SYMMETRIES AND TRANSPORT WITH QUASIPERIODIC DRIVING*

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We generalize recent studies of particle transport to the case of quasiperiodic potentials with quasiperiodic driving. We obtain the relevant set of space-time symmetries and the way of violating them in order to obtain directed transport. Numerical results confirm the predicted rectification for the dissipationless case.

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1. Introduction

The possibility to rectify transport with the help of fluctuations has been discussed for several years with respect to *e.g.* molecular motors and other nonequilibrium processes in biological systems [1], electrical currents in superlattices [2], and voltages in Josephson junction coupled systems [3, 4]. The fluctuations have zero mean value. The most simple underlying mathematical models correspond to a classical particle moving in a space-periodic but asymmetric (ratchet) potential and allowed to study the resulting *directed current* in great detail [5] (for a review see [6]). A recently elaborated symmetry approach to this problem established a clear relationship between directed currents and broken space-time symmetries [7–9]. The essential step was to separate the unavoidable correlations in the fluctuations from the uncorrelated ones. This is easily obtained by replacing the fluctuations as a superposition of ac driving fields and uncorrelated white noise. An important consequence is that the symmetries may be broken either by violating the reflection symmetry of the potential in space or by violating the shift symmetry of the ac fields. Thus, a particle may display a directed motion also in the case of a space-symmetric potential. Another interesting result

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is the persistence of directed currents in the Hamiltonian limit of systems exposed to ac fields but decoupled from the heat bath [7, 8] as well as in the case of absence of noise (deterministic ratchets) [10]. The underlying concept of symmetry analysis and breaking has been successfully applied to other cases, *e.g.* to obtain a transverse magnetization of driven quantum spins without and with interaction [11] and an energy transport in spatially extended systems [12].

While the symmetry analysis is straightforward for external fields which are periodic in time, the generalization to fields which vary quasiperiodically in time is less obvious. The adiabatic limit of slowly varying fields allows to conclude that the way of symmetry breaking, which leads to directed transport, is not substantially altered [13]. However dynamical symmetries may loose to some extend their meaning in this limit. A pioneering breakthrough was the symmetry study of an overdamped Josephson junction with a two-frequency quasiperiodic drive by Neumann and Pikovsky [14]. In this work, we will extend their analysis to the general case of a system with an N-frequency drive, and also to spatial potentials which vary quasiperiodically. Our study will incorporate both the case with damping and the dissipationless (Hamiltonian) limit. We will provide with classifications of various quasiperiodic functions in terms of their discrete symmetries relevant for studying the appearance of directed transport. We will finally present numerical results for the Hamiltonian case which support our symmetry studies.

2. Symmetry analysis

In this section we will present our results on the analysis of space-time symmetries of the following equation:

$$m\ddot{x} + \gamma\dot{x} - f(x) - E(t) - \xi(t) = 0\,. \tag{1}$$

Here $\xi(t)$ is a Gaussian white noise, $E(t)$ is an external time-dependent field and $f(x)$ is the force generated by a spatially dependent potential $U'(x) = -f(x)$, m is the mass of a particle located at coordinate x, and γ a coefficient characterizing the strength of an assumed external friction force. The two functions $E(t)$ and $f(x)$ are assumed to be bounded and of zero average. In what follows all symmetries considered will contain discrete operations in time, which transform a given realization of $\xi(t)$ into an equivalent realization with the same statistical weight [6,9]. Consequently we can skip this term from further symmetry considerations. The essential point is that we are left with a deterministic equation. A possible symmetry operation which leaves this equation invariant will in general transform a given trajectory (solution) in phase space into another trajectory (solution). The question we are going to investigate is whether (1) allows for the generation of a nonzero average current

$$J = \langle \dot{x} \rangle \neq 0\,, \tag{2}$$

where the average is done with respect to time. Consequently we are looking for symmetry operations which will change the sign of $\dot{x}$. If such a symmetry is identified under certain conditions, and if the corresponding symmetry related trajectories have the same statistical weight, their contributions to an average current will annihilate each other. As a result, in such a case the predicted current will be zero. By violating the conditions for the symmetry to be in place, we expect to observe a nonzero average current (for details see also [13]).

