Some new aspects of Lévy walks and flights: directed transport, manipulation through flights and population exchange

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Abstract

Lévy flights and walks have been shown to arise in a broad spectrum of areas, leading to anomalous diffusion. Here we investigate their central role in some dynamical phenomena encountered in Hamiltonian systems with a mixed phase space. In particular we discuss, within the continuous time random walk (CTRW) framework, the possibility to obtain currents in Hamiltonian systems and how to manipulate them, and the effect of population exchange between islands of stability. The latter can be viewed as the classical counterpart of chaos assisted tunneling.

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1 Introduction

Lévy flights and walks are stochastic processes which provide a framework for the description and analysis of anomalous random walks in physics, chemistry, astronomy, biology and economics [1]. It has been recognized that Lévy-stable laws, which govern these processes, provide an intiutive and powerful way in analyzing diffusion phenomena [1-6]. The so called "Lévy statistics" has been used, for instance, to describe tracer diffusion in rotating turbulent flow [2], in living polymer networks [3], and in electro-magnetically driven flows [4], diffusion of adsorbed nanoclusters [5], and in subrecoil laser cooling [6].

The characteristic of anomalous diffusion is usually based on the time evolution of a mean-squared displacement (msd) $\sigma^2(t) = \langle [r(t) - r(0)]^2 \rangle \sim t^\gamma$ which corresponds to a non-Brownian behavior for $\gamma \neq 1$ [7]. Superdiffusion, ($\gamma > 1$), is a generic behavior in the case of extended Hamiltonian systems with a mixed phase space [8]. The presence of complex boundaries between chaotic and regular regions in phase space makes a complete separation into chaotic and regular regions impossible. Due to sticky barriers (formed by *cantori*), a particle trajectory can be trapped for a long time near the corresponding regular islands and produce long unidirectional flights with a constant velocity. This velocity is equal to the corresponding winding number of the island, and leads to a superdiffusion behaviour [7].

In this paper we consider some phenomena, which have attracted recently attention: current rectification in AC-driven Hamiltonian systems (Hamiltonian *ratchets*) [9-13], manipulation of Hamiltonian systems through control of flights [14], and chaos-assisted population exchange between islands of stability [15]. We investigate these phenomena in the framework of the continuous time random walk (CTRW) and demonstrate that Lévy walks play a crucial role in the dynamics which underlies these phenomena.

The paper is organized as follows. In the next section we review briefly the CTRW formalism for Lévy walks and show their relationship to Hamiltonian kinetics. Section III is devoted to the effect of current rectification in Hamiltonian ratchet. We show that directed current is induced by the breaking of symmetry of the Lévy walks. In Section IV we demonstrate how the idea of currents through flights can be used for manipulation in dynamical system. In Section V we investigate an effect which we call chaos-assisted population exchange between symmetry related regular islands. We end with a brief summary in Section VI.

2 Lévy walks in terms of CTRW framework

2.1 Fligths and walks

Let us start by considering a trajectory of a particle given by

$$r(t) = r(t_i) + \Delta r_i, \qquad t \in (t_i, t_{i+1}), \tag{1}$$

where $r(t)$ denotes the location of the particle at time t, and t_i is a time at which the particle jumps over a distance Δr_i. We assume that the jumps occur with a finite velocity v_i [7]

$$r(t) = r(t_i) + v_i(t - t_i), \qquad t \in (t_i, t_{i+1}). \tag{2}$$

We restrict ourselves to the case of jumps with a constant velocity, $v_i = \pm v$. Next we introduce a heavy tailed probability distribution function (pdf), $\psi(t)$, for the duration of a single jump, which we refer to as an elementary step, or single motion event,

$$\psi(t) \sim t^{-\alpha}, \qquad \alpha > 1. \tag{3}$$

We will be mainly interested in the range $1 < \alpha < 3$, as we discuss below [7].

