Spatial Gliding, Temporal Trapping, and Anomalous Transport

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Abstract

We introduce and study a model of stochastic dynamics in a phase space governed by Markovian laws of motion and filled with random traps. The resulting (non-Markovian) stochastic *trapped motion* alternates intermittently between periods of *spatial gliding*, where the motion 'glides' the underlying Markovian dynamics, and periods of *temporal trapping*, where the motion is halted in random traps.

We investigate the asymptotics and scaling limits of this model. We prove that when the random trappings are heavy tailed then Mittag-Leffler functions and probability laws emerge and govern the functional structure and statistics of the system, and that the time flow has a random fractal structure whose fractal exponent (dimension) is determined by the 'heaviness' of the trappings.

We study the effect of random trapping on general Lévy dynamics. We prove that subjecting Lévy dynamics to heavy tailed trapping will always result in: (i) sub-diffusive behavior - when the underlying Lévy dynamics are of finite variance; and, (ii) space-time fractal behavior - when the underlying Lévy dynamics are scale-invariant.

Furthermore, we explore the issue of first exit times. To that end, a general Feynman-Kac framework for trapped processes is developed, and a method of transforming 'trapped' Feynman-Kac equations to 'standard' ones is established. The study of first exit times enables us to quantitatively connect macroscopic observations to microscopic behavior in general Markovian dynamics subjected to random trapping. In the case of Lévy dynamics, first exit times from balls are computed, and the relationships between their statistics and the statistics of the trapped motion are derived.

Keywords: Anomalous transport, temporal trapping, fractal time, Mittag-Leffler functions and laws, Lévy dynamics.

1 Introduction

Anomalous transport has been reported in a large variety of systems both experimentally and theoretically [?]-[?]. The anomaly, defined relative to ordinary diffusion, results in either sub-diffusive or super-diffusive behavior and is usually related to flights and/or trapping events in the motion dynamics. The flights and the trapping events appear intermittently in the trajectories of the moving particles. From these trajectories one can extract the basic motional rules [?]-[?].

Examples for anomalous transport include cases such as the motion of tracer particles in rotating flows [?], turbulent diffusion [?], and motion of particles subject to Hamiltonian dynamics [?] - where flights dominate; and cases such as advection-diffusion of contaminants in water catchments [?], and transport in amorphous semiconductors [?]-[?] - where the trapping dominates.

Of particular interest is the combined contribution of both flights and trappings. This has been previously investigated using fractional kinetics equations for Lévy flights and for heavy-tailed trapping events [?]-[?], and also within the continuous time random walk framework for Lévy walks [?]. In [?]-[?] the flights are instantaneous - i.e; the particle 'jumps' the Lévy flight in zero time, whereas in [?] the particle has a finite (constant) velocity and 'travels' the Lévy walk in this velocity [?],[?].

Here we revisit the problem from a different point of view: the transport alternates between periods of motion - where the particle follows general Markovian dynamics, and trapping events - where the motion is halted for a random period of time (drawn from a general probability law). The transition from motion to halting is Markovian, i.e; it occurs randomly according to some trapping rate. The underlying Markovian dynamics and the halting periods are henceforth referred to, respectively, as the '*free motion*' and the '*trapping durations*'. The combined transport is called the '*trapped motion*'. Although the free motion and the trapping are Markovian, the resulting trapped motion - due to the arbitrarily-distributed trapping durations - is *non*-Markovian.

Pictorially, one can imagine stochastic motion in phase space governed, on the one hand, by Markovian dynamics, but filled, on the other hand, by random traps. The trajectories alternate intermittently between *spatial gliding* - 'riding' the Markovian dynamics - and *temporal trapping*.

We begin our exploration, in section 2, with a formal setting of the model described above.