2.1. The case of periodic functions $E(t)$ and $f(x)$

We briefly review the symmetry analysis for the case of periodic functions $E(t) = E(t+T)$ and $f(x) = f(x+\lambda)$. As a prerequisite note that a periodic function $g(z) = g(z+z_p)$ with zero mean can have three more discrete symmetries. It can be symmetric $g_{\rm s}(z) = g_{\rm s}(-z)$, antisymmetric $g_{\rm a}(z) = -g_{\rm a}(-z)$ around certain argument values (which are for simplicity put to zero here), and can be shift symmetric $g_{\rm sh}(z) = -g_{\rm sh}(z+z_p/2)$. A given function $g(z)$ may either have none of these symmetries, exactly one or all three. One of the simplest ways to break all three symmetries is to choose $g(z) = g_1\cos(\omega z) + g_2\cos(2\omega z+\delta)$. If both g_1 and g_2 are nonzero, $g \neq g_{\rm sh}$. For $\delta = m\pi$ it follows $g = g_{\rm s}$, and for $\delta = (1/2+m)\pi$ it follows $g = g_{\rm a}$ (here m is an integer). For all other values of δ the function has none of the above symmetries.

The following current-relevant symmetries can be identified for (1) [7–9]:

$$\begin{aligned}
\hat{S}_a &: \; x \to -x\,, \;\; t \to t + \frac{T}{2}\,, && \text{if } \{f_{\rm a}\,,\; E_{\rm sh}\}\,; \\
\hat{S}_b &: \; x \to x\,, \;\; t \to -t\,, && \text{if } \{E_{\rm s}\,,\; \gamma = 0\}\,; \\
\hat{S}_c &: \; x \to x + \frac{\lambda}{2}\,, \;\; t \to -t\,, && \text{if } \{f_{\rm sh}\,,\; E_{\rm a}\,,\; m = 0\}\,.
\end{aligned} \tag{3}$$

By a proper choice of the functions $f(x)$ and $E(t)$ conditions for all symmetries are violated, and consequently a nonzero current is expected. This has been confirmed by many numerical studies [7, 8, 16], and even recently by experimental observations of the rectified motion of cold atoms in optical lattices [15]. These results show that typically the largest possible values of the directed current are obtained in the underdamped limit close to the Hamiltonian case $\gamma = 0$ [7, 8, 16]. In this limit the dynamical mechanisms have been studied [13, 16]. As a result the main contributions are due to nonlinear resonances [16], and efficient sum rules [17] allow to account for the average current value.

2.2. The case of quasiperiodic functions

Here we follow and generalize the symmetry analysis approach done in [14]. We consider a quasiperiodic function $g(z)$ to be of the form

$$g(z) \equiv \tilde{g}(z_1, z_2, \ldots, z_N)\,, \ \frac{dz_i}{dz} = \Omega_i\,, \tag{4}$$

where all ratios Ω_i/Ω_j are irrational if $i \neq j$ and $\tilde{g}(z_1, z_2, \ldots, z_i+2\pi, \ldots, z_N) = \tilde{g}(z_1, z_2, \ldots, z_i, \ldots, z_N)$ for any i. Such function may have numerous symmetries. With respect to the following symmetry analysis of the equation of motion we will list here only those symmetries of $\tilde{g}$ which are of relevance. It can be symmetric $\tilde{g}_{\rm s}(z_1, z_2, \ldots, z_N) = \tilde{g}_{\rm s}(-z_1, -z_2, \ldots, -z_N)$, antisymmetric $\tilde{g}_{\rm a}(z_1, z_2, \ldots, z_N) = -\tilde{g}_{\rm a}(-z_1, -z_2, \ldots, -z_N)$. It can be also shift-symmetric for a given set of indices $\tilde{g}_{{\rm sh},\{i,j,\ldots,m\}}$ which means that $\tilde{g}$ changes sign when a shift by π is performed in the direction of each $z_i, z_j, \ldots, z_m$ only, leaving the other variables unchanged.

The relevant symmetry properties of $\tilde{g}$ are thus studied on the compact space of variables $\{z_1, z_2, \ldots, z_N\}$. The irrationality of the frequency ratios guarantees that in the course of evolution of z this compact space is densily scanned by these variables with uniform density in the limit of large z. At the same time we note that it is always possible to find a large enough value Z such that

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau (g(z+Z) - g(z))^2 {\rm d}z \ < \ \varepsilon \tag{5}$$

with (arbitrarily) small absolute value of ε. For a given value of ε this defines a quasiperiod Z of the function $g(z)$.

