Deterministic Walks in Random Environments

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Abstract

Deterministic walks in random environments (DWRE) occupy an intermediate position between purely random (generated by random trials) and purely deterministic (generated by deterministic dynamical systems, e.g., by maps) models of diffusion. These models combine deterministic and probabilistic features. We review general properties of DWRE and demonstrate that, to a large extent, their dynamics and their statistics can be analyzed consecutively and separately. We also show that orbits of one dimensional walks in rigid environments with non-constant rigidity almost surely visit each site infinitely many times.

Key words: Random walks, Deterministic walks, Random environments, Cellular automata, Lorentz lattice gases
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1 Introduction

Transport properties of the systems of particles are traditionally described by stochastic models. By far the most popular one is the classical model of a diffusion given by the diffusion equation. The well known fact that a fundamental solution of the diffusion equation is also a transition probabilities function for the Wiener process allows one to build a bridge between the invertible microdynamics and noninvertible macrodynamics. This relation was exploited in the first rigorous derivation of the time noninvertible macrodynamics (governed by the diffusion equation) from the time invertible macrodynamics (governed by Newton's equations) [1]. That paper dealt with the special case of a two-dimensional Lorentz gas, in which the configuration of immovable particles

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(scatterers) is assumed to be periodic, and a free path of the moving particle is assumed to be bounded from above by some constant (so called, finite horizon condition).

This is just about the only result of this type available even though some attempts were made to extend this approach to another systems. The technique developed in [2] was elaborated in [3], which allowed for various generalizations [4] including a proof of the existence of a (normal) diffusion for a periodic Lorentz gas in higher ($d \geq 3$) dimensions [5], and a proof of existence of shear and bulk viscosities in the periodic two disks fluid [6].

All these results were obtained under the condition of boundedness of a free path (finite horizon). In fact, without this condition there is no normal diffusion (diffusion coefficient diverges) in a periodic Lorentz gas (see [7] and the much more detailed analysis in [8]).

An extremely interesting and important problem is to analyze the Lorentz gas with random distribution of scatterers and to derive the corresponding hydrodynamic (diffusion in this case) equation. Currently this problem seems to be completely out of reach for the existing techniques in the theory of random walks in random environments.

It is worthwhile to mention that the (kinetic!) Boltzmann equation has been rigorously derived [9] for the Lorentz gas with random distribution of scatterers in the Boltzmann-Grad limit. However, the studies of hydrodynamic behavior require an analysis for large time scales contrary to the short time kinetic stage of evolution governed by the Boltzmann equation.

A natural strategy to attack the Lorentz gas with random distribution of scatterers is to advance in this respect the theory or random walks in random environments. This direction is developing in the long series of papers by Sinai and collaborators starting with the paper [10] which is extensively cited in this volume.

Another approach to an attack on the Lorentz gas with random distribution of scatterers has been launched in [11]. It has been suggested to simplify the model and to consider the Lorentz gas on a lattice with scatterers randomly distributed along its vertices. These models sometimes are called (deterministic) Lorentz lattice gas (LLG).

It appeared that the dynamics of LLG is quite different from the Lorentz gas. They are much closer in this respect to the Ehrenfest's wind-tree model. However, LLGs are found to be very interesting because they allow for numerous important (for various applications) modifications and generalizations. Moreover, in many branches of science models mathematically equivalent to LLGs were independently introduced and extensively studied. LLGs form a subclass of a general class of systems which we call Deterministic Walks in Random Environments (DWRE).

Traditionally there are two classes of models which describe the motion (e.g., a diffusion) of some objects in a medium. One of these classes is formed by purely stochastic processes, which include classical random walks, Levy walks, etc. In this class an outcome of a random trial (a throw of possibly sophisticated dice) determines the next move of the particle. Another class is formed by purely deterministic models, where a dynamical system (Baker or another map) is applied to decide where the particle should move. Both these types of models are analyzed in many articles in this volume.

