Stable regions and singular trajectories in chaotic soft wall billiards

Ariel Kaplan, Nir Friedman, Mikkel Andersen, and Nir Davidson

Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel

Abstract

We present numerical and experimental results for the development of islands of stability in atom-optics billiards with soft walls. As the walls are soften, stable regions appear near singular periodic trajectories in converging (focusing) and dispersing billiards, and are surrounded by areas of "stickiness" in phase-space. The size of these islands depends on the softness of the potential in a very sensitive way.

Key words: billiards, soft-wall, atom-optics, cold atoms
PACS: 05.45.Ac, 32.80.Pj, 42.50.Vk

1 Introduction

Kolmogorov-Arnold-Moser (KAM) theory provides a framework for the understanding of an integrable Hamiltonian system which is subject to a perturbation [1]. As the strength of the perturbation is increased areas of phase-space became chaotic and eventually only islands of stability, surrounded by a chaotic "sea", survive. The opposite question is of interest as well: What will happen to a completely chaotic system when a perturbation is applied? will it remain chaotic or will islands of stability appear? The appearance of KAM islands is of great importance: First, for initial conditions falling inside the island, there will be finite range oscillations in the the tail of the correlation function. In addition, these islands are usually accompanied by areas of "stickiness" where particles belonging to the chaotic region of phase space spend an anomalous amount of time, and hence they affect the temporal correlation function even for initial conditions in the chaotic "sea". Indeed, it was argued that this stickiness can cause a power-law decay of correlations, instead of an exponential one, and thus affect the transport properties of the whole system even for a relatively small island size[2].

For the billiard system (i.e. a particle moving freely in a bounded region and reflecting elastically from the boundary), one such perturbation is making the walls "soft", as opposed to the infinitely steep potential of the ideal case. Since the billiard is a widely used paradigm for understanding the microscopic foundations of statistical mechanics, and noting that real classical molecules move in smooth potentials, we conclude that the study of soft-wall billiards is an important step towards sustaining these foundations and, in particular, Boltzmann's ergodic hypothesis.

For a certain kind of dispersing billiards it was theoretically proven[3,4] that when the wall becomes soft, an island appears near a singular, tangent trajectory (see details in section 3). More recently, and partly inspired by the numerical results presented herein, the appearance of an additional island, this time near a corner, was also shown[5]. Experimentally, some conjectures were made about the role played by the softness of walls in semi-conductor quantum dots. It was considered to be responsible for the algebraic tail in the distribution of the length of trajectories, which gives rise to the fractal nature of conductance fluctuations[6–9].

Stable islands immersed in a chaotic see play a role also in phenomena such as chaos assisted tunnelling[10] and anomalous diffusion[11]. In addition, it has been suggested[12] that the ability to manipulate the motional phase space of quantum particles, which have internal degrees of freedom, can be used as a way to increase the coherence time of spectroscopic measurements.

Observations of the formation of stable islands in a stadium shaped soft atom-optics billiard were reported in Ref. [13]. In this article, we present additional results of numerical and experimental studies of soft wall atom-optics billiards. Three unifying concepts are found in our results: First, islands of stability arise around singular trajectories (tangencies, corners) as predicted by Refs. [3–5]. Next, the size of these islands depend on the wall's softness in a very sensitive way. Finally, "stickiness" regions surround the stable islands and affect the decay properties of the billiards. In section 2, we present our numerical and experimental studies of the stability induced in an atom-optics billiard of the Bunimovich type[14] by making the walls soft. In section 3, we present results of numerical simulations for the dynamics of atoms inside totally dispersing billiards, which exhibit softness induced stability around singular trajectories.

2 Islands of stability in converging billiards

The atom-optics billiard is a recently developed experimental system in which billiard dynamics can be studied[15,16]. As described in detail in Ref. [16], the billiard is realized by the use of a laser beam which is rapidly (100 kHz)