In section 3 we turn to investigate the free-time process - the time periods in which the particle is in free motion, and derive its asymptotics and its *scaling limits*. When scaling the free time process the *Mittag-Leffler* probability laws emerge naturally, and turn out to govern the statistics of scaling limits. We prove that there are, essentially, two trapping categories - each leading to dramatically different limiting behavior: (i) light trapping - where the trapping durations are of finite mean; and, (ii) heavy trapping - where the trapping durations are heavy-tailed. In the 'light category' the behavior is highly regular, yielding a Law of Large Numbers scaling limit, and Gaussian fluctuations around the limit. The 'heavy category', on the other hand, yields a much more interesting behavior: the scaling limit is a *random fractal time*, whose fractal exponent (dimension) is determined by the 'heaviness' of the trapping durations [**?**]. (There is yet another, third, trapping category in which permanent trapping may occur. This category, however, leads to a degenerate limiting behavior.)

In section 4 we study the issue of the *first exit times* of the trapped motion, and derive asymptotic results regarding their behavior. We calculate explicitly the relationships between the statistics of the trapping duration, the first exit times of the free motion, and the first exit times of the trapped motion. These relationships enable use to connect *macroscopic* observations - the exit times of the trapped trajectories, to *microscopic* behavior - the characteristics of the trapping mechanism (trapping rate and duration).

In section 5 we proceed to investigate the effect of heavy-tailed trapping on general *Lévy dynamics*, and, in particular, on Wiener dynamics (Brownian motion) and on scale invariant Lévy dynamics. We prove that the linear time dependencies and exponential functional structure - which govern the statistics of Lévy dynamics - change to non-linear *power-law* time dependencies and *Mittag-Leffler* functional structures. Moreover, we prove that if the underlying Lévy dynamics has finite variance, then heavy-tailed trapping will always result in a *sub-diffusive* behavior with explicitly computable characteristics. In the case of Wiener and scale invariant Lévy dynamics we calculate the relationships between: (i) the moments of the displacement of the trapped motion (only in the Wiener case); (ii) the statistics of its first exit times from balls; and, (iii) the 'heaviness' of the trapping durations.

In section **??** we study the scaling limits of trapped Lévy dynamics. We focus on the case where both the Lévy dynamics and the trapping durations are heavy-tailed - yielding a fractal behavior in both the space and the time coordinates (each with a different characteristic fractal exponent). We compute the Fourier transforms and tail probabilities of the trapped motion, and the tail probabilities of the first exit times from balls. For both the motion and its first exit times, the space-time fractal behavior results in a space and time power-law behavior of the tail probabilities.

The issue of first exit times is a very special case of the, so called, *Feynman-*

Kac framework. We conclude, in section **??**, with the development of a Feynman-Kac framework for a general trapped motion model: arbitrary Markovian free-motion dynamics, and arbitrary state-dependent trapping rates and durations. We establish an explicit mapping reducing 'trapped' Feynman-Kac equations to standard, 'free', Feynman-Kac equations. This, in turn, enables us to transform problems regarding hard non-Markovian trapped motions to analogous problems regarding their underlying, simpler, Markovian free motions.

2 The model

Consider the stochastic *trapped motion* of a particle, taking place in the d-dimensional Euclidean space $\mathbb{R}^d$ (or, alternatively, in a general phase space), alternating intermittently between periods of *free motion* - during which the particle follows general Markovian-type dynamics, and *trapping events* - during which the motion is halted. When in motion, trapping occurs randomly according to a Markovian *trapping rate* r ($r > 0$). When trapped, the motion is halted for a random period of time, called the *trapping duration*, and denoted by S. The motion and the trapping are independent (i.e; de-coupled), and the trapping durations are independent and identically distributed. Hence, the particle's trajectory alternates between periods of free Markovian motion of exponentially-distributed duration, and trapping periods of S-long duration. The resulting trapped motion is, clearly, *non*-Markovian.

The trapping characteristic

The 'trapping mechanism' is characterized by the *trapping characteristic* $\Phi(\omega)$, $\omega \geq 0$, defined by:

$$\Phi(\omega) = r\left(1 - \mathbf{E}\left[\exp\{-\omega S\}\right]\right) \ , \tag{1}$$

where $\mathbf{E}[\cdot]$ denotes the expectation functional. Note that the trapping characteristic Φ is identical to the *Lévy characteristic* of a compound Poisson process $P = (P(t))_{t\geq 0}$ with jump-rate r and jump-size S: $\mathbf{E}[\exp\{-\omega P(t)\}] = \exp\{-\Phi(\omega)t\}$, $\omega \geq 0$.