Consequently, the spatio-temporal pdf, which describes the space-time coupling inherent to Lévy walks is [5]

$$\psi(r,t) = \delta(|r| - tv)\psi(t). \tag{4}$$

We further introduce the $\Psi(r,t)$, which describes the probability density to move a distance r in one direction at a constant velocity in a elementary motion event, without necessarily stopping,

$$\Psi(r,t) = \frac{1}{2}\delta(|r| - tv)\int_t^\infty \psi(t^{'})dt^{'}. \tag{5}$$

$\psi(r,t)$ and $\Psi(r,t)$ are the two relevant ingredients required for a full description of the motion. The motion consists of a sequence of motion events, and the propagator $P(r,t)$ can be written in the following way [7],

$$P(r,t) = \Psi(r,t) + \int_0^t dt^{'} \int_{-\infty}^{\infty} dr^{'}\psi(r^{'},t^{'})\Psi(r - r^{'}, t - t^{'}) + \ldots. \tag{6}$$

The propagator includes all the possible ways of motion that bring the particle to r at t. The first term in Eq.(6) denotes the contribution of reaching location r in time t in a single motion event, Eq.(5). The second term is the probability density to reach r at time t with one stop, and so on, to include all combinations of motion events. In the Fourier-Laplace space ($r \to k, t \to u$) the convolution integrals are simplifed and the series in Eq.(6) can be written in a closed form as

$$P(k,u) = \frac{\Psi(k,u)}{1 - \psi(k,u)}. \tag{7}$$

Based on the propagator in Eq. (6), the time evolution of the msd is found to be[7]

$$\langle r^2(t) \sim \begin{cases} t^2 \quad , & 1 < \alpha < 2 \\ t^{4-\alpha}, & 2 < \alpha < 3, \\ t \quad , & \alpha > 3 \end{cases} . \tag{8}$$

Note in particular the result $\langle r^2(t)\rangle \sim t^{4-\alpha}$, which corresponds to superdiffusive Lévy walks and is often found for mixed phase space Hamiltonians [16]. For this intermediate enhanced diffusion regime, $2 < \alpha < 3$, the asymptotic behavior of the propagator is given by [16]

$$P(r,t) \sim t^{-1/(\alpha-1)}f(\xi), \quad \xi = |r|/t^{1/(\alpha-1)}, \tag{9}$$

where ξ is the scaling variable and $f(\xi)$ is a function that shows scaling

$$f(\xi) = \begin{cases} t^{-1(\alpha-1)}L_{\alpha-1}(-c\xi), & |r| < vt \\ \delta(|r| - vt)t^{-\alpha+1} \quad , & |r| \simeq vt, \\ 0 \quad , & |r| > vt \end{cases} . \tag{10}$$

Here $L_\alpha(x)$ is a Lévy stable distribution. The cutoff at $|r| = vt$ is a consequence of the finite constant velocity and leads to peaks at the outermost wings due to the first term in Eq.(6).

The above mentioned framework can be used now for the analysis of Hamiltonian dynamics in the case of two symmetry-related regular islands, R_+ and R_-, embedded in the fully chaotic area in phase space . These are responsible for ballistic fligths with opposite velocities determined by their winding numbers, $v_+ = -v_-$. The pdf $\psi(t)$ introduced above is basically the pdf for sticking times which displays power-law asymptotics due to hierarchical nature of near-island region [16].

In Fig.1 we show the Poincarè section, a trajectory and the propagator $P(x,t)$ for a fixed time for a Hamiltonian which describes the classical motion of a particle in a spatially standing wave with a modulating amplitude $Ecos^2(\omega t)$ with a modulation period $T = 2\pi/\omega$ [17]