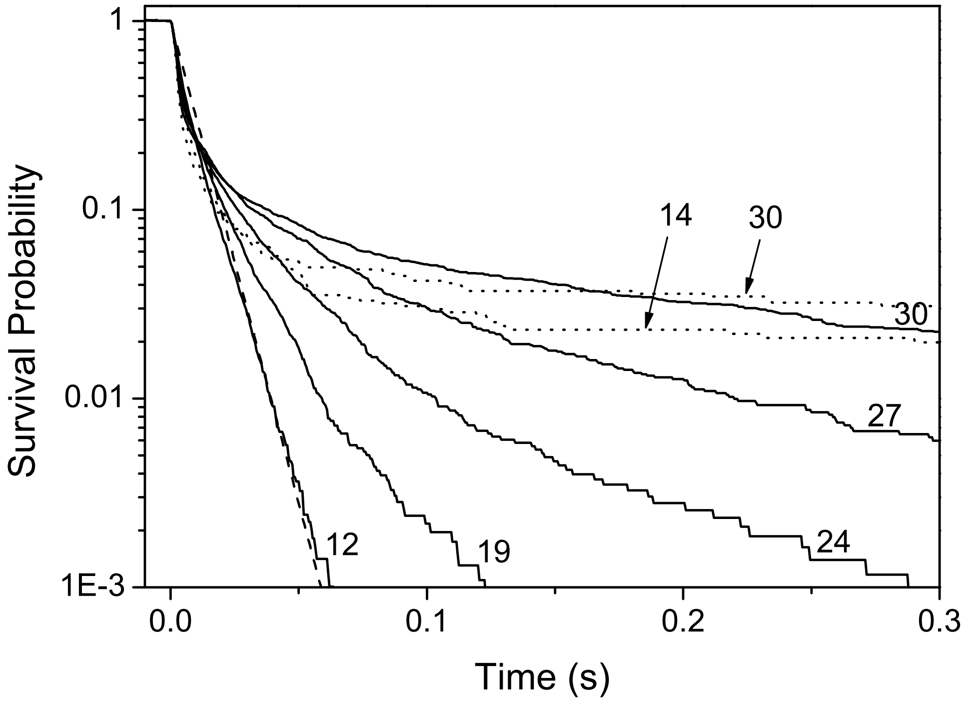

Fig. 1. Numerical simulations for the decay, through a small hole, from a tilted Bunimovich stadium (solid lines) and a circular billiard (dotted lines) with the parameters specified in the text, for several values of the softness parameter. Both billiards have the same area and hole size. The numbers adjacent to the lines indicate the value (in microns) of the softness parameter w_0. The simulations for the stadium show a monotonic slowing down in the decay and increase in stability as the walls are made softer. For the circular billiard, there is almost no change in the decay curve as a consequence of the higher softness. The dashed line shows exp(-t/τ_c), where τ_c is the escape time calculated for the experimental parameters.

scanned using two perpendicular acousto-optic scanners (AOSs). The laser frequency is tuned above the atomic resonance (the D_2 line of ^{85}Rb in our experiment), hence it applies a repulsive force on the atoms. By controlling the deflection angle of both AOSs, we create the required billiard shapes which confine the atoms in the transverse direction. The instantaneous potential is given by the dipole potential of the laser gaussian beam:

$$U\left(x,y,t\right)=U_0\exp\left[-2\left(\left(x-x_0\left(t\right)\right)^2+\left(y-y_0\left(t\right)\right)^2\right)/w_0^2\right], \qquad (1)$$

where the curve $\left(x_0\left(t\right),y_0\left(t\right)\right)$ is the shape of the ideal billiard, along which the center of the Gaussian laser beam scans. w_0 is the laser beam waist and U_0 is the instantaneous potential height. The time averaged potential is kept constant for different values of w_0 by changing the laser power. In such way, w_0 serves as the softness control parameter, and is experimentally controlled by the use of a a telescope with a variable magnification, located prior to the AOS's such that w_0 can be changed without affecting the billiard's size and

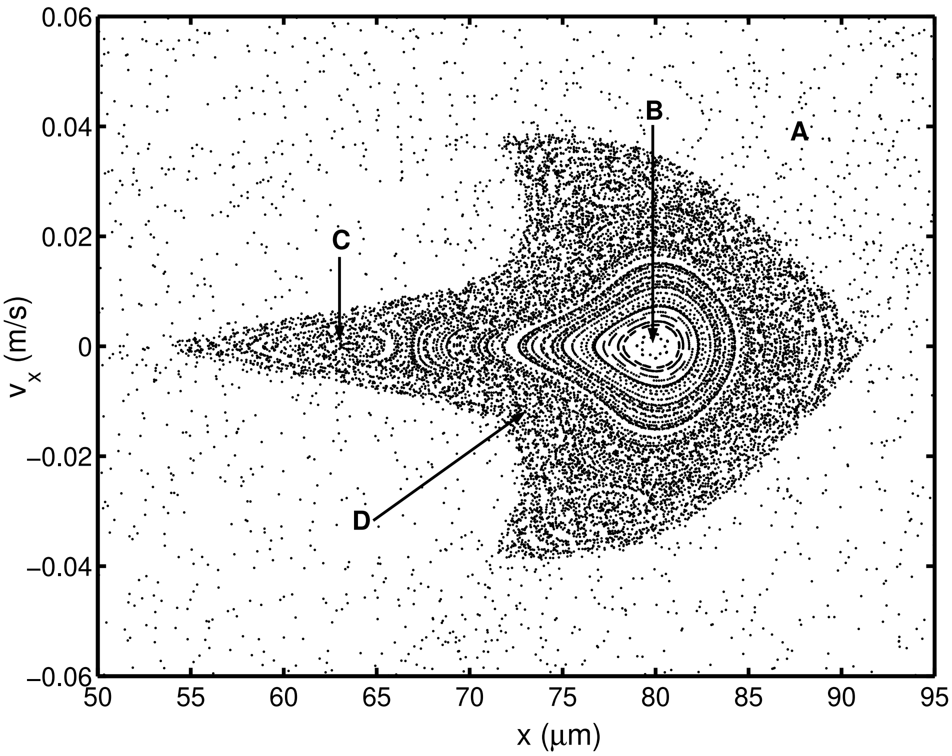

Fig. 2. Poincaré surface of section for monoenergetic atoms confined in a soft ($w_0 = 24\mu$m) tilted-stadium shaped atom-optics billiard with parameters as in Fig. 1. The Figure shows $\sim 9\%$ of the total phase space area. A phase space area corresponding to chaotic trajectories is marked "A", one of period 2 trajectories inside an island "B", an area of higher periodicity trajectories "C" and one of "sticky" trajectories "D".

shape. Fast enough scanning of the beam results in an effective time-averaged potential wall [17]. Photon scattering from the laser beam is greatly suppressed due to the large detuning, and collisions between the atoms are very rare for the range of densities used, hence motion of the atoms between reflections from the light walls can be regarded as strictly ballistic[16].