Subordination

Let $\xi = (\xi(t))_{t\geq 0}$ denote the trajectory of the (underlying) free Markovian motion, and let $X = (X(t))_{t\geq 0}$ denote the trajectory of the trapped motion. The trapped motion can be represented as a *subordination* of the free motion. Indeed, let $T(t)$ denote the free, un-halted, time the particle spent in motion during the interval $[0, t]$. Then

$$X(t) = \xi(T(t)) \quad ; \quad t \geq 0 \ , \tag{2}$$

where the subordinating process $T = (T(t))_{t\geq 0}$ and the free motion process $\xi = (\xi(t))_{t\geq 0}$ are independent (de-coupled).

We call the subordinating process T the *free-time*. The subordinating process is also referred to [**?**]-[**?**] as the *operational time*.

Note that $0 \leq T(t) \leq t$, and that the derivative of T - the free-time's 'speed' - alternates between the values 0 and 1: spending exponentially distributed durations at the value 1, and S-long durations at the value 0.

Heavy tailed trapping

A non-negative valued random variable Y is *α-heavy tailed with amplitude a* if its distribution tails are, asymptotically, of the form

$$\mathbf{P}(Y > y) \approx \frac{a}{\Gamma(1-\alpha)} \cdot \frac{1}{y^{\alpha}} \qquad (y \to \infty), \tag{3}$$

where $0 < \alpha < 1$ and $a > 0$ (α being the order, or the 'heaviness', of the tail, and a being its amplitude).

In the sequel, we shall put much emphasis on the case where the trapping durations are heavy tailed. In this case the following connection, between the tail-structure of the trapping duration and the behavior of the trapping characteristic near the origin, holds:

$$\begin{gathered} S \text{ is } \alpha\text{-heavy tailed with amplitude } a \\ \Longleftrightarrow \\ \Phi(\omega) \approx ra \cdot \omega^{\alpha} \qquad (\omega \to 0). \end{gathered} \tag{4}$$

The proof of (4) follows immediately from the definition of the trapping characteristic (1) and Karamata's Tauberian theorem for random variables ([**?**], corollary 8.1.7).

3 The free-time process and the Mittag-Leffler scaling limits

The role the free-time $T = (T(t))_{t \geq 0}$ plays in the analysis of the trapped process $X = (X(t))_{t \geq 0}$ is of key importance. This section is devoted to the study of the subordinating free-time process.

We begin our investigation with the analysis of the free-time's moments:

$$\mathbf{E}\left[T(t)^m\right] \quad ; \quad m = 1, 2, \cdots, \tag{5}$$

and Moment Generating Function (MGF):

$$\mathbf{E}\left[\exp\{\theta T(t)\}\right] \quad ; \quad \theta \in \mathbb{R} \tag{6}$$

(note that, since $0 \leq T(t) \leq t$, both the moments and the MGF indeed converge and are hence well defined).

A direct computation of (5) and (6) is hard. However, their Laplace transforms, as functions of time, are given explicitly by the following theorem:

Theorem 1 *The Laplace transform of the m^{th} moment of the free-time is given by ($\omega \geq 0$):*

$$\int_0^\infty \exp\{-\omega t\} \boldsymbol{E}\left[T(t)^m\right] dt = \frac{m!}{\omega \cdot (\omega + \Phi(\omega))^m} \ . \tag{7}$$

The Laplace transform of the MGF of the free-time is given by ($\omega \geq 0$, $\theta < \omega + \Phi(\omega)$):

$$\int_0^\infty \exp\{-\omega t\} \boldsymbol{E}\left[\exp\{\theta T(t)\}\right] dt = \frac{\omega + \Phi(\omega)}{\omega\left(\omega + \Phi(\omega) - \theta\right)} \ . \tag{8}$$

The proof of theorem 1 is brought in the appendix.

