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SESSION "STATISTICAL MECHANICS, KINETICS AND QUANTUM THEORY OF CONDENSED MATTER"

Wave Propagation in Nonlinear Disordered Media[1]

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Abstract—We analyze mechanisms and regimes of wave packet spreading in nonlinear disordered media. We discuss resonance probabilities, and predict a dynamical crossover from strong to weak chaos. The crossover is controlled by the ratio of nonlinear frequency shifts and the average eigenvalue spacing of eigenstates of the linear equations within one localization volume. We consider generalized models in higher lattice dimensions and obtain critical values for the nonlinearity power, the dimension, and norm density, which influence possible dynamical outcomes in a qualitative way.

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1. INTRODUCTION

Wave localization in linear disordered systems was obtained by P.W. Anderson [1] and experimentally observed for light [2] and ultracold atoms [3] recently. A number of studies was recently devoted to the ease of interplay of nonlinearity and disorder [4–12], showing that wave packets may spread way beyond the limits set by the linear theory. In this paper we will discuss the mechanisms of wave packet spreading in nonlinear disordered systems. More specifically, we will consider cases when (i) the corresponding linear wave equations yield Anderson localization, (ii) the localization length is bounded from above by a finite value, (iii) the nonlinearity is compact in real space and therefore does not induce long range interactions between eigenstates of the linear equations. We will analyze the chaotic dynamics which is at the heart of the observed destruction of Anderson localization. We obtain an intermediate sub diffusive regime of strong chaos and an even slower asymptotic regime of weak chaos.

We consider the disordered discrete nonlinear Schrödinger equation (DNLS)

$$i\dot{\psi}_i = \epsilon_l\psi_l + \beta|\psi_l|^2\psi_l - \psi_{l+1} - \psi_{l-1}. \qquad (1)$$

with complex variables ψ_l, lattice site indices l and nonlinearity strength $\beta \geq 0$. The random on-site energies ϵ_l are chosen uniformly from the interval $[-W/2, W/2]$, with W denoting the disorder strength. Eqs. (1) conserve the energy and the norm $S = \sum_l |\psi_l|^2$.

For $\beta = 0$ with $\psi_l = A_l \exp(-i\lambda t)$ Eq. (1) is reduced to the linear eigenvalue problem $\lambda A_l = \epsilon_l A_l - A_{l-1} - A_{l+1}$. The normalized eigenvectors $A_{\nu,l}$ ($\sum_l A_{\nu,l}^2 = 1$) are the NMs, and the eigenvalues λ_ν are the frequencies of the NMs. The width of the eigenfrequency spectrum λ_ν is $\Delta = W + 4$ with $\lambda_\nu \in [-2 - W/2, 2 + W/2]$.

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The asymptotic spatial decay of an eigenvector is given by $A_{\nu,l} \sim e^{-l/\xi(\lambda_\nu)}$ where $\xi(\lambda_\nu)$ is the localization length and $\xi(\lambda_\nu) \approx 24(4 - \lambda_\nu^2)/W^2$ for weak disorder $W \leq 4$ [1, 13]. The localization volume (spatial extend) V of the NM is on average of the order of $3\xi(0)$ for weak disorder, and tends to $V = 1$ in the limit of strong disorder. The average spacing d of eigenvalues of NMs within the range of a localization volume is therefore of the order of $d \approx \Delta/V$, which becomes $d \approx \Delta W^2/300$ for weak disorder. The two scales $d \leq \Delta$ are expected to determine the packet evolution details in the presence of nonlinearity.

2. ADDING NONLINEARITY

The equations of motion of (1) in normal mode space read

$$i\dot{\phi}_\nu = \lambda_\nu\phi_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3}\phi^*_{\nu_1}\phi_{\nu_2}\phi_{\nu_3}, \qquad (2)$$

with the overlap integral $I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_l A_{\nu,l}A_{\nu_1,l}A_{\nu_2,l}A_{\nu_3,l}$. The variables ϕ_ν determine the complex time-dependent amplitudes of the NMs.

The frequency shift of a single site oscillator induced by the nonlinearity is $\delta_l = \beta|\psi_l|^2$.

We order the NMs in space by increasing value of the center-of-norm coordinate $X_\nu = \sum_l l A_{\nu,l}^2$. We analyze normalized distributions $n_\nu \geq 0$ using the second moment $m_2 = \sum_\nu (\nu - \bar{\nu})^2 n_\nu$, which quantifies the wave packet's degree of spreading and the participation number $P = 1/\sum_\nu n_\nu^2$, which measures the number of the strongest excited sites in n_ν. Here $\bar{\nu} = \sum_\nu \nu n_\nu$. We follow norm density distributions $n_\nu \equiv |\varphi_\nu|^2/\sum_\mu |\varphi_\mu|^2$. The second moment m_2 is sensitive to the distance of the tails of a distribution from the center, while the participation number P is a measure of the inhomogeneity of the distribution, being insensitive to any spatial correlations.