The loading scheme of cold atoms into the billiard was described in [13]. Briefly, laser cooled ^{85}Rb atoms are loaded from a magneto optical trap into a red-detuned one-dimensional optical lattice which is laterally shifted (250 μm) from the billiard's location. The atoms are then transferred from the lattice into the billiard by pushing them with a pulse of a strong on-resonance beam which is perpendicular to both the lattice and the billiard beams. Simultaneously with the pushing, the lattice beams are adiabatically switched-off, resulting in additional cooling[18]. Further reduction of the radial velocity spread and especially a decrease in the number of very slow atoms is achieved by capturing only a central velocity group of atoms into the billiard, which is turned on at the proper time after the push. Typically, $3 \cdot 10^5$ cold ^{85}Rb atoms are loaded into the billiard, with a typical velocity of ~ 100 mm/s and

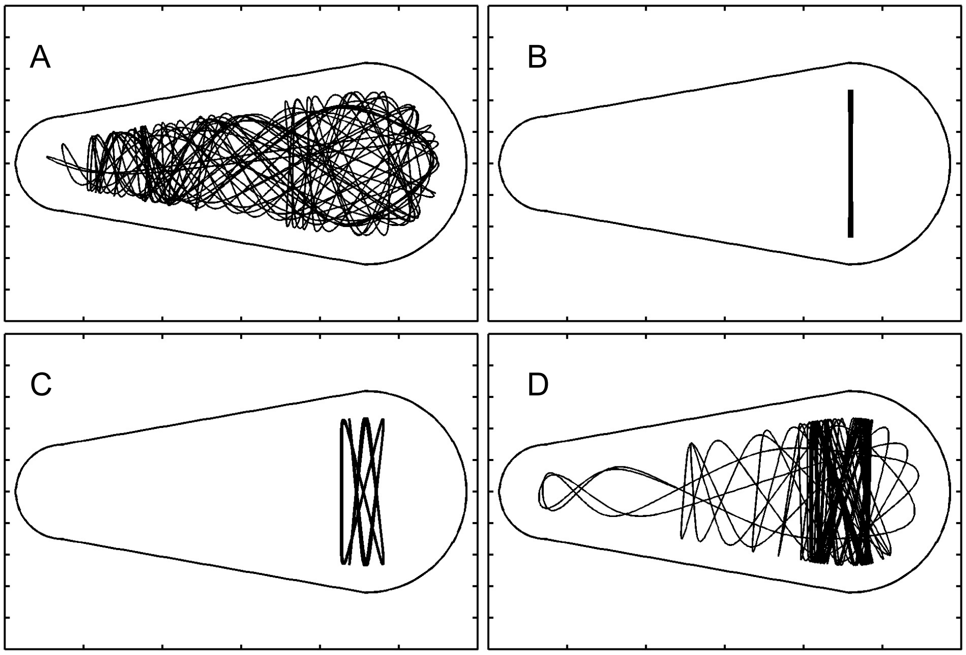

Fig. 3. Typical trajectories for atoms confined in a soft ($w_0 = 24\mu$m) tilted-stadium shaped atom-optics billiard with parameters as in Fig. 1. (A) A chaotic trajectory, corresponding to the area marked "A" in Fig. 2. (B) A periodic trajectory of period 2 and belonging to the central island, marked "B" in Fig. 2. (C) A high periodicity trajectory (Smaller islands surrounding the central one and denoted "C" in Fig. 2). (D) A "Sticky" trajectory, spending in the island's vicinity a relative long time ("D" in Fig. 2).

a velocity spread of $\sim$ 20 mm/s. The cooling in the longitudinal direction ensures that the system can be approximated as a two dimensional one.

As a way to probe the nature of the dynamics of the atoms confined by such an "atom-optics" billiard we measure the decay of the number of confined atoms through a small hole on the boundary. A purely exponential decay is expected for chaotic motion[19–21], with a decay time given by $\tau_c = \pi A/vL$, where A is the billiard's area, v the atomic velocity and L the length of the hole [19]. A slower algebraic decay usually characterizes a stable or partly stable phase space. In our system, the hole is produced by switching off one of the AOSs for ~ 1 μs every scan cycle, synchronously with the scan. The number of atoms remaining in the trap is measured using fluorescence detection[17]. The ratio of the number of trapped atoms with and without the hole, as a function of time, is the main data of our experiments.

First, we study a tilted Bunimovich stadium, which is chaotic and ergodic for the ideal hard-wall case. We used a "tilted" stadium, to avoid the long segments of regular motion associated with "bouncing ball" trajectories in the original stadium[22]. It is composed of two semicircles of different radii (64 μm and 31 μm), connected by two non-parallel straight lines (192 μm long).