$$H_s(p,x,t) = \frac{p^2}{2} + Ecos(x)cos^2(\omega t). \tag{11}$$

For the set of parameters, $E = 1$ and $\omega = 0.53$, there are only two symmetry-related regular islands, which lie at the borders of the chaotic layer and have winding numbers $v_\pm = \pm 3\omega$. We found numerically that the pdf $\psi(t)$ for a single motion duration near a ballistic island follows an asymptotic power-law behavior $\psi(t) \sim t^{-\alpha}$ with the exponent $\alpha \simeq 2.35$ [18]. Thus, here we have a Lévy walk process for which we expect an enhanced anomalous diffusion regime with the exponent $\gamma \simeq 1.65$ for msd. The evolution of the msd, $\langle x^2 \rangle \sim t^\gamma$, obtained from numerical simulation, gives the value of the exponent $\gamma \simeq 1.6$ (see Fig.1d), close to the prediction in Eq. (8) for $2 < \alpha < 3$.

In the general case of Hamiltonian systems with a chaotic layer which includes several types of regular islands with different winding numbers (including a localized one, i. e. with $v = 0$) it is also possible to give an adequate description of the kinetics in terms of a generalized Lévy walk model [13]. The case with an additional localized island, $v = 0$, can be investigated in the framework described in the next section.

2.2 Ballistic-Localized Motion Dichotomy

Anamolaus diffusion in general may exhibit complicated trajectory patterns and the case of a constant velocity with only random directions is an oversimplification of reality. An improvement of the model can be achieved by assuming an intermittent behavior of alternating localization and ballistic motion [19].

For the description of this dichotomous behavior we consider the pdfs of the ballistic motion, $\psi_b(t)$, and of the localization periods, $\psi_l(t)$, both obeying power laws,

$$\psi_b(t) \sim t^{-\alpha_b}, \qquad \psi_l(t) \sim t^{-\alpha_l}. \tag{12}$$

In analogy Eq.(4) and Eq.(5) we introduce the spatio-temporal distribution functions $\psi_l(r,t)$ and $\Psi_l(r,t)$,

$$\psi_l(r,t) = \delta(r)\psi_l(t), \quad \Psi_l(r,t) = \delta(r)\int_t^\infty \psi_l(t')dt', \tag{13}$$

where the latter quantity is the probability for not having moved until time t. For the ballistic intervals we use Eq.(4) and Eq.(5). For the propagator, one has to take into account that the particle may be initiated either in a localization mode l or in a ballistic motion b. Thus,

$$P(r,t) = g_b[\Psi_b(r,t) + \int_0^t dt' \int_{-\infty}^{\infty} dr' \psi_b(r',t')\Psi_l(r-r',t-t') + ...] +$$
$$g_l[\Psi_l(r,t) + \int_0^t dt' \int_{-\infty}^{\infty} dr' \psi_l(r',t')\Psi_b(r-r',t-t') + ...], \tag{14}$$

where g_b and g_l are weight factors defined by initial conditions. In Fourier-Laplace space

$$P(k,u) = g_b \frac{\Psi_b(k,u) + \Psi_l(k,u)\psi_b(k,u)}{1-\psi_b(k,u)\psi_l(k,u)} +$$
$$g_l \frac{\Psi_l(k,u) + \Psi_b(k,u)\psi_l(k,u)}{1-\psi_b(k,u)\psi_l(k,u)}. \tag{15}$$

The msd exponent γ depends on α_b and α_l according to

$$\gamma = \begin{cases} 4 + min\{\alpha_b, \alpha_l] - \alpha_b, & 1 < \alpha_b < 2 \\ 4 + min\{\alpha_l, 2\} - min\{3, \alpha_b\}, & \alpha_b > 2 \end{cases}. \tag{16}$$

Limiting the range of α_b to that of the intermediate enchanced diffusion [16], $2 < \alpha_b < 3$, Eq. (16) gives

$$\langle r^2(t) \rangle \sim \begin{cases} t^{2+\alpha_l-\alpha_b}, & \alpha_l < 1 \\ t^{4-\alpha_b}, & \alpha_l > 1 \end{cases}. \tag{17}$$

Depending on the two exponents α_b and α_l the diffusion is dispersive, normal or enhanced [19].