3. FROM STRONG TO WEAK CHAOS

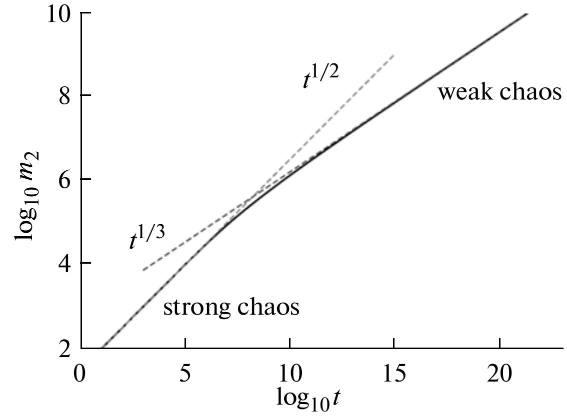
Consider a wave packet at $t = 0$ which has norm density n and size L. If $\beta n \geq \Delta$, a substantial part of the wave packet will be selftrapped [7, 9]. This is due to nonlinear frequency shifts, which will tune the excited sites immediately out of resonance with the nonexcited neighborhood [14]. If now $\beta n < \Delta$, selftrapping is avoided, and the wave packet can start to spread. For $L < V$, the packet will spread over the localization volume during the time $\tau_{lin} \approx 2\pi/d$ (even for $\beta = 0$). After that, the new norm density will drop down to $\beta n(\tau_{lin}) \approx nL/V$. For $L > V$ the norm density will not change appreciably up to τ_{lin}, $n(\tau_{lin}) \approx n$. The nonlinear frequency shift $\beta n(\tau_{lin})$ can be now compared with the average spacing d. If $\beta n(\tau_{lin}) > d$, all NMs in the packet are resonantly interacting with each other. This regime will be coined strong chaos. If instead $\beta n(\tau_{lin}) < d$, NMs are weakly interacting with each other. This regime will be coined weak chaos.

A NM with index μ in a layer of width V in the cold exterior, which borders the packet, is either incoherently *heated* by the packet, or *resonantly excited* by some particular NM from a layer with width V inside the packet. Chaos is a combined result, of resonances and nonintegrability. Let us estimate the number of resonant modes in the packet for the DNLS model. Excluding secular interactions, the amplitude of a NM with $|\varphi_\nu|^2 = n_\nu$ is modified by a triplet of other modes $\vec{\mu} \equiv (\mu_1, \mu_2, \mu_3)$ in first order in β as

$$\left|\varphi_\nu^{(1)}\right| = \beta\sqrt{n_{\mu_1}n_{\mu_2}n_{\mu_3}}R^{-1}_{\nu,\vec{\mu}}, \quad R_{\nu,\vec{\mu}} \sim \left|\frac{\overrightarrow{d\lambda}}{I_{\nu,\mu_1,\mu_2,\mu_3}}\right|, \tag{3}$$

where $\overrightarrow{d\lambda} = \lambda_\nu + \lambda_{\mu_1} - \lambda_{\mu_2} - \lambda_{\mu_3}$. The perturbation approach breaks down, and resonances set in, when $\sqrt{n_\nu} < \left|\phi_\nu^{(1)}\right|$. Since all considered NMs belong to the packet, we assume their norms to be equal to n.

Collecting $R_{\nu,\vec{\mu}_0} = \min_{\vec{\mu}} R_{\nu,\vec{\mu}}$ for many ν and many disorder reahzations, we obtain the probability density distribution $\mathcal{W}(R_{\nu,\vec{\mu}_0})$. The main result is that $\mathcal{W}(R_{\nu,\vec{\mu}_0} \longrightarrow 0) \longrightarrow C(W) \neq 0$ [9]. For the cases studied, the constant C drops with increasing disorder strength W. As a simple approximation, we may use $\mathcal{W}(R) \approx Ce^{-CR}$.



$m_2(t)$ in a log-log plot according to (5) (black solid line). Dashed lines—power laws for strong and weak chaos.