To emphasize even more the importance of DWRE it is worthwhile to mention that these models naturally appear in much more sophisticated and relevant models of statistical mechanics than the Lorentz gas. It is well known that the main deficiency of the Lorentz gas is the lack of a local thermal equilibrium, which is a fundamental assumption of irreversible thermodynamics. To overcome this deficiency an elegant model of a Lorentz gas with rotating scatterers was recently introduced [12]. This model is much more complicated to analyze than the classical Lorentz gas. However, in some cases the Lorentz gas with rotating scatterers can be (exactly) reduced to some exactly solvable models of DWRE [13].

In this paper we briefly review some existing results and state a new general result of the theory of DWRE (Sect. 2). Then (Sect. 3) we describe in detail the exactly solvable class of one-dimensional DWDE. This class includes dynamical models of sub-, super- and normal-diffusion. The corresponding model with the sub-diffusion behavior can be viewed as a deterministic analog of the Ornstein-Uhlenbeck process, while the model with super-diffusion describes an interesting phenomenon of (eventual) propagation in a random media. By the eventual propagation we mean a type of the particle's motion which consists of two stages. At the first stage the particle experience a kind of irregular (random) motion without any preferable direction of propagation. At the second stage the particle (still going back and forth) advance (in average) into one direction. The limiting case of this is the ballistic one which occurs with the probability zero. In Sect. 4 we prove some new results on diffusion in a more general class of one-dimensional DWRE. Some concluding remarks are given in Sect. 5.

2 Definition and Some Properties of DWRE

Deterministic walks in random environments (DWRE) can be defined on any graph G. These systems are discrete in time. At each time step a moving object (particle, signal, wave, ant, read/write head of a Turing machine, etc., depending in which branch of science the corresponding model is considered) hops from a vertex $g \in G$ to another vertex g' of this graph. The choice of the vertex g' is completely determined by the scatterer (local scattering rule, scattering matrix, etc.) $S(g)$ which *currently* occupies the vertex g and by the edge of G along which the particle came to g. The collection of all scatterers in all vertices of G forms an environment in which the particle moves.

DWRE consists of two subclasses of models. The first subclass is formed by the models with fixed environment. This means that a particle which arrives at the vertex g at any moment of time always encounters one and the same scatterer. The second (much more rich) subclass consists of models which allow for a feedback of the moving object on the environment by changing a type of scatterer at the vertex g upon visits to this vertex. (Only a number of hits matters, not the directions along which the particle came to the given vertex.) Here there are two possibilities. Either the particle first gets scattered according to the scatterer $S(g)$ and then the type of the scatterer at the vertex g changes or at first the scatterer $S(g)$ changes into (another or the same) scatterer $S'(g)$, and then the particle gets scattered (at the vertex g) by the scatterer $S'(g)$. Dynamics in both cases remains completely similar. Therefore, in what follows we stick with the first option. (If $S(g)$ is always identical to $S'(g)$ at any moment of time and at any vertex $g \in G$, then we are dealing with DWRE with fixed environment.)

It is easy to see that DWRE with fixed environment are formal (discrete) analogs of the Lorentz gas. It is enough (as in the Lorentz gas) to consider motion of a single particle, because different moving particles do not interact.

On the contrary, in DWRE with changing (upon collisions with particles) environment the moving particles do interact via environment. Therefore dynamics of the one particle and of many-particles models are usually quite different [14,15]. While both classes of DWRE are considered in physics [11,16,17], the applications in other areas (biology, computer science, etc.) mostly deal with models with changing environments (see [18] and references therein). In particular, DWRE with changing environment serve as models of parallel computing.

For the sake of simplicity, we consider in this paper only the (most interesting) case when the graph G is a lattice, i.e., the simplest nondirected graph. Another simplifying assumption we impose is to allow the particle to hop only to the neighboring vertices of G.

Example 1 Let G be the square lattice. Then in the corresponding DWRE there are at most 4^4 different scatterers. Indeed each scatterer defines along which direction (out of four compass directions) the particle must leave the vertex g if it came to g along some (again one out of four) compass direction. (In a great majority of publications devoted to DWRE, including all numerical investigations of these models, just two different scatterers in any model were considered.)