In the case where the trapping duration is heavy tailed, the asymptotic behavior (as $t \to \infty$) of the free-time can be deduced from the Laplace transforms of theorem 1. Indeed, set $U_m(t)$ to be the primitive of $\mathbf{E}[T(t)^m]$ (i.e; $U_m(t) = \int_0^t \mathbf{E}[T(s)^m]\, ds$), and observe the following chain of equivalences:

$$S \text{ is } \alpha\text{-heavy tailed with amplitude } a$$

$\Longleftrightarrow$ (using (4))

$$\frac{m!}{\omega \cdot (\omega + \Phi(\omega))^m} \approx \frac{m!}{(ar)^m} \cdot \frac{1}{\omega^{1+\alpha m}} \qquad (\omega \to 0)$$

$\Longleftrightarrow$ (applying Karamata's Tauberian theorem for functions ([**?**], theorem 1.7.1) to (7))

$$U_m(t) \approx \frac{m!}{\Gamma(2 + \alpha m) \cdot (ar)^m} \cdot t^{1+\alpha m} \qquad (t \to \infty)$$

Hence, differentiating $U_m(t)$, we can conclude that $\forall m = 1, 2, \cdots$ the following statements are equivalent:

(i) The trapping duration S is α-heavy tailed with amplitude a.

(ii) The m^{th} moment of the free-time is given, asymptotically, by:

$$\mathbf{E}\left[T(t)^m\right] \approx \frac{m!}{\Gamma(1 + \alpha m)} \cdot \left(\frac{t^\alpha}{ar}\right)^m \qquad (t \to \infty). \tag{9}$$

The asymptotic structure of the moments of the free-time is of a special form - it bears the "fingerprints" of the *Mittag-Leffler* probability laws. Before dwelling into this matter, let us have a quick review of this special class of laws.

3.1 Mittag-Leffler functions and laws

The *Mittag-Leffler function* of order α, $0 \leq \alpha \leq 1$, is a generalization of the exponential function (see [**?**], chapter XVIII, or the more recent [**?**] and references therein) and is defined by the power series:

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(1+\alpha m)} \ . \tag{10}$$

When $\alpha = 0$ then the right hand side of (10) is a geometric progression and hence $E_0(z) = 1/(1-z)$. When $\alpha = 1$ then $\Gamma(1+\alpha m) = m!$ and hence the exponential function is retrieved: $E_1(z) = \exp\{z\}$.

Both $E_0(z)$ and $E_1(z)$ are MGFs of probability distributions. Indeed, if M_0 is a unit-mean exponential random variable, then:

$$\mathbf{E}\left[\exp\{zM_0\}\right] = \frac{1}{1-z} = E_0(z) \ .$$

And, if M_1 is a degenerate random variable centered at 1 (i.e; $M_1 \equiv 1$), then:

$$\mathbf{E}\left[\exp\{zM_1\}\right] = \exp\{z\} = E_1(z) \ .$$

It turns out that also for $0 < \alpha < 1$ the Mittag-Leffler function $E_\alpha(z)$ is a MGF of a probability distribution and therefore: $\forall 0 \leq \alpha \leq 1$ there exists a non-negative random variable M_α such that

$$\mathbf{E}\left[\exp\{zM_\alpha\}\right] = E_\alpha(z) \ , \tag{11}$$

or, equivalently,

$$\mathbf{E}\left[M_\alpha^m\right] = \frac{m!}{\Gamma(1+\alpha m)} \quad ; \quad m = 1, 2, \cdots \ . \tag{12}$$

The probability law of the random variable M_α, $0 \leq \alpha \leq 1$, is called *Mittag-Leffler of order* α.

Feller discovered (see [**?**]) that for $0 < \alpha < 1$ the following, remarkable, representation holds:

$$M_\alpha \stackrel{d}{=} \frac{1}{(Y_\alpha)^\alpha} \ , \tag{13}$$

where $\stackrel{d}{=}$ denotes equality in distribution (law), and where Y_α is a non-negative scale-invariant Lévy random variable of order α ($0 < \alpha < 1$) with unit amplitude: $\mathbf{E}[\exp\{-\omega Y_\alpha\}] = \exp\{-\omega^\alpha\}$, $\omega \geq 0$.