A question arises now whether it is possible to manipulate a Hamiltonian system in a way that the anomalous walks, which are a key ingredient in the system's dynamics, have a preferred direction without a trivial bias.

3 Directed currents in Hamiltonian ratchets and Lévy walks

Following the *ratchet* idea (see review in Ref. [20]), it is possible to get a DC current in a Hamiltonian system using a *zero-mean* external drive [10-13]. In order to achive this we must break time/spatial reversal symmetry of the system [10]. While the necessary condition for the presence or absence of a DC current is clearly connected to the above mentioned absence or presence of symmetries, the value and direction of the current in the chaotic layer are determined by microscopic dynamical mechanisms.

Here we would like to demonstrate that Lévy walks are the main origin of rectification in driven Hamiltonian ratchets. Another possibility is that the purely diffusive transport inside the layer also contributes to a current [11]. We will show below that this contribution is negligible in the case under consideration. Recently it has been demonstrated that breaking symmetry of Hamiltonian systems leads to an asymmetry of the resonance island structure in the chaotic layer [12,13]. This leads, in turn, to a strong symmetry breaking of flights which the particle performs in the sticking regime. This disymmetrization breaks the balance between motion in opposite directions, and results in a DC current inside the stochastic layer. Another mechanism of the symmetry breaking leading to asymmetric distribution of flights has been proposed in Ref. [21]

As an example we consider the Hamiltonian in Ref.[11],

$$H(p,x,t) = \frac{p^2}{2} + V(x,t), \tag{18}$$

$$\partial_x V(x,t) = \frac{\pi}{2.88}[cos(2\pi x) + 0.6cos(4\pi x + 0.4) - 2.3sin(2\pi t) - 1.38sin(4\pi t + 0.7)] \tag{19}$$

For the detection of the relevant ballistic islands we used the Poincarè section (Fig.2a) and the propagator $P(x,t)$ for a fixed time (Fig.2b). The peaks in the function $P(x,t)$ correspond to flights near regular islands, and their positions are determined by corresponding winding numbers (see Eq. (10)). These islands contribute to transport in positive and negative directions asymmetrically (see resonances R_5 and R_6 in Fig.2a),which results in the appearance of a finite positive current. In order to illustrate this behavior we have considered the motion of an ensemble of $N = 10^4$ particles which are initially distributed uniformly along the line $p = 0$. In Fig.2c we show the locations of the ensemble after 20000 periods of the external temporal driving (solid lines). For this ensemble we obtain a current of magnitude $J \approx 0.081$ which is in good agreement with the result in Ref.[11]. Then we remove all unidirectional flights, connected with the sticking to resonances R_5 and R_6, with duration t_c of more than 10 periods. For this purpose we integrate the equations of motion for the Hamiltonian and after time t_c we check the velocity of the particle displacement. If this velocity exceeds $v_c = 1.22$ (see Fig.2b) then the event is considered as a flight and is removed from the trajectory. The result for the modified ensemble of particles is shown in Fig.2c (dotted line) and displays an almost fully symmetric propagator. The estimate of the mean current in this case is $J \approx -0.008$. Using the same filtering technique we numerically calculate the pdf for single flights with a velocity exceeding $v_c = 1.22$ (see Fig. 3b). This pdf has a power-like tail, Eq.(3), with the exponent $\alpha \approx 2.6$, which corresponds to a Lévy walk.

Thus, in the case of the Hamiltonian mentioned above, the mechanism of current rectification is breaking of the symmetry of the flights. These asymmetric flights exist due to the presence of asymmetric resonance islands with nonzero winding numbers.

4 Manipulation of Hamiltonian system by symmetry breaking of the flights

In this section we will show how understanding the symmetry breaking (SB) mechanism for directed motion, discussed in the previous section, naturally leads to a new way to manipulate classical Hamiltonian systems.