The deterministic character of dynamics in DWRE is generated by deterministic evolution of the environment. These models can be interpreted as many-dimensional Turing machines, where at each site g of the lattice G there is (fixed from the very beginning) sequence of scatterers (program) $S(g) = \{S_i(g)\}$. Hence, upon the ith visit to the vertex g the particle (read/write head of a Turing machine) gets scattered according to the scatterer $S_i(g)$. At time $t = 0$ the programs (sequences of scatterers) are assumed to be randomly distributed over the vertices of the graph G. Therefore, they form a random environment in which the particle moves. However, each program $\{S_i(g)\}$ is deterministically organized (one does not flip a coin to figure out what the next scatterer $S_{i+1}(g)$ should be).

Example 2 Let G be again the square lattice. Consider two types of scatterers formed by the segments aligned along diagonals of G (left and right mirrors [11]). Let initially these mirrors are independently distributed over vertices of G with probabilities p_r and p_ℓ, $p_r + p_\ell = 1$. Suppose that upon collision with a particle right mirror becomes left mirror and vice versa. Then one gets a flipping mirror model [11] where each program $S_i(g)$ is a two-periodic sequence. Thus the programs differ only by their phases (types of initial scatterers $S_1(g)$, $g \in G$).

Therefore, from the general point of view, DWRE form a class of deterministic cellular automata. Hence, these models are just purely dynamical systems. This approach is not very fruitful though. Moreover, no rigorous results at all about these systems were obtained with this formal approach. On the contrary, by treating these models as hybrid ones, which have both deterministic (dynamics) and random (environment) features it was possible to develop a rather rich theory of these systems (especially in one dimension).

By adopting this approach we assume that at any moment of time t the graph G is partitioned into two regions, $G = G_D(t) \cup G_R(t)$, where the deterministic region $G_D(t)$ is formed by all the vertices already (to the moment t) visited by the particle and the random region $G_R(t) = G \backslash G_D(t)$. Observe that already the first visit to a vertex makes this vertex to belong forever to the deterministic region. Even without changing the type of the scatterer the particle would know which type is there in each of its returns to this vertex. Dynamics of DWRE can be viewed as a process of growth of the deterministic region into the random region. Therefore this dynamics is essentially intermittent because deterministic pieces of each unbounded orbit are surrounded by the pieces where the particle explores an unknown random region. Obviously, bounded periodic orbits eventually belong to a deterministic region.

Therefore in Deterministic Walks in Random Environments it is possible, in a sense, to separate deterministic and stochastic features in the evolution of these models. We call the following statement the fundamental theorem of the theory of DWRE. It is interesting that this almost obvious fact has not been observed and stated so far despite the numerous theoretical and computational studies of various concrete models of DWRE.

Each orbit (solution) of any model of DWRE is a broken line, i.e., it is continuous and consists of straight segments. (We assume here that all edges of the graph G, where the corresponding model is defined are straight segments.) By the structure of an orbit we mean a sequence of such segments.

Theorem 1 (Fundamental theorem of the theory of DWRE) *Consider any model of DWRE on some graph G. Then the structure of orbits (solutions) of such model does not depend on the probability distribution of scatterers among the vertices $g \in G$.*

PROOF. Consider two models of DWRE which are defined on the same graph G and admit the same scatterers which change upon collisions with the particle in the same way and differ only by probability distributions of these scatterers among the vertices of G. Then all possible initial configurations of scatterers on G are the same for both models. Moreover, the time evolutions of both models will be the same for identical initial configurations of scatterers in G. Therefore, the corresponding orbits of these two models of DWRE will have the same structure, i.e., they will be geometrically identical. For each orbit of the first model there is the corresponding orbit of the second model and vice versa. Finally recall that any invariant set of a dynamical system consists of its orbits.

Corollary 1 *Probabilities of some geometrically identical orbits and of some invariant sets in the models of DWRE considered in* Theorem 1 *can be equal. However, if the probabilities of any two corresponding (consisting of geometrically identical orbits) invariant sets are equal then the two models of DWRE are identical.*

In this paper we mostly deal with one dimensional DWRE. Therefore, for the sake of brevity, we will give a formal definition of these systems only for the simplest case of the one-dimensional lattice.