Now, returning back to the asymptotic behavior (as $t \to \infty$) of the free-time;

Combining (9) together with the moment characterization of the Mittag-Leffler laws (12) we can conclude that the trapping duration S is α-heavy tailed with amplitude a if and only if the MGF of the free-time is given, asymptotically, by the Mittag-Leffler function:

$$\mathbf{E}\left[\exp\{\theta T(t)\}\right] \approx E_{\alpha}\left(\frac{\theta t^{\alpha}}{ar}\right) \qquad (t \to \infty). \tag{14}$$

This, in turn, implies a probabilistic limit theorem: if the trapping duration S is α-heavy tailed with amplitude a then $T(t)/t^{\alpha}$ converges, in law, as $t \to \infty$, to the limit $(1/ar) \cdot M_{\alpha}$, where M_{α} is Mittag-Leffler of order α.

The more 'fundamental' way to obtain and understand the asymptotic emergence of the Mittag-Leffler limit laws is by the use of *scaling*, as we shall see in the following subsection.

3.2 The Mittag-Leffler scaling limits

We categorize the trapping duration into *trapping classes* $\{\mathcal{S}_{\alpha}\}_{0 \le \alpha \le 1}$ defined as follows:

Infinitely-heavy trapping ($\alpha = 0$): $S \in \mathcal{S}_0$ if and only if the trapping duration has an atom at infinity: $\mathbf{P}(S = \infty) > 0$;

Heavy trapping ($0 < \alpha < 1$): $S \in \mathcal{S}_{\alpha}$ if and only if the trapping duration is α-heavy tailed;

Light trapping ($\alpha = 1$): $S \in \mathcal{S}_1$ if and only if the trapping duration has finite mean: $\mathbf{E}[S] < \infty$;

and set:

$$c_{\alpha} = \begin{cases} r\mathbf{P}\left(S = \infty\right) & \alpha = 0 \\ ra & 0 < \alpha < 1 \\ 1 + r\mathbf{E}\left[S\right] & \alpha = 1 \end{cases}, \tag{15}$$

where, in the case $0 < \alpha < 1$, a stands for the trapping duration's tail amplitude.

Having defined the trapping classes' categorization, we are ready to state the *scaling limit theorem*:

Theorem 2 *Given that $S \in \mathcal{S}_{\alpha}$ introduce the scaled sequence*

$$T_n(t) = \frac{1}{n^{\alpha}} T(nt) \quad ; \quad n = 1, 2, \cdots. \tag{16}$$

Then; $T_n(t)$ converges, in law, to the scaling limit

$$T_{\infty}(t) \overset{d}{=} \frac{t^{\alpha}}{c_{\alpha}} \cdot M_{\alpha} \ , \tag{17}$$

where c_{α} is given by (15) and M_{α} is Mittag-Leffler of order α.

The proof of theorem 2 is brought in the appendix.

The scaling limits

In the case $\alpha = 0$ we have obtained that $T_\infty(t)$ is exponentially distributed with rate $c_0 = r\mathbf{P}(S = \infty)$. The rate c_0 has a simple probabilistic meaning: while r is the rate of trapping, $r\mathbf{P}(S = \infty)$ is the rate of *permanent trapping*.

In the case $\alpha = 1$ we have obtained a *deterministic*, i.e; non-random, scaling limit:

$$T_\infty(t) = \frac{1}{1 + r\mathbf{E}\left[S\right]} \cdot t \ . \tag{18}$$

That is, the scaled sequence $\{T(nt)/n\}_{n=0}^{\infty}$ satisfies a Law of Large Numbers (LLN). The LLN limit is nothing but a slowing down, by the factor $c_1 = 1 + r\mathbf{E}[S]$, of the standard 'clock time' t.

Both extremal trapping classes $\mathcal{S}_0$ and $\mathcal{S}_1$ - corresponding, respectively, to 'infinitely-heavy trapping' and to 'light trapping' - have very wide basins of attraction (in the space of distributions of the trapping durations). However these classes yield trivial limiting behavior. On the other hand, the intermediate trapping classes $\mathcal{S}_\alpha$, $0 < \alpha < 1$, yield non-trivial limiting behavior: the scaling limits are random and *fractal* [**?**]. The fractal behavior of the scaling limit is manifested in the power law t^α, and the fractal exponent (dimension) is the order of 'heaviness' of the trapping tail α. The basin of attraction of each intermediate trapping class $\mathcal{S}_\alpha$ is, however, very narrow - consisting only of α-heavy tailed distributions.