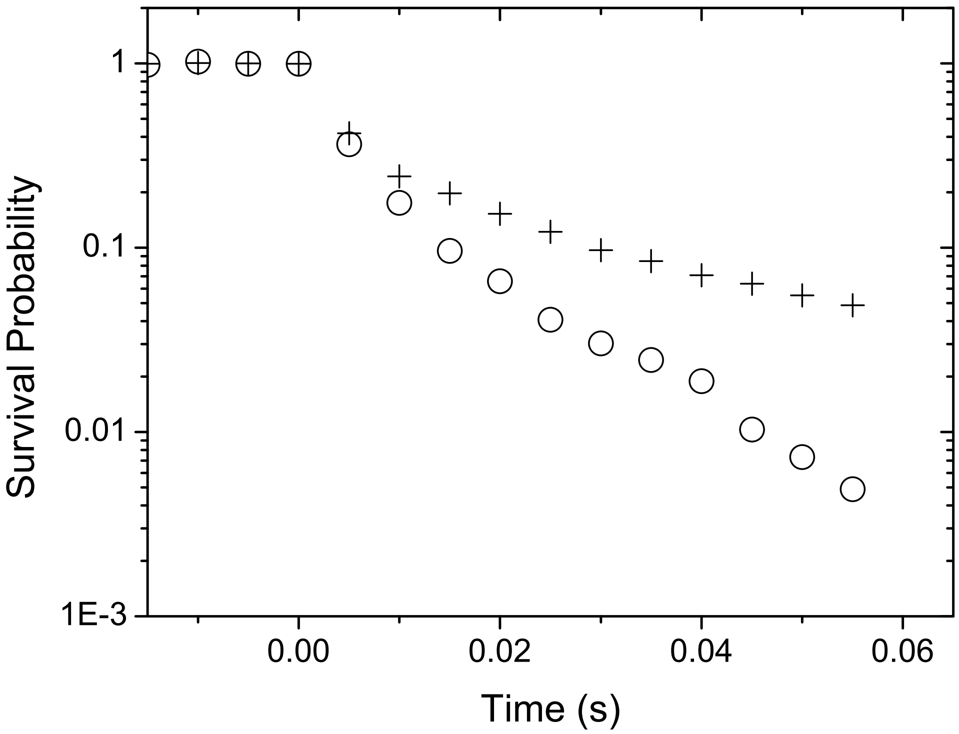

Fig. 4. Experimental results for the decay of cold atoms from a tilted-stadium shaped atom-optics billiard, with two different values for the softness parameter: $w_0 = 14.5\mu$m ($\circ$), and $w_0 = 24\mu$m ($+$). The hole is located inside the big semicircle. The smoothening of the potential wall causes a growth in stability, and a slowing down in the decay curve.

The results of numerical simulations for the decay in the number of confined atoms, through a small hole in the boundary, are presented in Fig. 1. For the stadium with $w_0 = 12$ μm a nearly exponential decay is seen, as expected for the ideal hard-wall billiard. As the walls are made softer, a monotonic slowing down in the decay is observed. We compare this behavior with that of a circle with the same area and hole size, which exhibits integrable motion. In the ideal hard-wall case the existence of nearly periodic trajectories results in many time scales for the decay through a small hole on the boundary, and then yield an algebraic decay[19]. Simulations for the decay from a circular billiard (dotted lines in Fig. 1) show that the decay is almost unaffected by the change in softness parameter, as expected.

Figure 2 shows the Poincaré surface of section obtained from the results of numerical simulations for classical trajectories of Rb atoms inside the tilted-stadium billiard, with $w_0 = 24\mu$m. For clarity, we assume a monoenergetic ensemble (with $v = 120$ mm/s), a two-dimensional system, and no gravity. However, including the experimental velocity spread, three dimensional motion and gravity have only a small effect on the results. In analogy to our experimental setup, we use for our simulation a scanning gaussian potential, with a scan frequency of 100 kHz[23]. The dimensions of the billiard are equal