Without any loss of generality, one-dimensional regular lattice can be identified with the set of integers $\mathbb{Z}$. The particle moves with the unit speed along the lattice $\mathbb{Z}$, i.e., $v(t) = 1$ or $v(t) = -1$ at each moment of time t. Denote by $z(t)$ position of the particle at time t. Then the position of the particle at the next moment of time is determined by $v(t)$ and by the type of scatterer located at the site $z(t)$. Recall that we consider a discrete time. Hence it is enough to specify that $v(t)$ is the velocity with which the particle approaches a site $z(t)$.

There are only 2^2 possible scatterers on $\mathbb{Z}$, which we will denote by BS, FS, LS, and RS. The backward scatterer BS always changes the velocity of the particle to the opposite one. In other words, if BS is located at a site $z(t) \in \mathbb{Z}$, then $v(t_+) = -v(t)$. The forward scatterer FS is the trivial one, which does not change the velocity of the particle, i.e., $v(t+1) = v(t)$ if at the site $z(t)$ was the forward scatterer. The other two scatterers, LS and RS, which will be referred to as the left and the right scatterer respectively, are the semi-transparent ones. It means that LS (RS) sends all scattered particles to the left (right), i.e., if a LS (RS) is located at a site $z(t) \in \mathbb{Z}$ then $z(t+1) = z(t)-1$ ($z(t+1) = z(t)+1$).

We are ready now to exactly define the dynamics of our system. With each vertex $z \in \mathbb{Z}$ is associated a semi-infinite sequence $\mathbb{S}(z) = \{S_i(z)\}_{i=0}^{\infty}$ of scattering operators, where $S_0(z)$ is the scattering operator for vertex z at time 0. To capture the time evolution of the scattering operator at a site, we define a second sequence of scattering operators $\mathbb{S}(z,t)$ such that if vertex z has experienced $\tau(z,t)$ scattering events or particle visits in the time interval $(0,t)$, we have

$$\mathbb{S}(z,t) = \{S_n(z,t)\}_{n=0}^{\infty} = \left\{S_{\tau(z,t)}(z), S_{\tau(z,t)+1}(z), S_{\tau(z,t)+2}(z) \cdots \right\}.$$

Consider the shift U which acts on the space of semi-infinite sequences $\Omega = \{\omega_i\}_{i=0}^{\infty}$ as $U\{\omega_0, \omega_1, \omega_2, \ldots\} = \{\omega_1, \omega_2, \omega_3, \ldots\}$. We can now write

$$\mathbb{S}(z,t) = \{S_n(z,t)\}_{n=0}^{\infty} = U^{\tau(z,t)}\mathbb{S}(z,0).$$

If a scattering event occurs at the time t at vertex z, then we have

$$\begin{aligned} v(t+1) &= S_0(z,t)v(t) \\ \mathbb{S}(z',t+1) &= \mathbb{S}(z',t) \quad \text{for } z' \neq z \\ \mathbb{S}(z,t+1) &= U\mathbb{S}(z,t). \end{aligned} \tag{1}$$

There is a big class of DWRE for which it was possible to develop in one dimension rather complete rigorous theory. This class is called walks in rigid environments [19]. In these models each site z of the lattice is characterized by a parameter $r(z)$ which is called a local rigidity of an environment.

A type of scatterer at the vertex z changes after the $r(z)$th visit by particle to z. In other words, the particle must hit (collide with) a scatterer at the vertex z exactly $r(z)$ times in order to change the type of this scatterer to another one.

Walks in rigid environments generalize all the models of DWRE studied numerically to this time. Moreover, walks in rigid environments, in a sense, interpolate between the models with fixed environments and the models with flipping environments, where the type of the scatterer changes each time when the particle visits the corresponding vertex [11,20]. Indeed, $r(z) \equiv 1$ in the models with flipping environments and $r(z) \equiv \infty$ if the environment is fixed.