Fluctuations around the LLN limit

When the trapping duration has finite mean, i.e; when $S \in \mathcal{S}_1$, then the scaling limit is deterministic. Hence, in this case it is natural to study the *fluctuations* of the scaled sequence $\{T(nt)/n\}_{n=0}^{\infty}$ around its LLN scaling limit (18):

$$Z_n(t) = \sqrt{n}\left(T_n(t) - T_\infty(t)\right) = \sqrt{n}\left(\frac{1}{n}T(nt) - \frac{t}{c_1}\right) \ . \tag{19}$$

The scaling limit of the sequence of fluctuations $\{Z_n(t)\}_{n=0}^{\infty}$ is given by the following Central Limit Theorem:

Theorem 3 *Assume that the trapping duration has finite variance. Then; $Z_n(t)$ converges, in law, to the limit $Z_\infty(t)$ which is* Gaussian *with mean* 0 *and variance*

$$\frac{r\boldsymbol{E}\left[S^2\right]}{\left(1 + r\boldsymbol{E}\left[S\right]\right)^3} \cdot t \ . \tag{20}$$

The proof of theorem 3 is brought in the appendix.

4 First exit times

Let $D \subset \mathbb{R}^d$ be an open and connected domain containing the origin, and let $\tau(X)$ and $\tau(\xi)$ denote, respectively, the first exit times, from the domain D, of the trapped and free motions (starting at the origin).

The connection between the 'free' first exit time $\tau(\xi)$ and the 'trapped' first exit time $\tau(X)$ is given by the following formula:

$$\mathbf{E}\left[\exp\{-\omega\cdot\tau(X)\}\right] = \mathbf{E}\left[\exp\{-(\omega+\Phi(\omega))\cdot\tau(\xi)\}\right] \ . \tag{21}$$

That is, the transformation $\tau(\xi) \leftrightarrow \tau(X)$ is equivalent, in Laplace space, to the non-linear transformation

$$\omega \longleftrightarrow \omega + \Phi(\omega) \ . \tag{22}$$

Formula (21) follows, as a corollary, from the general Feynman-Kac framework which we shall develop in section **??**.

An immediate consequence of formula (21) is that: (i) if both S and $\tau(\xi)$ have finite mean then

$$\mathbf{E}\left[\tau(X)\right] = \left(1 + r\mathbf{E}\left[S\right]\right)\cdot\mathbf{E}\left[\tau(\xi)\right] \ ; \tag{23}$$

and, (ii) if both S and $\tau(\xi)$ have finite second moment then

$$\mathbf{E}\left[\tau(X)^2\right] = \left(1 + r\mathbf{E}\left[S\right]\right)^2\cdot\mathbf{E}\left[\tau(\xi)^2\right] + \left(r\mathbf{E}\left[S^2\right]\right)\cdot\mathbf{E}\left[\tau(\xi)\right] \ . \tag{24}$$

Note that, when passing from free motion to trapped motion, the mean exit time increases by the factor $c_1 = 1 + r\mathbf{E}[S]$ which we already encountered in the Law of Large Numbers (18).as the factor *slowing down* the standard 'clock time' t.

Combining formula (21) together with Karamata's Tauberian theorem for random variables ([**?**], corollary 8.1.7) yields:

Theorem 4

(a) Assume that $\mathbf{E}[\tau(\xi)] < \infty$. Then; the trapping duration S is α-heavy tailed with amplitude a if and only if the 'trapped' exit time $\tau(X)$ is α-heavy tailed with amplitude $ra{\cdot}\mathbf{E}[\tau(\xi)]$.

(b) Assume that the 'free' exit time $\tau(\xi)$ is β-heavy tailed with amplitude b. Then; the trapping duration S is α-heavy tailed with amplitude a if and only if the 'trapped' exit time $\tau(X)$ is $\alpha\beta$-heavy tailed with amplitude $b\,(ra)^{\beta}$.

