}, \phi) = \sum_{k \in \mathbb{Z}} c_k \; e^{ik\left(\hat{\phi} - \phi\right)}$$

$$c_k \leq 0 \qquad \forall k \neq 0$$

Theorem

All cost functions in the Holevo class lead to the same optimal measurement.

A generalized Holevo class of cost functions (I)

First requirement: **invariant cost functions**

$$\begin{cases} c(h\hat{g}, hg) = c(\hat{g}, g) & \forall h \in \mathbf{G} \\ c(\hat{g}h, gh) = c(\hat{g}, g) & \forall h \in \mathbf{G} \end{cases}$$

This requirement is satisfied if and only if

$$c(\hat{g}, g) = \sum_{\sigma \in \mathrm{Irr}(\mathbf{G})} a_\sigma \, \chi^\sigma(\hat{g} g^{-1}) \qquad \chi^\sigma(h) := \mathrm{Tr}[U^\sigma(h)]$$

A generalized Holevo class of cost functions (II)

Second requirement: **negative Fourier coefficients**

$$a_\sigma \leq 0 \qquad \forall \sigma \neq \sigma_0$$

$\sigma_0 =$ trivial representation, $\qquad U^{\sigma_0}(g) = 1 \qquad \forall g$

A little technicality: Wedderburn decomposition

Decomposing the Hilbert space:

$$\mathscr{H} = \bigoplus_{\mu \in \mathsf{S}} \mathscr{H}_\mu \otimes \mathbb{C}^{m_\mu}$$

Decomposing a pure state:

$$|\Psi\rangle = \bigoplus_{\mu \in \mathsf{S}} c_\mu \, |\Psi_\mu\rangle\!\rangle$$

Decomposing the unitaries:

$$U_g = \bigoplus_{\mu \in \mathsf{S}} U_g^\mu \otimes \mathbb{1}_{m_\mu}$$

The role of the external reference system

Adding an ancilla increases the dimension of the multiplicity spaces m_μ:

$$U_g \otimes \mathbb{1}_\mathcal{R} = \bigoplus_{\mu \in \mathsf{S}} U_g^\mu \otimes \left(\mathbb{1}_{m_\mu} \otimes \mathbb{1}_\mathcal{R}\right)$$

If we exploit a reference system the multiplicity becomes $m_\mu^\mathcal{R} = m_\mu d_\mathcal{R}$.

If $m_\mu \geq d_\mu$ there is no need of any ancilla!

We can exploit the **entanglement between the virtual subsystems** that appear in the Wedderburn decomposition.

The form of the optimal input states

For any compact group

and for any cost function in the generalized Holevo class

the optimal input state has the form

$$|\Psi\rangle = \bigoplus_{\mu \in \mathsf{S}} \frac{c_\mu}{\sqrt{d_\mu}} |W_\mu\rangle\rangle$$

where $|W_\mu\rangle\rangle/\sqrt{d_\mu}$ is a maximally entangled state

Entanglement between representation spaces and multiplicity spaces is the real key ingredient for optimal estimation (!!)

A first simplification

One only needs to take a state of the optimal form:

$$|\Psi\rangle = \bigoplus_{\mu \in \mathsf{S}} \frac{c_\mu}{\sqrt{d_\mu}} \, |W_\mu\rangle\rangle$$

and then to optimize the coefficients c_μ

Looking for the best estimation strategy...