We consider first the walks in rigid environments with constant rigidity, i.e., $r(z) = r$ for all vertices z, where $1 \leq r \leq \infty$ is a positive integer. Again for simplicity we discuss here only one-dimensional lattices.

Let Sc be the set of all possible scatterers on $\mathbb{Z}$. Then $Sc = \{BS, FS, LS, RS\}$. Walks in rigid environments are defined by three objects:

(i) a set $\hat{S} \subset Sc$ of scatterers, which we call the set of allowed scatterers, i.e., only the scatterers from $\hat{S}$ may ever appear in a given model of DWRE;
(ii) an integer valued function $r(z)$, $z \in \mathbb{Z}$, which is called a rigidity;
(iii) a collection of functions $e(z) : \hat{S} \to \hat{S}$, $z \in \mathbb{Z}$, determining the sequence in which scatterers change each other in the vertez z.

Define a function a_z which counts how many times vertex z has been visited since the last change of the type of scatterer in z as

$$\begin{aligned} a_z(s,i) &= (s, i+1), \quad \text{if } 0 \leq i < r(z) - 1 \\ a_z(s,i) &= (e(s), 0), \quad \text{if } i = r(z) - 1, \end{aligned} \tag{2}$$

where $s \in \hat{S}$ and i is an integer. We will call i an index of the corresponding scatterer.

Denote by $s(z)$ a type of a scatterer that is located at the site $z \in \mathbb{Z}$. The type of scatterer at z may change in the course of dynamics if $r(z) \geq 1$. Let $z(t)$ be the vertex where the particle is located at time t. By $(s(z))_t$ we denote the type of a scatterer located at any site $z \in \mathbb{Z}$ at a moment of time t. A symbol $s(z(t))$ will be referred to as a type of scatterer located at a moment t at the site where the particle sits at this moment.

The configuration space of our system is $W = S_r^{\mathbb{Z}} \times \mathbb{Z}$, where $S_r^{\mathbb{Z}}$ is a configuration of scatterers (together with a number of visits occurred to a site $z \in \mathbb{Z}$ while a present type of scatterer was located at z), and the second factor $\mathbb{Z}$ corresponds to the position of the particle. The phase space of this model is $\tilde{\Omega} = W \times \{-1, 1\}$.

The equations governing the dynamics read as

$$\begin{aligned} v(t+1) &= g(v(t), s(z(t))), \\ z(t+1) &= z(t) + v(t+1), \\ (s(z(t+1)), i) &= (s(z(t)), i), \quad \text{if } z \neq z(t) \\ ((s(z(t)), i) &= a_z(s(z(t)), i), \quad \text{if } z = z(t), \end{aligned} \tag{3}$$

where the function $g(v(t), s(z(t)))$ is completely defined by the type of scatterer $S(z(t))$ located at z at the moment t. For any concrete model (where the set of allowed scatterers is already defined) it can be simply written up. However, the formulas for "abstract" scatterers become rather cumbersome and are omitted.

In DWRE both the symmetries of the graph G and of the allowed scatterers $\hat{S}$ are of importance. For instance, some models of DWRE on two-dimensional square lattice $\mathbb{Z}^2$ with different sets $(\hat{S})$ of allowed scatterers can be reduced to percolation problems on different graphs [21]. The importance of a "mutual" symmetry of a lattice (graph) and of scatterers becomes especially transparent in one-dimensional walks in rigid environments.

3 Walks in Rigid Environments

Consider walks in rigid environments on $\mathbb{Z}$. It is natural to assume that probability distribution of scatterers is translationally invariant. Therefore, for our purposes the only nontrivial symmetry of the one-dimensional lattice $\mathbb{Z}$ is its reflection about the origin. (Recall that we always assume that the particle starts its motion at the origin of the lattice.)

For $\mathbb{Z}$, two of the four possible scatterers (BS and FS) are also invariant under reflection, while the other two scatterers (LS and RS) are not. It is therefore natural to consider two following models of DWRE, one with $\hat{S} = \{BS, FS\}$, and another with $\hat{S} = \{LS, RS\}$. We will refer to the first (second) model as to NOS- (OS-) model, i.e., the model with oriented (non-oriented) scatterers. Formally, the dynamics of these two models is defined by the relations (3). We will also describe it less formally.