(c) Assume that $\mathbf{E}[S] < \infty$. Then; the 'free' exit time $\tau(\xi)$ is β-heavy tailed with amplitude b if and only if the 'trapped' exit time $\tau(X)$ is β-heavy tailed with amplitude $b\left(1 + r\mathbf{E}\left[S\right]\right)^{\beta}$.

(d) Assume that the trapping duration S is α-heavy tailed with amplitude a. Then; the 'free' exit time $\tau(\xi)$ is β-heavy tailed with amplitude b if and only if the 'trapped' exit time $\tau(X)$ is $\alpha\beta$-heavy tailed with amplitude $b\,(ra)^{\beta}$.

The proof of theorem 4 is brought in the appendix.

The relationships between the behavior of the trapping duration S, the 'free' first exit time $\tau(\xi)$, and the 'trapped' first exit time $\tau(X)$, are summarized in the following table (the constants c_α are defined in (15), and $Y \sim (\alpha; a)$ is a shorthand for "Y is α-heavy tailed with amplitude a" (3)):

	$\mathbf{E}[\tau(\xi)] < \infty$	$\tau(\xi) \sim (\beta; b)$
$\mathbf{E}[S] < \infty$	$\mathbf{E}[\tau(X)] = c_1 \mathbf{E}[\tau(\xi)]$	$\tau(X) \sim (\beta; bc_1^{\beta})$
$S \sim (\alpha; a)$	$\tau(X) \sim (\alpha; c_\alpha \mathbf{E}[\tau(\xi)])$	$\tau(X) \sim (\alpha\beta; bc_\alpha^{\beta})$

This table is most helpful for deducing *micro-behavior* from *macro-observations*. We explain; consider the case where the particle's trajectory can not be observed directly (hence making it unable to measure the trappings), but first exit times can be detected. Comparing 'free' measurements to 'trapped' measurements, and using the above table, immediately reveals the trapping structure.

For example; assume that measurements conclude that the 'free' exit time is β-heavy tailed with amplitude b, while the 'trapped' exit time is γ-heavy tailed with amplitude c. Then, two cases are possible: (i) if $\beta = \gamma$ then the trapping duration has finite mean and

$$r\mathbf{E}\,[S] = \frac{c}{b} - 1 \;;$$

and, (ii) if $\beta \neq \gamma$ then the trapping duration is α-heavy tailed with amplitude a, where

$$\alpha = \frac{\gamma}{\beta} \quad ; \quad ra = \left(\frac{c}{b}\right)^{1/\beta} \;.$$

5 Trapped Lévy motion

In this section we study trapped Lévy motions, i.e; we investigate the effects of random trapping in the case where the underlying stochastic motion follows general Lévy dynamics. We assume that the free motion $\xi = (\xi(t))_{t \geq 0}$ is a d-dimensional Lévy process with Lévy characteristic $\Psi(\theta)$, $\theta \in \mathbb{R}^d$;

$$\mathbf{E}\,[\exp\{i\theta \cdot \xi(t)\}] = \exp\{-\Psi(\theta) \cdot t\} \;. \tag{25}$$

Note that if the Lévy process ξ has zero mean and finite variance, then it's *mean square displacement* is given by

$$\mathbf{E}\left[|\xi(t)|^2\right] = \mathbf{tr}\,(\Delta\Psi(0)) \cdot t \tag{26}$$

where $\mathbf{tr}\,(\cdot)$ is the trace operator and $\Delta\Psi$ is the Laplacian of Ψ.

We set $\tau(l)$ to be the first time the trapped motion X (starting at the origin) exits a ball of radius l, $l > 0$, centered at the origin:

$$\tau(l) = \inf\left\{t \geq 0 \;\middle|\; \big| \; | \; {\scriptstyle ||X(t)| \geq l \;.(27)}\right.$$

For a general Lévy motion ξ the following statements are equivalent:

(i) The trapping duration S is α-heavy tailed with amplitude a.