In the previous section we have shown that directed particles drift stems from asymmetry in the structure of the ballistic islands with positive and negative winding numbers. Based on this, we expect that controlling the manifold overlap in phase space, one can control the directed transport by tuning the value of the velocity. We now describe a possible way to manipulate a particle through SB during a finite time interval t_{SB}. In order to do this we use two features of the system: (i) the possibility to temporarily remove the barriers in phase space (formed by invariant KAM-tori) between different invariant manifolds, and (ii) the sticky nature of the regular islands. Namely, one can remove the barriers from the phase space during a time interval t_{SB} and then restore them. This can be viewed as an act of a *demon* [8]. Here the demon removes the barrier ("opens a door") at time t_{on} and restores the barrier ("closes the door") at time t_{off}. Due to the stickness property, the information required for the control the particle is the particle velocity only. This means that the "door" closes when the velocity of the particle is close to a desired winding number. The realization of this strategy for a one-particle system was demonstrated in Ref.[17].

Here we concentrate on the application of the SB strategy in the case of a statistical ensemble of particles. In this case, the SB can change the populations of particles on the different manifolds through the control of the KAM-tori barriers. Let us consider the example of the Hamiltonian H_s in Eq. (11).

We can break all relevant symmetries by switching on a second standing wave, shifted with respect to the main one:

$$\begin{aligned} H(p,x,t) &= H_s(p,x,t) \\ &+\Phi(t,t_{on},t_{off})E_2 cos(x+\phi)cos^2(\omega t+\tau), \end{aligned} \tag{20}$$

where ϕ and τ are spatial and temporal shift constants, $\Phi(t,t_{on},t_{off}) = \Theta(t-t_{on})\Theta(t_{off}-t)$ is a square pulse function, and t_{on} and t_{off} are the switching on and switching off times. The duration time for SB-force is therefore $t_{SB} = t_{off} - t_{on}$. The Hamiltonian in Eq.(20) losses the spatial and time reversal symmetries for $\phi \neq m\pi, m = 0, 1, ...$ and $\tau \neq lT/2, l = 0, 1, ...$

We consider the dynamics of an ensemble of particles with an initial Maxwellian distribution in p, and which is homogeneous in x, inside one spatial period of the potential

$$\rho(p,x,0) = \frac{1}{2\pi}\sqrt{\frac{\beta}{2\pi}}\mathrm{e}^{-\frac{\beta}{2}p^2}\Theta(x)\Theta(2\pi - x), \tag{21}$$

with $\beta = 10$.

Under the influence of the main standing wave, $E_1\cos(x)\cos^2(\omega t)$, the ensemble performs diffusive spreading with no drift, as shown in Section 2. Now we show that using SB for a finite duration t_{SB}, i. e. a pulse of a second force, Eq. (20), it is possible to chip off a small fraction of the particles from the an initial "cloud" and transport it in a preasiggned direction. Namely, a small fraction (compared to the initial ensemble) of particles can be locked into the manifold with a nonzero drift. After switching off the pulse of the second standing wave the particles move in the prescribed direction with a velocity of the corresponding manifold. This is a kind of tweezers, which chip off some fraction of particles. In Fig.3 we show an example of the realization of this strategy. SB induces an overlap of the main chaotic layer with the thin upper ballistic layer. The number of chipped particles can be controlled by varying of t_{SB}. For example, for $t_{SB} = 10T$, the chipped fraction is about 3% of the initial cloud and for $t_{SB} = 20T$ it is about 7%. The direction of the motion of chipped fraction is controlled by the values of the shift constants. For example, direction inversion (mirroring the layer overlap) can be obtained by a simple shift inversion $\phi \to -\phi$ or $\tau \to T - \tau$.

It should be mentioned that this manipulation can not be achieved by standard techniques using an external bias, since this will lead to the total displacement of the ensemble. The SB approach, proposed here, provides therefore a new possibility to perform a nontrivial manipulation on a statistical ensembles using zero-mean external fields.