Both these models deal with two types of scatterers. The particle moves with unit velocity along the lattice $\mathbb{Z}$. At each integer moment of time it comes to some vertex $z(t) \in \mathbb{Z}$ and gets scattered by the scatterer located at this moment at $z(t)$. (The function $g(\cdot, \cdot)$ in (3) is easily specified by the type of this scatterer.) If the particle was scattered $r(z)$ consecutive times by this scatterer located at $z(t)$ (i.e., if the particle returned to this site with this very scatterer $r(z)$ times) then the type of the scatterer gets changed to another type.

We now specify initial conditions for our dynamical system. Without any loss of generality we can always assume that the particle starts at the origin with the initial velocity $v(0) = 1$. We also assume that indices of all scatterers were zero at $t = 0$.

According to Theorem 1 there is no need to specify the distribution of scatterers unless we are interested in statistical properties of these models. Therefore, we'll do it later.

In this section we consider models with constant rigidity, i.e., $r(z) = r$ for all $z \in \mathbb{Z}$, where $1 \leq r \leq \infty$ is an integer.

In case of infinite rigidity $r = \infty$ (i.e., when the environment is fixed) the dynamics of both OS- and NOS-models is very simple and similar. Indeed, the particle will oscillate between the two closest to the origin BS (in the NOS-model) with positive and non-positive coordinate respectively, or between the closest to the origin LS with positive coordinate and the closest to the origin RS with non-positive coordinate in the OS-model. Otherwise the particle will propagate to infinity with the velocity ± 1 if the entire positive (negative) semi-axis is occupied by FS (in the NOS-model) or by RS (LS) in the OS-model.

The dynamics of the OS-model is characterized qualitatively by the following statement [22].

Theorem 2 *In the OS-model with a constant rigidity r for any value $r < \infty$ the particle will visit each site $z \in \mathbb{Z}$ infinitely many times, unless a configuration of scatterers contains a positive infinite tail of RS or/and a negative infinity tail of LS.*

Therefore the dynamics of the OS-model is qualitatively the same for all values of rigidity r. Quite different situation is with the NOS-model [19].

Theorem 3 *Consider the NOS-model with constant rigidity $r < \infty$.*

(a) *If r is an even number then the particle visits all sites of the lattice infinitely many times, unless the configuration of scatterers has a positive tail of FS and/or a negative tail of FS. In the latter case the particle will eventually propagate in one direction with velocity $v = 1$ or $v = -1$.*
(b) *If r is an odd number then for all configurations of scatterers particle will visit any site not more than $3r$ times, and it will eventually propagate in one direction with random velocity*

Observe that qualitatively the behaviors of OS- and NOS-models occurred to be quite different. (It was studied in detail in [19,22].) Theorems 2 and 3 may suggest that the OS-model is quite similar to the NOS-model with even rigidity. However, it is not the case even qualitatively (see [19,22]).

The drastic differences between all three models (the OS-model and the NOS-model with even and odd rigidities) appear in their quantitative properties. In fact, these three models could be viewed as the simplest dynamical models of diffusion, sub- and super-diffusion.

So far we made no assumptions about distributions of scatterers in these models. Suppose now that the scatterers are distributed independently and identically along the sites of $\mathbb{Z}$.

Denote by $z_{\max}(t)$ and $z_{\min}(t)$ the sites with the maximal and the minimal coordinate respectively visited by the particle to a moment t.

Theorem 4 ([19,22]) *In the OS-model $Ez^2_{\max}(t)$, $Ez^2_{\min}(t)$ and $Ez^2(t)$ grow asymptotically as $t \to \infty$ linearly in t. In the NOS-model with even rigidity $Ez^2(t)$, $Ez^2_{\min}(t)$ and $Ez^2_{\max}(t)$ all grow as* const $\log t$. *Finally, in the NOS-model with odd rigidity they grow as* const t^2.