In order to make the symmetry analysis of the equation of motion transparent, we rewrite it (skipping the noise term) in the following form:

$$\begin{aligned} m\ddot{x} + \gamma\dot{x} - f(x) - E(\phi_1, \phi_2, \ldots, \phi_N) \ &= \ 0\,, \\ \dot{\phi}_1 \ &= \ \omega_1\,, \\ \dot{\phi}_2 \ &= \ \omega_2\,, \\ &\ \ \vdots \\ \dot{\phi}_N \ &= \ \omega_N\,. \end{aligned} \tag{6}$$

The function $f(x)$ is also assumed to be quasiperiodic with M corresponding spatial frequencies.

The following symmetries can be identified, which change the sign of $\langle \dot{x} \rangle$ and leave (6) unchanged:

$$\tilde{S}_a \,:\, x \to -x\,,\ \phi_{i,j,\ldots,m} \to \phi_{i,j,\ldots,m} + \pi\,,\ \text{if } \{f_{\rm a}\,,\ E_{{\rm sh},\{i,j,\ldots,m\}}\}\,, \quad (7)$$

$$\tilde{S}_b \,:\, x \to x\,, \qquad t \to -t\,, \qquad \text{if } \{E_{\rm s}\,,\ \gamma = 0\}\,. \quad (8)$$

The symmetry $\tilde{S}_a$ is actually a set of various symmetry operations which are defined by the given subset of indices $\{i, j, \ldots, m\}$.

The prediction then is that if for a given set of parameters any of the relevant symmetries (8) is fulfilled, the average current will be zero. If however the choice of functions $f(x)$ and $E(t)$ is such that the symmetries are violated, a nonzero current can be expected to appear.

3. Numerical studies

We considered three different cases for our numerical studies. The first one (case I) is with quasiperiodic $E(t)$ and periodic $f(x)$:

$$\begin{aligned} f(x) &= \sin(x)\,, \\ E(t) &= a\cos(\omega_1 t) + b\cos(\omega_2 t + \delta_1) + c\cos(2\omega_1 t + \delta_2)\,. \end{aligned} \quad (9)$$

The second case II is with quasiperiodic $f(x)$ and periodic $E(t)$:

$$\begin{aligned} f(x) &= \sin(x) + b\cos(\omega_2 x + \delta_1)\,, \\ E(t) &= a\cos(\omega_1 t) + c\cos(2\omega_1 t + \delta_2)\,. \end{aligned} \quad (10)$$

The third one (case III) is with both quasiperiodic $f(x)$ and $E(t)$:

$$\begin{aligned} f(x) &= \sin(x) + b\cos(\omega_2 x + \delta_1)\,, \\ E(t) &= a\cos(\omega_1 t) + c\cos(2\omega_1 t + \delta_2) + d\cos(\omega_3 t + \delta_3)\,. \end{aligned} \quad (11)$$

The numerical values of the parameters are:

$$\begin{aligned} a &= -0.33\,, \quad b = 0.24\,, \quad c = 0.123\,, \quad d = 0.2\,, \\ \delta_1 &= 1.2\,, \quad \delta_2 = 1\,, \quad \delta_3 = 0.75\,, \\ \omega_1 &= 1\,, \quad \omega_2 = \sqrt{5} - 1\,, \quad \omega_3 = \sqrt{2.5}\,. \end{aligned} \quad (12)$$

All three cases are chosen such that $\tilde{S}_{\rm a}$ and $\tilde{S}_b$ are violated. We will consider here only the Hamiltonian case $\gamma = 0$, $\xi = 0$ and $m = 1$. The numerical integration of (6) is done using a symplectic leap-frog algorithm with time step 0.01.