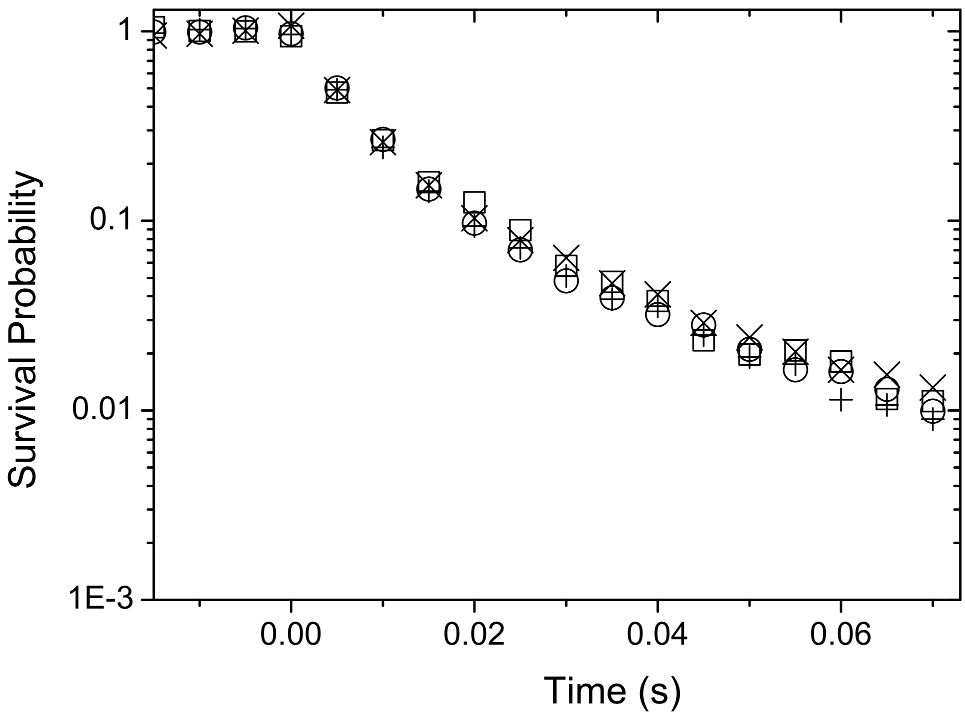

Fig. 5. Experimental results for the decay of cold atoms from a circle shaped atom-optics billiard, with different values for the softness parameter: $w_0 = 14.5\mu$m ($\circ$), $w_0 = 18.5\mu$m ($+$), $w_0 = 21\mu$m ($\Box$), $w_0 = 23.5\mu$m ($\times$).

to the experimental ones. The Poincaré surface of section shows v_x versus x at every trajectory intersection with the billiard's symmetry axis ($y = 0$), provided that $v_y > 0$. The smoothness of the wall results in a stability region that appears around the singular trajectory which connects the points where the big semicircle joins the straight lines. This fact is also confirmed by analyzing some typical trajectories (see Fig. 3). Additional islands (smaller than the central one) appear around it, and correspond to trajectories with higher periodicity, like the one shown in Fig. 3C. Around these islands there is a large area of "stickiness", where the trajectories spend a long time. Such a "sticky" trajectory is presented in Fig. 3D. The exact structure of the island and its vicinity depends on the softness parameter w_0 in a sensitive way[13]. In general, the size of the island and the stickiness region increase with the softness parameter. The slowing down in the decay is attributed mostly to the sticky trajectories, which in general cover a phase space volume greater than that of the island itself (see Fig. 2 of [13]).

In Fig. 4, experimental results for the decay from a tilted stadium shaped atom-optics billiard are presented, for two different values of the softness parameter ($w_0 = 14.5$ μm and $w_0 = 24$ μm). The experimental results confirm that the soft wall causes an increased stability, and a slowing down in the decay curve, in good agreement with the numerical results. We also observed[13] that when the hole includes the singular point where the semicircle meets the straight line, no effect for the change in w_0 is seen. In this case the stable island that is formed around the singular trajectory and the region of "stickiness" around it (see Fig. 3) are destroyed by the hole. By moving the hole we can thus experimentally map the location of the stable island.

Similar decay measurements for a circular atom-optics billiard (see Fig. 5) showed no dependence on w_0 in the range $14.5 - 24$ μm. We were not able to see any dependence on the hole position either.

3 Islands of stability in dispersive billiards

We studied the effect of a soft potential wall also in a dispersing billiard. The billiard that we studied was formed from the intersection of four arcs with different radii and it had a tangency, namely, a specific periodic orbit is tangent to the billiard's boundary at one point (see Fig. 7). In this billiard, the periodic orbit had a period of two. More generally, we consider a family of such billiards, with a variable parameter ζ, which is the vertical distance between the lower arc and the two-periodic orbit. For $\zeta = 0$ the periodic orbit is tangent to the lower arc. For $\zeta > 0$ the lower arc is below the periodic orbit, while for $\zeta < 0$ it is above and actually intercepts the orbit. A billiard with a two-periodic tangent orbit was chosen, to increase the size of the island. Every hard-wall billiard of this family is fully chaotic. This specific billiard shape was motivated by a theoretical study which predicts the formation of an elliptic island around such a tangency [4].

In Fig. 6, the Poincaré surface of section in the vicinity of the two-periodic orbit is presented, for a billiard with $\zeta = 40$ μm, and $w_0 = 24$ μm. An elliptic island is clearly observed. Typical trajectories for this case are shown in Fig. 7. In Fig. 7A, a typical chaotic trajectory is shown, corresponding to the areas in phase space marked as "A" in Fig. 6. A periodic trajectory of period 2 (corresponding to the central island in Fig 6, marked as "B") is shown in Fig. 7B. Also shown in Fig. 7 are a high-period periodic trajectory and a "sticky" one. We repeated these calculations for billiards with different values of ζ and w_0, and plotted the size of the island as a function of these two parameters in Fig. 8. It can be seen that the island becomes observable above some softness of the potential, and that its size initially increases with an increased softness. However, further softening reduces the size of the island. In general, higher values of ζ require higher values of w_0 to induce the stability. These results, although obtained for a strongly perturbed case, are in agreement with Refs. [3,4]. Some intuition can be gained by noting that the stable trajectory is slightly reflected by the lower arc. Since the stable trajectory is almost tangent to the lower arc, its perpendicular velocity is very low, and a small potential is already sufficient to reflect it. This small potential is provided at a certain distance from the lower