The probability $\mathcal{P}$ for a mode, which is excited to a norm n (the average norm density in the packet), to be resonant is then given by $\mathcal{P} = \int_0^{\beta n} \mathcal{W}(x)dx$ [9]. For $\beta n \longrightarrow 0$ it follows $\mathcal{P} \approx C\beta n$. The heating of the exterior mode should evolve according to $i\dot{\phi}_\mu \approx \lambda_\mu\phi_\mu + \mathcal{P}\beta n^{3/2}f(t)$. It follows with $m_2 \sim 1/n^2$:

$$m_2 \sim Dt, \quad D \sim \beta^2 n^2 (\mathcal{P}(\beta n))^2. \tag{4}$$

With $\mathcal{P} = 1 - e^{-C\beta n}$ it follows

$$\frac{1}{n^2} \sim \beta(1 - e^{-C\beta n})t^{1/2}. \tag{5}$$

The solution of this equation yields a crossover from subdiffusive spreading in the regime of strong chaos to subdiffusive spreading in the regime of weak chaos:

$$m_2 \sim (\beta^2 t)^{1/2}, \text{ strong chaos, } C\beta n > 1,$$

$$m_2 \sim (\beta^4 t)^{1/3}, \text{ weak chaos, } C\beta n < 1,$$

In figure we show the resulting time dependence of m_2 on t in a log-log plot, where we used $\beta = 1$, $C = 6.2$, $L = 20$ and $n(t = 10^2) = 1$. With $x = \log_{10}(t)$ and $y = \log_{10}(m_2)$ it is straightforward to calculate the zero of the third derivative $d^3y/dx^3 = 0$ to obtain the crossover position $C\beta n_c \approx 1.86$. The only characteristic frequency scale here is $1/C$. From the above discussion of the different spreading regimes it follows that $1/C \approx d$. In particular $C \approx 100/W^2$, $W \leq 4$ and $C \approx 1/W$, $W \gg 4$. Resonant growth can be excluded using the same arguments [8, 9]. Thus, a wave packet is trapped at its edges, and stays localized until the interior of the wave packet decoheres (thermalizes). On these (growing) time scales, the packet will be finally, able to incoherently excite the exterior and to extend its size.

4. GENERALIZATIONS

Let us consider $\mathcal{D}$-dimensional lattices with nonlinearity order $\sigma > 0$:

$$i\dot{\psi}_{\boldsymbol{l}} = \epsilon_{\boldsymbol{l}}\psi_{\boldsymbol{l}} - \beta|\psi_{\boldsymbol{l}}|^{\sigma}\psi_{\boldsymbol{l}} - \sum_{m \in D(l)} \psi_m. \quad (6)$$

Here $\boldsymbol{l}$ denotes an $\mathcal{D}$-dimensional lattice vector with integer components, and $m \in \mathcal{D}(\boldsymbol{l})$ defines its set of nearest neighbour lattice sites. We assume that all NMs are spatially localized (which can be obtained for strong enough disorder W). A wavepaeket with average norm n per excited mode has a second moment $m_2 \sim 1/n^{2/\mathcal{D}}$. The nonlinear frequency shift is proportional to $\beta n^{\sigma/2}$.

A straightforward generalization of the expected regimes of spreading leads to the following: selftrapping if $\beta n^{\sigma/2} > \Delta$, strong chaos if $\beta(n(\tau_{lin})^{\sigma/2} > d$, and weak chaos if $\beta(n(\tau_{lin})^{\sigma/2} < d$. Similar to the above we obtain a diffusion coefficient

$$\mathcal{D} \sim \beta^2 n^{\sigma}(\mathcal{P}(\beta n^{\sigma/2}))^2. \quad (7)$$

In both regimes of strong and weak chaos the spreading is subdiffusive [8]:

$$m_2 \sim (\beta^2 t)^{2/2 + \sigma\mathcal{D}}, \text{ strong chaos,}$$

$$m_2 \sim (\beta^4 t)^{1/1 + \sigma\mathcal{D}}, \text{ weak chaos,}$$

The number of resonances on the wave packet surface $N_{\mathrm{RS}} \sim \beta n^{\mathcal{D}(\sigma - 3) + 2/2\mathcal{D}}$. This number will grow with time for

$$\mathcal{D} > \mathcal{D}_c = \frac{1}{1 - \sigma/2}, \quad \sigma < 2. \quad (8)$$

Therefore, for these cases, the wave packet surface will not stay compact. Instead surface resonances will lead to a resonant leakage of excitations into the exterior. This process will increase the surface area, and therefore lead to even more surface resonances, which again increase the surface area, and so on. The wave packet will fragmentize, perhaps get a fractal-like structure, and lower its compactness index. The spreading of the wave packet will speed up, but will not anymore be due to pure incoherent transfer, instead it will become a complicated mixture of incoherent and coherent transfer processes. For such cases. Anderson localization will be destroyed quickly even in the tails of wave packets.

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