As is well known for the case of periodic functions $f(x)$ and $E(t)$, the strongest contributions to a nonzero current originate from the stochastic layer in the phase space. This layer appears because of the underlying nonintegrability of the driven system. It is bounded by invariant tori and encloses internal resonances (regular islands). For periodic functions $f(x)$ and $E(t)$ it is possible to visualize this stochastic layer using a stroboscopic Poincare map in time (after each period of the drive $E(T)$) by plotting $p = \dot{x}$ and $x \mathrm{mod} 2\pi$. The periodicity of the potential in x can be used to shift the x coordinate by multiples of the spatial period of $f(x)$ into a given stripe of length of the spatial period. Invariant tori will be then characterized by a certain winding number or average velocity on such a torus. The dynamical mechanism of rectification in the stochastic layer is due to desymmetrizations of probabilities to enter and stay in the fractal region of boundaries of the stochastic layer close to invariant tori with nonzero winding numbers [13, 16]. An efficient sum rule allows to compute the average current using the area of the stochastic layer, the areas of enclosed regular islands and their corresponding winding numbers (see [17] for details).

For quasiperiodic functions it is not that straightforward to study Poincare plots. Nevertheless we performed such an analysis, using for the stroboscopic period $2\pi/\omega_1$ in time, and still folding x into a stripe of length 2π.

In Fig. 1 we plot the $x(t)$ dependence for case I. We observe a nonzero average current, which is interrupted by long and rare quasiballistic flights due to the abovementioned dynamics in the boundary regions of the stochas-

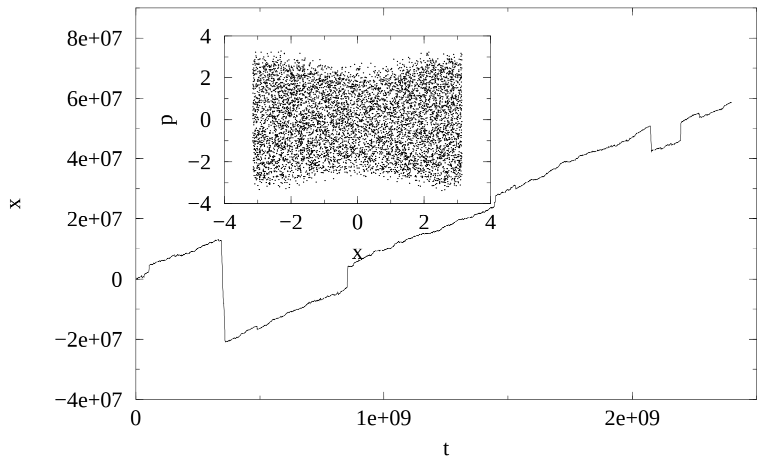

Fig. 1. $x(t)$ dependence for case I (see text for parameters). Note the large scales of both time and space. Inset: Poincare map for the trajectory. After each time 2π the coordinate $x \mathrm{mod} 2\pi$ and the velocity $p = \dot{x}$ are plotted.

tic layer. A rough estimate for the average velocity from these data suggests $J \approx 0.02 \ldots 0.03$. In the inset we show a Poincare map. Because of the quasiperiodicity of $E(t)$ our method is not capable of detecting possible regular islands embedded into the stochastic layer. The layer is bounded from above and below by invariant tori with average kinetic energies $T_+ = 3.47$ and $T_- = 3.24$, respectively. Assuming that embedded islands are not present, using a rough estimate for the layer width in p-direction $\Delta p = 5$ and using the sum rule $J_{\rm sr} = (T_+ - T_-)/\Delta p$ [17] we obtain an estimate for the current $J_{\rm sr} \approx 0.009$.

Case II is shown in Fig. 2. Here the current is roughly one order of magnitude larger $J \approx 0.2$. With the same assumptions as for case I the sum rule estimate yields $J_{\rm sr} \approx 0.088$ using $T_+ = 2.87$, $T_- = 2.43$ and $\Delta p = 5$.