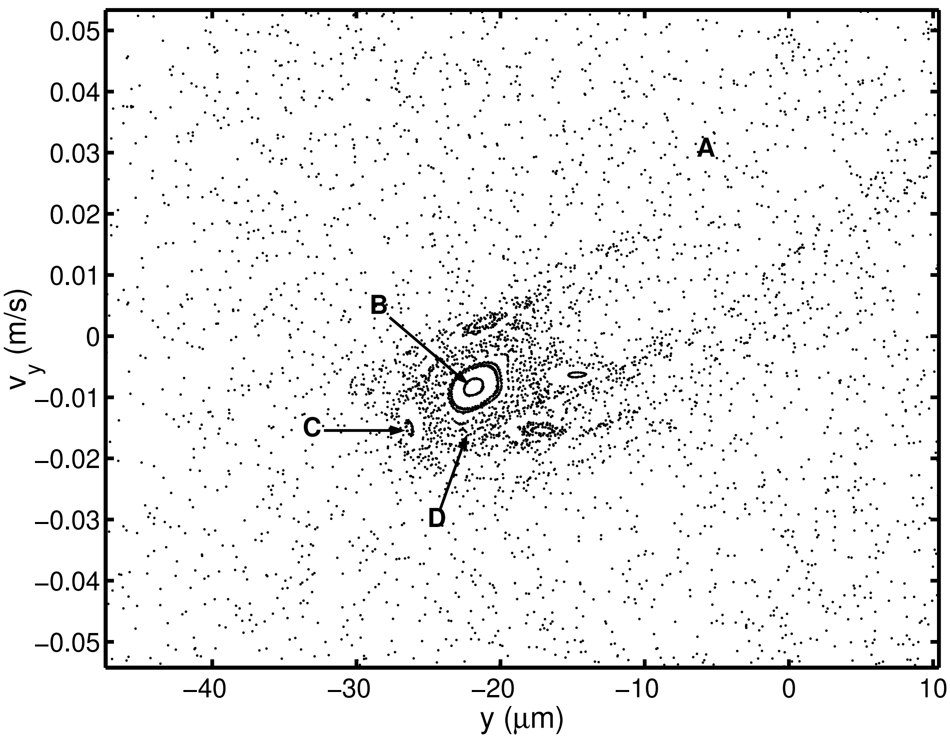

Fig. 6. Poincaré surface of section calculated for monoenergetic atoms confined in the soft dispersing billiard described in the text with $\zeta = 40$ μm and $w_0 = 24$ μm. The Figure shows $\sim 17\%$ of the total phase space area. Phase space area corresponding to chaotic trajectories are marked "A", those of period 2 trajectories inside an island "B", areas of higher periodicity trajectories "C" and "sticky" trajectories "D".

arc, depending on w_0. If ζ is increased, the trajectory will "feel" the lower arc (and hence will be stabilized) only for a larger value of w_0. However, if w_0 is further increased, the potential at the trajectory would become too high, thus "blocking" it and destroying the stable island. In additional simulations we observed that the center of the island moves towards larger y values when w_0 is increased, in good agreement with the above intuitive model.

We observed also another kind of stable islands in these billiards, around a different type of singular trajectories, i.e. trajectories that are supported by the corners of the billiard. An example for such a trajectory, and a sticky trajectory in the area surrounding this new island, are shown in Fig. 9. Our observations are consistent with the theoretical proof for the existence of this large islands, provided in Ref. [5]. There it was shown that, for a small perturbation, its existence depends both on the geometry of the corner and on the smooth potential behavior at the corners. Our results show the existence of this island also for a strong perturbation. In all cases where an island was found, we also found a sticky region near it with atoms spending arbitrary long times near the island.