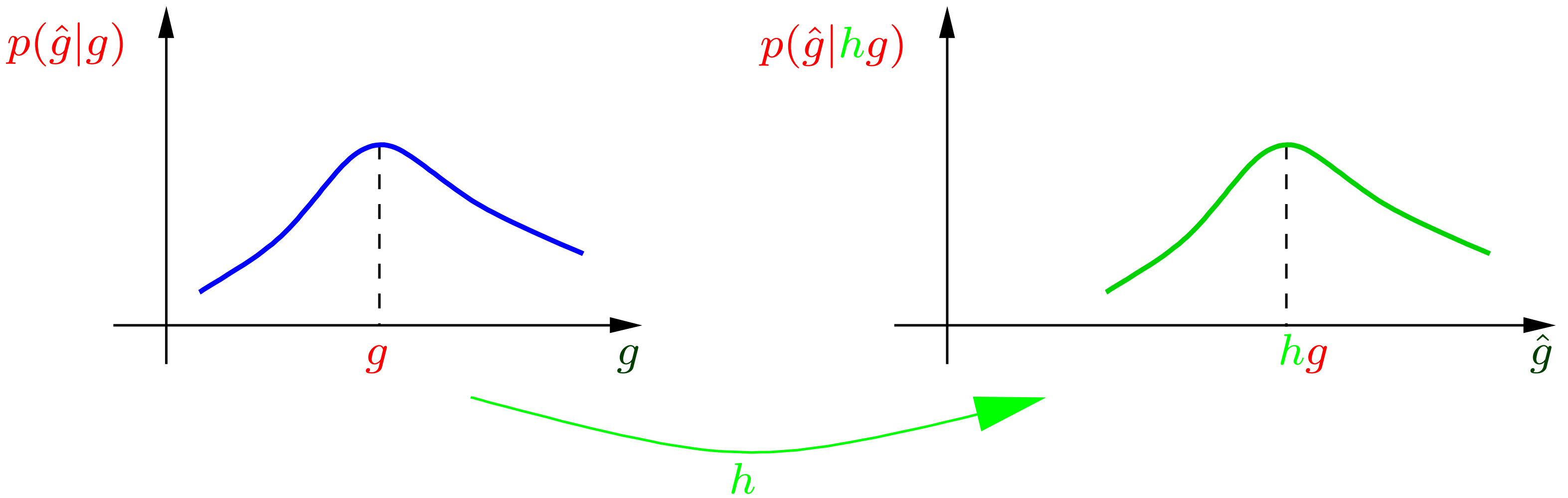
The most general estimation strategy allowed by Quantum Mechanics is described by a POVM $\mathbf{P}(\hat{\mathbf{g}})$

$$\begin{cases} P(\hat{g}) \geq 0 \qquad \forall \hat{g} \in \mathbf{G} & \text{(positivity)} \\ \int \mathrm{d}\,\hat{g}\; P(\hat{g}) = 1\!\!1 & \text{(normalization)} \end{cases}$$

Born rule for probabilities: $p(\hat{g}|g) = \mathrm{Tr}[\; P(\hat{g})\; U_g \rho U_g^\dagger\;]$

Covariant POVM's

$$p(\hat{g}|g) = p(h\hat{g}|hg) \qquad \forall h \in \mathbf{G}$$



Theorem (Holevo)

For any possible estimation strategy,
there is always a covariant POVM
with the same average cost

General form of a covariant POVM:

$$P(\hat{g}) = U_{\hat{g}} \ \Xi \ U_{\hat{g}}^{\dagger}$$

where $\Xi \geq 0$.

Theorem

Let be $|\Psi\rangle$ an input state of the optimal form.

For any compact group

and for any cost function in the generalized Holevo class

the optimal covariant POVM is

$$P(\hat{g}) = U_{\hat{g}}\ |\eta\rangle\langle\eta|\ U_{\hat{g}}^{\dagger}$$

where

$$|\eta\rangle = \bigoplus_{\mu \in \mathsf{S}} \sqrt{d_\mu}\ e^{i \arg(c_\mu)}\ |W_\mu\rangle\rangle$$

Optimizing the input state

Great simplification in the search for the optimal input state:

The average Bayes cost for the optimal POVM is

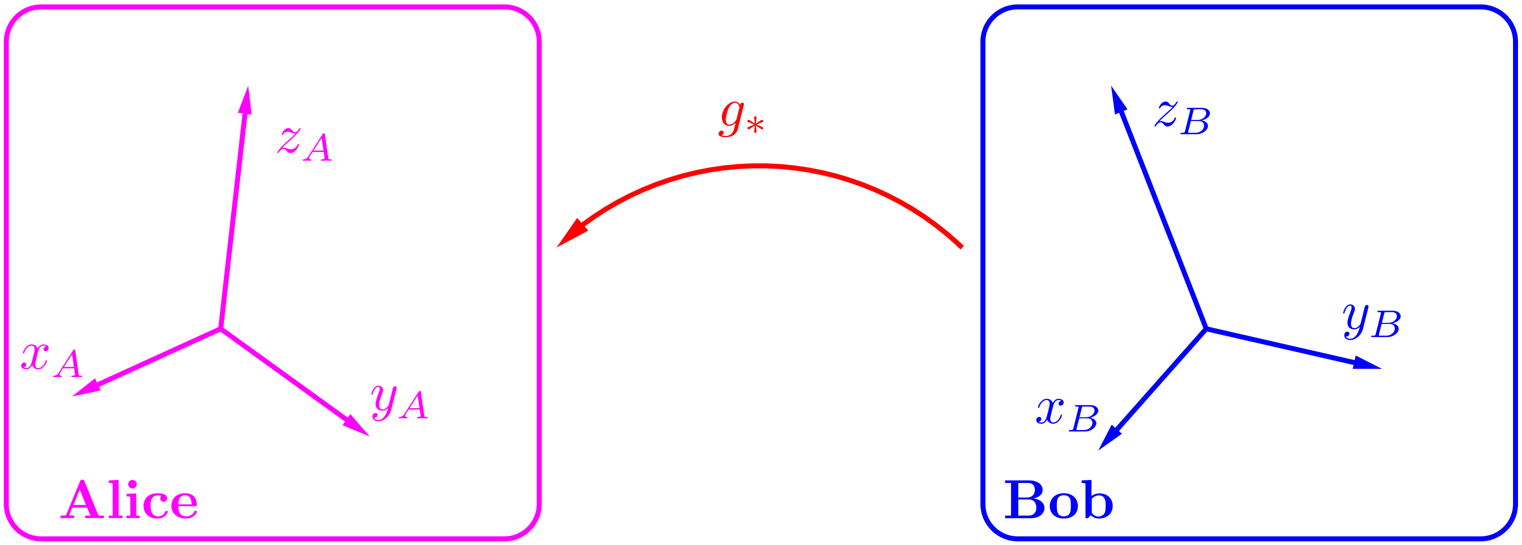
$$\langle\, C \,\rangle = a_0 \; + \sum_{\mu,\nu \in \mathsf{S}} |c_\mu|\; C_{\mu\nu}\; |c_\nu|$$

where $C_{\mu\nu}$ is a cost matrix.

We only need to find the minimum eigenvalue of the cost matrix and the corresponding eigenvector.

An application: reference frames alignment

Two distant parties want to align their Cartesian reference frames:



$x_A = g_* \, x_B$ $\quad$ $y_A = g_* \, y_B$ $\quad$ $z_A = g_* \, z_B$

Alignment with N spin $1/2$ particles

Step 1 Referring to **her** Cartesian frame, Alice prepares an entangled state

$$|A\ \rangle = c_1|\uparrow\uparrow\ldots\uparrow\rangle + c_2|\downarrow\uparrow\ldots\uparrow\rangle + \ldots$$

and sends it to Bob...

Step 2 With respect to **his** Cartesian frame, Bob receives the state

$$|A_{g_*}\rangle = c_1|\nearrow\nearrow\cdots\nearrow\rangle + c_2|\swarrow\nearrow\cdots\nearrow\rangle + \ldots$$

where

$$|\nearrow\rangle = U_{g_*}|\uparrow\ \rangle$$

In other words: $$|A_{g_*}\rangle = U_{g_*}^{\otimes N}\ |A\ \rangle$$

What is the best asymptotic performance?

The original claim was: $\mathcal{O}\left(1/N\right)$ (classical scaling)

A. Peres and P. F. Scudo, Phys. Rev. Lett. **87**, 167901 (2001).

Bagan et al, Phys. Rev. Lett. **87**, 257903 (2001).

Some years later... $\mathcal{O}\left(1/N^2\right)$ (typical quantum improvement)

Chiribella et al, Phys. Rev. Lett. **93**, 180503 (2004)

Bagan et al, Phys. Rev. A **70**, 030301(R) (2004)

M. Hayashi, quant-ph/0407053

Is this the final answer?

Yes

Using the theorem about optimal POVMs,

we can prove that the asymptotic scaling

$$\langle e \rangle \sim \frac{8\pi^2}{N^2}$$

is the ultimate precision limit

imposed by the laws of Quantum Mechanics

to the alignment of two Cartesian reference frames.

Conclusions (I)

- **Generalization of the Holevo class:**
  invariant cost functions with negative Fourier coefficients

- **How to use entanglement:**
  what really matters is **only** the entanglement between
  representation and multiplicity spaces

- **Optimal input states:**
  direct sum of maximally entangled states
  (w.r.t. bipartition representation/multiplicity spaces)

Conclusions (II)

- **Optimal estimation strategy:**
  same optimal POVM
  for any cost function in the generalized Holevo class

- **An application:**
  optimality proof for reference frames alignment

Reference: G. Chiribella, G. M. D'Ariano, and M. F. Sacchi,
Optimal estimation of group transformations using entanglement,
quant-ph/0506267.