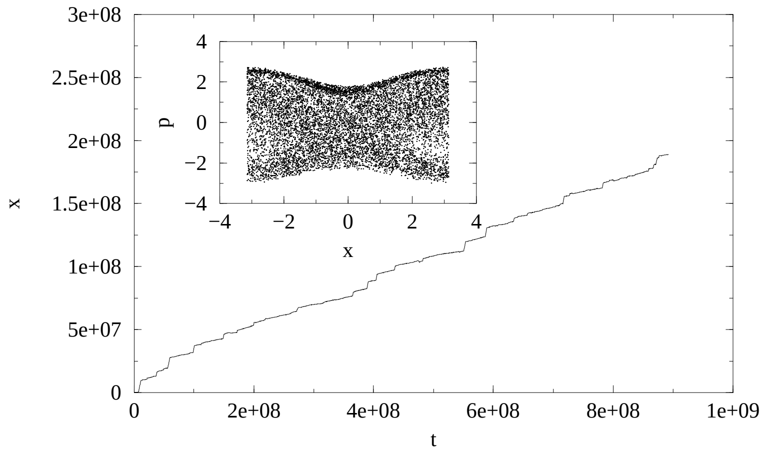

Fig. 2. $x(t)$ dependence for case II (see text for parameters). Note the large scales of both time and space. Inset: Poincare map for the trajectory. After each time 2π the coordinate $x \bmod 2\pi$ and the velocity $p = \dot{x}$ are plotted.

Finally case III is depicted in Fig. 3. Again the current is roughly one order of magnitude larger $J \approx 0.2$ compared to case I. With the same assumptions as for case I the sum rule estimate yields $J_{\rm sr} \approx 0.19$ using $T_+ = 4.74$, $T_- = 3.78$ and $\Delta p = 5$.

Note that for case I J is 2 times smaller than J_{sr}, while for case II it is 2 times larger.

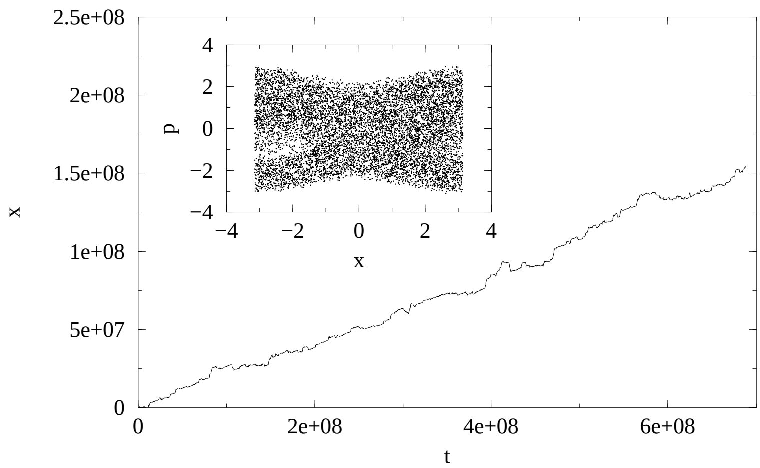

Fig. 3. $x(t)$ dependence for case III (see text for parameters). Note the large scales of both time and space. Inset: Poincare map for the trajectory. After each time 2π the coordinate $x \mathrm{mod} 2\pi$ and the velocity $p = \dot{x}$ are plotted.

4. Discussion

The obtained symmetry analysis and its predictions concerning directed transport have been nicely confirmed in the Hamiltonian limit. In addition we tested various situations where at least one of the symmetries is not violated, and checked that the averaged current is indeed zero. We also tested numerically the case of weak damping and stochastic forcing. In analogy to [13] we found that the dissipation is limiting the time of duration of quasiballistic flights and leads to a reduction of the averaged current value.

A number of problems are still to be investigated. First we did not observe fractality in the $x(t)$ curves with respect to ballistic flights, as compared to the case of periodic functions [16]. Instead we observe the absence of intermediate ballistic flight times. We find only very long flights (see Figs. 1–3) and relatively short flights which are 2–3 orders of magnitude shorter. This implies that the structure of the boundaries of the stochastic layer is significantly altered as compared to the case of periodic functions.

Second we note some discrepancy between the estimated current value $J_{\rm sr}$ using the sum rule [17] and the numerically observed one J, although the orders of magnitude coincide. The main reason may be the assumption that we can neglect possible regular islands embedded in the stochastic layer. In fact these islands most probably exist, but we have not found so far an easy numerical method to reliably extract them from the data.

Finally we expect these systems to be characterized by several competing length and time scales. The characteristic structure of the phase space flow of our system implies certain length and time scales, which are related to the spatial and temporal periods in the case of strictly periodic functions. Taking an ε much smaller in (5), we will obtain some large but finite second space and time scales, on which the functions $f(x)$ and $E(t)$ will be nearly periodic. We speculate that these competing scales are the origin of the disappearance of fractality in the phase space flow on certain times and spatial distances. These issues will be studied in more detail in future work.

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