We performed decay simulations for the scattering billiard, by opening a hole

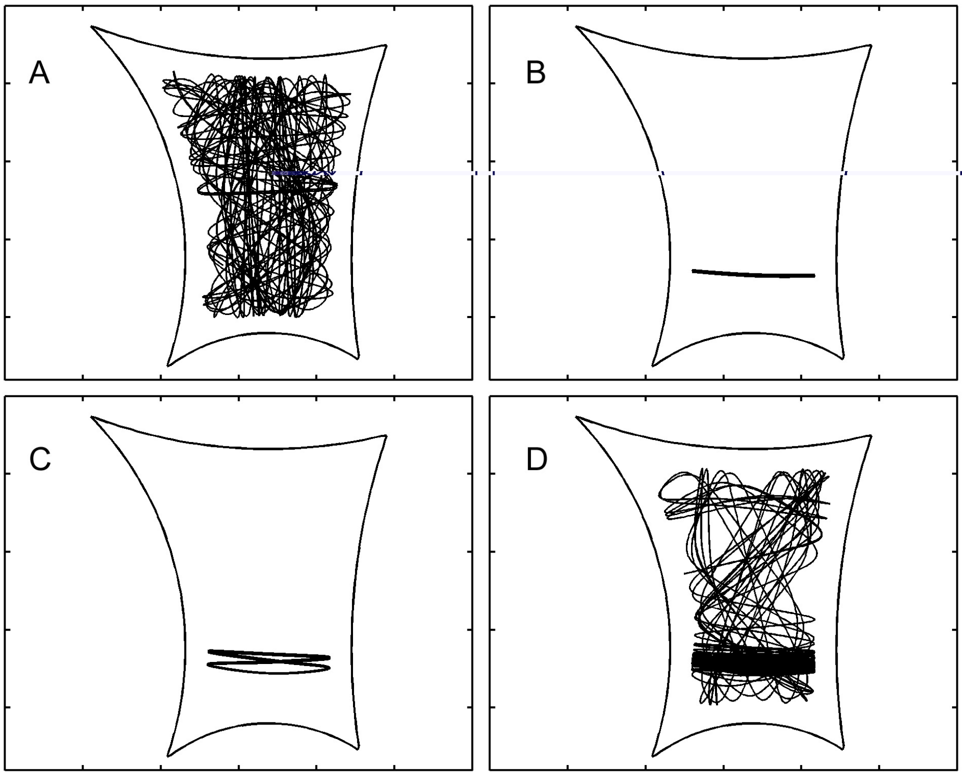

Fig. 7. Typical trajectories in a soft dispersive billiard, with the parameters as in Fig. 6, and softness parameter $w_0 = 24\ \mu$m. (A) A chaotic trajectory. (B) A periodic trajectory of period 2. (C) A high periodicity trajectory. (D) A "Sticky" trajectory.

on the boundary, and counting the number of remaining atoms as a function of time. Here, as in the case of the Stadium billiard, an exponential decay over 3 orders of magnitude was observed for $w_0 = 8\ \mu$m. In cases where an island develops, a slowing in the decay was observed when the hole did not include a reflection point of the periodic orbit. This slowing down in the decay was similar to that for the stadium billiard of Fig. 1.

4 Summary

In this paper we showed, numerically and experimentally, that stable islands are created in the phase space of chaotic billiards, either dispersive or focusing, when the walls are softened. Using cold atoms in atom-optics billiards, an island of stability was demonstrated in a tilted Bunimovich stadium billiard, around the periodic trajectory that connects the points where the big semicircle joins the straight lines. For a dispersive billiard, we confirmed numerically the existence of a stable island around a periodic orbit which is tangent to the billiard's wall. In addition, we found numerical evidence, for the first time, for the formation of a stable island near a "corner" trajectory. In all three cases,

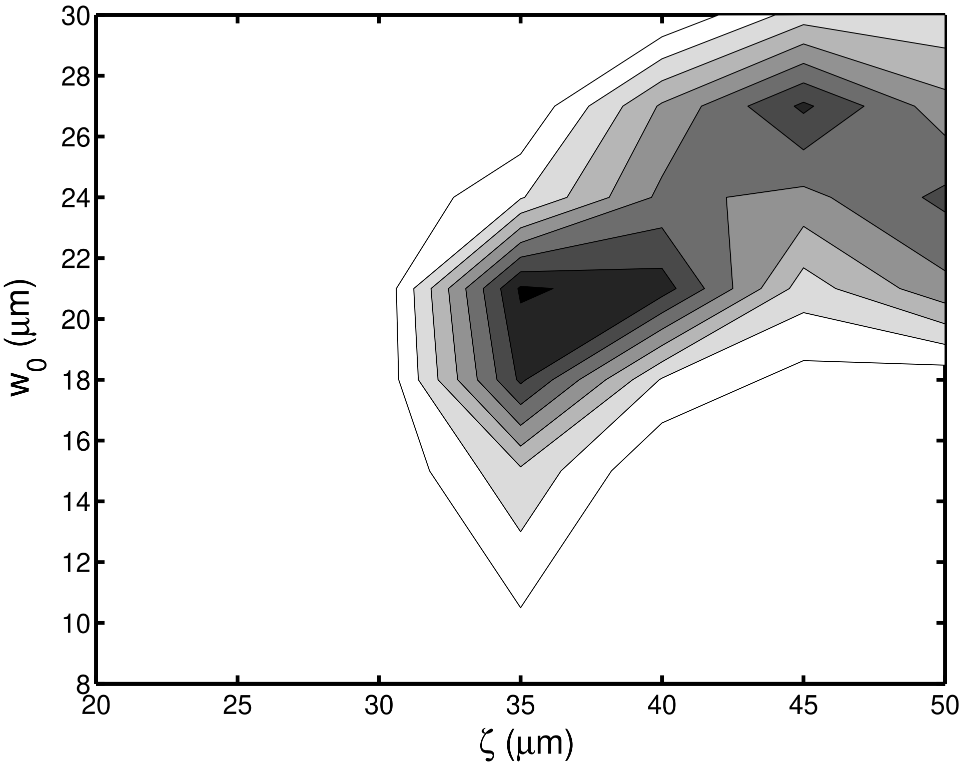

Fig. 8. Island size in the soft wall dispersing billiard, as a function of the softness parameter, w_0, and the shape parameter, ζ. (White - no island with an area greater than 1.6×10^{-3} of the total phase space area, black - a large island, $\geq 4 \times 10^{-2}$ of the total phase space).