5 Population exchange between islands of stability in the background of Lévy evaporation

Let us consider again the situation where there are only two symmetry-related islands, R_- and R_+, embedded in the chaotic sea. A particle, initially trapped near one of these islands, moves away from its initial location and performs a random walk before it sticks again to an island. Chaotic diffusion can be viewed as some "communication channel" between the islands R_- and R_+. In the case of an ensemble of particles, initially prepared near one of the islands, this channel provides a possibility for population exchange between the islands. How would the particles redistribute due to this chaos-assisted "communication" between the islands? This problem is interesting in the context of the chaos-assisted tunneling effect, which has been observed recently in cold-atom experiments [22,23]. Here we look at the classical counterpart of the effect.

As a model we use the Hamiltonian in Eq.(11) with parameters taken from the experiment in Ref. [22] ($\omega = \pi, E = 21$). The corresponding Poincarè section has two symmetry-related islands, R_+ and R_-, $v_\pm = \pm 2\pi$ (inset in Fig.4) [14].

We are interested in the relaxation dynamics from the initially prepared asymmetric state, where all particles are located around R_+. For noninteracting particles, the dynamics of the ensemble of particles is equivalent to the one obtained from a long trajectory of a single particle. We check the flight status of a particle using a filtering technique described earlier in Section 3. If the particle performs a flight with a duration of at least $2T$, we consider the last visited coordinate, $(x(t); p(t))$, as the initial point for a particle from the initial ensemble (see Fig.4). Collecting S "pieces" of the *single particle* trajectory , we can obtain information on the behaviour of the *ensemble* of S particles. As in Ref.[22] we are interested in the average velocity

$$V(nT) = \frac{1}{ST}\sum_{j=1}^{S}(x_j(nT) - x_j((n-1)T)) \tag{22}$$

where S is number of particles in the ensemble. The evolution of the average velocity $V(nT)$ (for an ensemble with $S = 10^4$ particles) is shown in Figs.4b,c.

The characteristic time for the population exchange mediated by chaotic diffusion corresponds to the first minimum in the time dependence of mean ensemble velocity. This occurs at $t_{exch} \simeq 9T$. For time $t \sim t_{exch}$ a fraction of the particles $\Omega(M)$ accumulates near the opposite island R_-. Then this process is reiterated and a small fraction of particles $\Omega(\Omega(M))$ reaccumulates near R_+ at time $2t_{exch}$ which corresponds to the local maximum in the velocity $V(nT)$.

The exchange of populations takes place on the background of a slow process during which particles "evaporate" from the vicinity of R_+ into the chaotic area. This process is governed by the survival probability $\Psi(t)$, i. e. probability that the particle initially located in the vicinity of the island will stay there during time t. The probability $\Psi(t)$ can be expressed through the escape time pdf $\psi_{esc}(t)$ [16]

$$\Psi(t) = \int_t^\infty \psi_{esc}(t)dt. \tag{23}$$

Assuming a power law $\psi_{esc} \sim t^{-\alpha_{esc}}$ and taking into account the relationship between exponent α_{esc} and exponent for sticking time pdf $\alpha, \alpha_{esc} = \alpha + 1$ [16] we obtain

$$\Psi(t) \sim t^{-\alpha_{esc}+1} = t^{-\alpha+2}. \tag{24}$$

For the parameters chosen here we obtain $\alpha = 2.6$ (see inset in Fig.4c). Thus, we are again confronted with the appearance of Lévy walks. Slow evaporation of particles from the vicinity of islands is governed by a power-law $\Psi(t) \sim t^{-0.6}$ due to anomalously long sticking near regular islands.

The population exchange can be defined as a transport between two symmetry-related islands through a chaotic diffusion. Leaving one of the islands, particle performs a random walk in a chaotic area before being trapped near another island. This process is determined by the properties of the chaotic diffusion, the pdf $\psi(t)$ which is characteristic to the island itself is not sufficient for the exchange.