All these three models are completely solvable and all quantities of interest can be exactly computed (see [19,22]). For instance, the average velocity of propagation in the NOS-model with odd rigidity equals $\langle v\rangle = (r(1+2q))^{-1}$, where q is probability to have a backscatterer BS in any given site.

The NOS-model with even rigidity can be considered as a deterministic analog of the Ornstein-Uhlenbeck process. Actually in this model also there is a "force" at the origin $z = 0$ which "attracts" the particle. This "force" has, of course, a dynamical origin. In fact, almost all orbits (asymptotically) approach the configuration consisting of all backscatterers with the particle ("predominantly") sitting in a neighborhood of the origin (see the details in [19,22]).

4 Walks in Environments with Non-Constant Rigidity

Walks in rigid environments with constant rigidity form (in one dimension) completely solvable models. In this section we study walks in environments with non-constant rigidity. Although this model is too general to be completely solvable, qualitatively it can be quite well understood. In fact the regions between the local maxima of rigidity form a kind of wells where the particle spends most of time. From this point the behavior of walks in rigid environments with non-constant rigidity resembles one of Sinai's model of random walks in random environments [10].

Consider a (finite or infinite) collection of positive integers $r = \{r_1 < r_2 <$

$\cdots < r_k\}$. We assume that the rigidities $r(z)$ of vertices $z \in \mathbb{Z}$ are i.i.d.'s which assume values r_i, $1 \leq i \leq k$, with probabilities $p_i \geq 0$, $\sum_{i=1}^{k} p_i = 1$.

Observe that now a model from the very beginning is defined in probabilistic terms. The assumption of i.i.d.'s can be essentially weakened. (Actually only some very general properties of regularity of the distribution of rigidities are needed. Again we sacrifice here generality to simplicity.)

Theorem 5 *Let all four possible scatterers on $\mathbb{Z}$ be allowed, i.e., $\hat{S} = \{BS, FS, LS, RS\}$. Then in the corresponding model with i.i.d. rigidities the particle with probability one visits all sites of the lattice $\mathbb{Z}$, infinitely many times provided that $k > 1$, i.e., rigidity is not constant.*

PROOF. We outline a sketch of the proof only for the case $\hat{S} = \{BS, FS\}$. The general proof is, in fact, not much more difficult, but rather much more cumbersome because of variety of different cases one must consider. The analysis is very similar to the one for a constant rigidity [19,22]. We need to analyze only what happens on boundaries between clusters of sites with constant rigidity, because the dynamics within such clusters is already completely known [19,22].

Let $\{r(z)\}$ is the sequence of rigidities in the vertices $z \in \mathbb{Z}$. This sequence can be viewed as the function of the discrete variable z.

Consider the set of local maxima of $r(z)$, which we denote $LM(r)$. Clearly $LM(r)$ consists of segments Δ_j, $j = 0, \pm 1, \pm 2$, of consecutive vertices with constant rigidities R_j. We enumerate these segments consecutively (in both directions) by assigning index 0 to the segment which either intersect $z = 0$ or (if such segment of local maxima of rigidity does not exist) to the one with minimal positive coordinate. We call a cluster contained in $LM(r)$ open if it does not contain backscatterers. Non open clusters in $LM(r)$ are called closed.

We recall now how the particle moves within clusters with a constant rigidity [19,22]. (We also assume that $r > 1$ because the case $r = 1$ [23], although the simplest one has some non-essential differences to the general case of odd rigidity r. The case $r = 1$ can be easily included but it requires at each situation a simple special consideration, which distracts the general line of reasoning, and makes the things much more cumbersome).

Suppose that the particle moves within a segment of sites with a constant rigidity r. Let the closest to the particle (from the left and from the right) backscatterers in this segment are located at the vertices $z_1, z_2 \in \mathbb{Z}, z_1 < z_2$. At the moment when the particle penetrates through one of such backscatterers (say z_2) this backscatterer becomes a forward scatterer with index 0. At the same moment in all vertices z, $z_1 < z < z_2$, there are backscatterers with indices zero, while at z_1 there is backscatterer with index $r/2$ in case if r is even, or with index $(r+1)/2$ if r is odd [19].