(ii) The Fourier transform of X is given, asymptotically, by:

$$\mathbf{E}\left[\exp\{i\theta \cdot X(t)\}\right] \approx E_\alpha\left(-\frac{1}{ar}\Psi(\theta)\cdot t^\alpha\right) \qquad (t \to \infty). \tag{28}$$

And, if ξ has zero mean and finite variance,

(iii) The mean square displacement of X is given, asymptotically, by:

$$\mathbf{E}\left[|X(t)|^2\right] \approx \frac{\mathbf{tr}\,(\Delta\Psi(0))}{ar\Gamma(1+\alpha)}\cdot t^\alpha \qquad (t \to \infty). \tag{29}$$

This equivalence follows straightforwardly, using conditioning, from the subordination formula (2) and the asymptotic behavior of the free-time.

The effect of heavy-tailed trapping

It is illuminating to compare equations (25)-(26) with equations (28)-(29), to see the effect of heavy-tailed trapping:

* The linear time dependence in (25)-(26) is replaced by the non-linear time dependence t^α in (28)-(29).
* In Fourier space the exponential functional structure $\exp(\cdot)$ in (25) is replaced by the Mittag-Leffler functional structure $E_\alpha(\cdot)$ in (28).

Sub-diffusive behavior

A stochastic process whose mean square displacement displays a power law asymptotic behavior of the type $c \cdot t^\alpha$ (as $t \to \infty$, where c is a positive constant), is said to be *sub-diffusive of order* α ($0 < \alpha < 1$). Hence, 29 enables us to assert that:

Subjecting a Lévy process, with zero mean and finite variance, to heavy tailed trapping yields a sub-diffusive behavior.

5.1 Wiener dynamics

We turn now to the case where the Lévy process is Wiener, i.e; when $\xi = (\xi(t))_{t\geq 0}$ is a d-dimensional Brownian Motion. In this case the following statements are equivalent:

(i) The trapping duration S is α-heavy tailed with amplitude a.

(ii) The moments of the square displacement are given, asymptotically, by $(m = 1, 2, \cdots)$:

$$\mathbf{E}\left[|X(t)|^{2m}\right] \approx \frac{\Gamma\left(d/2+m\right)}{\Gamma\left(d/2\right)} \cdot \frac{m!}{\Gamma(1+\alpha m)} \cdot \left(\frac{2t^{\alpha}}{ar}\right)^{m} \qquad (t \to \infty). \quad (30)$$

(iii) The first exit times $\tau(l)$ are α-heavy tailed with amplitudes arl^2/d $(l > 0)$.

The proof of equivalence is brought in the appendix. Note that, taking $m = 1$ in (30), gives the mean square displacement:

$$\mathbf{E}\left[|X(t)|^{2}\right] \approx \frac{d}{ar\Gamma(1+\alpha)} \cdot t^{\alpha} \qquad (t \to \infty), \quad (31)$$

which, in turn, agrees with (29).

5.2 Symmetric scale-invariant Lévy dynamics

Brownian Motion is a special case of *symmetric* and *scale-invariant* Lévy dynamics. The class of a symmetric and scale invariant Lévy processes corresponds to the class of Lévy characteristics of the form

$$\Psi(\theta) = b\left(|\theta_1|^{\beta} + \cdots |\theta_d|^{\beta}\right) \ , \quad (32)$$

where $b > 0$ and $0 < \beta \leq 2$. The parameter β is the *Lévy exponent* of the process, and b is its amplitude. A Lévy process with Lévy characteristic of the form (32) is called *symmetric β-stable* with amplitude b.

When $\beta = 2$ (and $b = \frac{1}{2}$) we return to the case of Brownian Motion. When $0 < \beta < 2$ we are in the "pure Lévy" domain: the distributions are non-Gaussian and heavy tailed, their variances diverge (and so do their means when the Lévy exponent is in the range $0 < \beta \leq 1$), and the trajectories are purely discontinuous. The divergence of the variance of the free motion ξ implies, in turn, that the mean square displacement (let alone the higher order moments of the square displacement) is not defined. However, the first exit times of (centered) balls are well defined, and they even have *finite means*. Let us denote by $\mu_{d,\beta}$ the mean time to exit of a symmetric β-stable Lévy process with unit amplitude, from the unit ball. For a non-Gaussian symmetric β-stable Lévy process with amplitude b the following statements are equivalent:

(i) The trapping duration S is α-heavy tailed with amplitude a.