Using the velocity gate technique (with an accuracy 3% and duration T) we obtain the pdf for the duration of random diffusion events between two consecutive flights in *opposite* directions (see Fig.4b). This pdf has a unique maximum near t_{exch}. We assume that this pdf which accounts for events between flights in *opposite* direction contains the information about t_{exch} which is observed experimentally. Interestingly, one faces here two processes which dominate the exchange on different times scales: the fast population exchange at early times and the slow evaporation at long times. We conclude that the classically observed population exchange is mainly a short time effect.

It should be noted that the value of the characteristic time for population exchange in the classical case (in the unit of driving period T) is close to the time observed in real cold-atom systems [22].

6 Conclusion

We have studied three phenomena in Hamiltonian systems that exemplify some new aspects of Lévy walks. We have found that directed transport inside the main chaotic layer is determined by breaking the symmetry of Lévy walks which are generated due to the presence of resonance islands with nonzero winding numbers. We have shown that symmetry breaking of resonance structures for finite time durations provides a new tool for manipulating and directing dynamical systems. Finally, we have found the effect of fast chaos assisted population exchange between islands of stability. This takes place on the background of a slow "evaporating" process which is governed by the anomalous character of the escape times from the islands and leads to enhanced diffusion.

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FIGURE CAPTIONS

1. (a) Poincarè section p vs x for the Hamiltonian system in Eq.(11). (b) $x(t)$ vs t for the same Hamiltonian. Insets show enlargements of x(t) (upper left) and of Poincarè section (right) for a single ballistic flight. (c) The propagator for a fixed time, $t = 100T$, for the Hamiltonian, Eq.(11), $E = 1$, $\omega = 0.53$. (d) Mean square displacement for the anomalous diffusion within the chaotic layer averaged over 10^4 trajectories. Straight line corresponds to $\langle x^2 \rangle \propto t^\gamma$, $\gamma = 1.6$.

2. (a) Poincarè section p vs x for the system in Eq.(18-19). We indicate by the arrows the islands of relevant resonances. (b) The propagator $P(x,t)$ for given time $t = 100T$ for Hamiltonian Eq.(18-19). The dotted line corresponds to $x = 100Tv_c$, where $v_c = 1.22$. (c) Probability distribution function $P(x)$ of the system in Eq.(18-19) after $20000T$. Initially, 10^4 trajectories were started uniformly over $x, x \in [0,1]$ on the line $p = 0$. The mean current, which is determined from $J = \frac{1}{100T}\sum_{i=1}^{N} x_i(t = 100T)$, is $J \approx 0.081$ for the full dynamical evolution (solid line) and $J \approx -0.008$ for evolution without flights (dotted line). The central peak corresponds to the resonance R_1, which is localized. Inset shows long individual trajectories with (i) and without (ii) flights. The central peak corresponds to the resonance R_1, which is localized.

3. The spatial distribution of an ensemble of particle $N = 10^4$ (see text) for the Hamiltonian in Eq.(20) ($E_1 = 0.5$, $E_2 = 0.1$, $T = 2\pi$, $\phi = 1.2$, $\tau = 0.8$) (a) just before and (b) after the action of the SB force ($t_{on} = 200T$, $t_{off} = 220T$). The inset is an enlargement of the additional peak which corresponds to the chipped fraction of the directed particles.

4. (a) Poincarè section for the Hamiltonian in Eq.(11), $E = 21$, $\omega = \pi$. White area shows the ensemble located around the island R_+. (b) Time dependence of the ensemble averaged velocity, Eq.(22). Initial conditions as in Fig.4(a). The straight line corresponds to the asymptotic decay due to evaporation, $\Psi(t) \sim t^{-0.6}$. The superimposed dotted curve is the pdf for the duration of random walk events between the consecutive flights in opposite directions. (c) Time dependence of the ensemble averaged velocity $V(nT)$, Eq. (22), for short times scale. The inset shown the pdf for sticking times, $\psi(t)$, near a ballistic island. All plotted quantities are dimensionless.