We assume for simplicity that $k < \infty$. The case $k = \infty$, although quite similar, is more complicated. Consider all closed clusters with maximal rigidity r_k. The segments between such clusters will be called wells by analogy with [10]. (Observe, though, that on the contrary to [10] we consider all wells, not just the narrowest ones.)

Suppose that the particle is located in some well W. Denote the closed clusters with rigidity r_k which define this well by $L(W)$ and $R(W)$, where L and R stand for "left" and "right" respectively.

Let $L(W)$, $(R(W))$ contains more backscatterers than $R(W)$ $(L(W))$. Then, it is easy to see that the particle will eventually escape this well through the cluster $R(W)$ $(L(W))$. Before this escape the particle will visit all sites in the well W. (Obviously a number of visits to different vertices within W varies.)

After escaping W the particle will stay in a bigger well $W' \supset W$. Obviously $L(W') = L(W)$ or $R(W') = R(W)$. Let $R(W') = R(W)$. Then the number of backscatterers in $R(W')$ is strictly less than was the number of backscatterers in $R(W)$.

Again the particle will escape the well W' through the cluster $L(W')$ or $R(W')$ which has fewer backscatterers and will then move within another well W' and so on. Therefore, it is clear that the particle eventually will pass through the cluster $R(W)$.

This process of growth of a well where the particle moves is intermittent with the process of shrinking of such well. The well shrinks after the particle crosses r_k times one of the segments with maximal rigidity r_k located inside this well. Indeed, according to dynamics of the walks in rigid environments with constant rigidity [19,22] at this moment the entire this segment consists of backscatterers.

It remains to show that all orbits of our model are unbounded, i.e., the particle cannot be trapped in any well. Suppose that, on the contrary, the particle is trapped in some segment $\Delta \subset \mathbb{Z}$. Denote by $\Delta' \subset \Delta$ the collection of all vertices in Δ which the particle visits infinitely many times. Let $z_\ell \in \Delta'$ be the site with the minimal coordinate in Δ'. The scatterer located in z_ℓ has a finite rigidity. Therefore during any $r_k + 1$ consecutive visits to z_ℓ at least once at this site was a forward scatterer. Then the particle must visit the site $z_\ell - 1$ infinitely many times. Therefore, we came to the contradiction with the assumption that z_ℓ is the site with minimal coordinate in Δ'.

5 Concluding Remarks

Deterministic walks in random environments have very rich dynamics. Besides that they can mimic well both (purely) random and deterministic walks, DWRE demonstrate some quite unusual types of evolution. One of the most striking examples of this kind is an ultimate propagation of the particle in a random media. This phenomenon reminds the famous gliders in Conway's game of life. However, while in the game of life gliders appear just as very special solutions, in some classes of DWRE all orbits demonstrate such behavior. It is worthwhile to mention that this phenomenon is not a one-dimensional one. It exists as well in higher-dimensional DWRE, e.g., in the Ant model on the triangular lattice [23], where the particle, after some period of a seemingly random walk, starts to propagate in one direction along some strip in this lattice.

In the DWRE triangular random lattices the effect similar to Andersen's localization occur. Namely on such random lattices the particle's orbits become (with positive probability) bounded [24].

It has already been mentioned that dynamics of DWRE can be viewed as a process of growth of deterministic (already visited) region into the random (never visited) one. Therefore, for DWRE one can write a master equation only for probabilities of the first visit to a given site, but not for probabilities to be at a given site at time t [23–27].

In continuous limit [25–27] fluctuations (generated by the randomness of the environment) in the dynamics of DWRE disappear. It occurs because in DWRE (on the contrary to the classical biased random walk [28]) there is no appropriate scaling of the probability distribution of the environment [27]. Therefore, the most interesting behavior demonstrate deterministic walks in discrete random environments rather than in continuous ones.

Luckily one dimensional walks in rigid environments with constant rigidity are found to be completely solvable [19,22]. Therefore, these models can be easily tested against random walks in the analysis of experimental data. The advantage of the walks in rigid environments is that their parameter (rigidity) is nonrandom and has a natural physical interpretation.

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