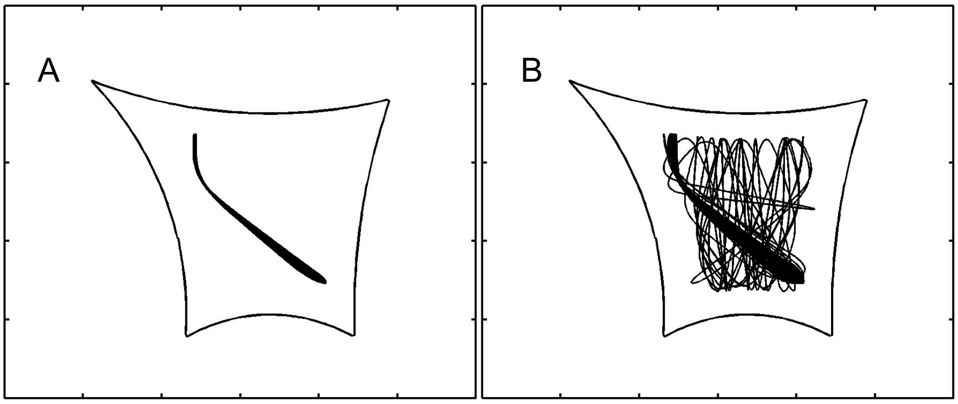

Fig. 9. Corner trajectories. (A) A periodic trajectory of period 2. (B) A "Sticky" trajectory.

the elliptic islands appear around trajectories which are singular in the "ideal" infinitely hard wall billiard. The orbits are singular in the sense that they do not have a smooth limit when approaching the singularity from both sides.

Our numerical results show that the appearance of a stable island in a soft-wall billiard is in general accompanied by areas of "stickiness" surrounding it.

Although the size of the stable island is a sensitive function of softness, not always showing a monotonic behavior, we confirmed numerically and experimentally that the "sticky" trajectories modify this behavior and, in general, introduce a monotonic slowing down in the decay of the number of trapped particles, as the walls are made softer.

An interesting question in this context is wether the existence of a singularity in the hard-wall billiard is a necessary and sufficient condition for stability in the soft-wall one. More generally, one can ask to what extent is the relation between the existence of a singularity in a chaotic system and the emergence of stability when this system is perturbed, a universal feature. From the practical point of view, the softness of the wall can be used as a control parameter for the structure of phase space, in particular in the context of controlling chaos[24]. Finally, we note that since stable and "sticky" regions follow the softness of the walls, and since any physically realizable potential is inherently soft, the physical realization of a truly chaotic and ergodic billiard is questioned.

5 Acknoledgements

We acknowledge helpful discussions with V. Rom-Kedar and support from the Israel Science Foundation, the Minerva Foundation, the United States-Israel Binational Science Foundation and Foundation Antorchas. M. F. Andersen acknowledges support from the Nachemsohn Dansk-Israelsk Studienfond.

References

[1] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).

[2] G. M. Zaslavsky, Physics Today **52**, 39 (1999).

[3] D. Turaev and V. Rom-Kedar, Nonlinearity **11**, 575 (1998).

[4] V. Rom-Kedar and D. Turaev, Physica D **130**, 187 (1999).

[5] D. Turaev and V. Rom-Kedar, nlin.CD/0209061 (2002).

[6] R. Ketzmerick, Phys. Rev. B **54**, 10841 (1996).

[7] A. Sachrajda et. al., Phys. Rev. Lett. **80**, 1948 (1998).

[8] T. Ouchterlony et. al., Eur. Phys. J. B. **10**, 361 (1999).

[9] A. Sachrajda and R. Ketzmerick, Phys. Rev. Lett. **83**, 1075 (1999).

[10] D. A. Steck, W. H. Oskay, and M. G. Raizen., Science **293**, 274 (2001).

[11] J. Klafter, M. F. Shlesinger, and G. Zumofen, Phys. Today **49**,33 (1996).

[12] N. Davidson, H. J. Lee, C. S. Adams, M. Kasevich, and S. Chu, Phys. Rev. Lett. **74**, 1311 (1995).

[13] A. Kaplan, N. Friedman, M. Andersen, and N. Davidson, Phys. Rev. Lett. 87, 274101 (2001)

[14] L. A. Bunimovich, Commun. Math. Phys. **65**, 295 (1979).

[15] V. Milner, J. L. Hanssen, W. C. Campbell, and M. G. Raizen, Phys. Rev. Lett. **86**, 1514 (2001).

[16] N. Friedman, A. Kaplan, D. Carasso, and N. Davidson, Phys. Rev. Lett. **86**, 1518 (2001).

[17] N. Friedman, L. Khaykovich, R. Ozeri, and N. Davidson, Phys. Rev. A **61**, 031403(R) (2000).

[18] S. L. Winoto, M. T. DePue, N. E. Bramall, and D. S. Weiss, Phys. Rev. A **59**, R19 (1999).

[19] W. Bauer and G. F. Bertsch, Phys. Rev. Lett. **65**, 2213 (1990).

[20] H. Alt *et al.*, Phys. Rev. E **53**, 2217 (1996).

[21] V. B. Kokshenev and M. C. Nemes, Physica A **275**, 70 (2000).

[22] F. Vivaldi, G. Casati, and I. Guarneri, Phys. Rev. Let. **51**, 727 (1983).

[23] We found our results to be independent of the scanning rate, for rates ≥ 20 kHz.

[24] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).