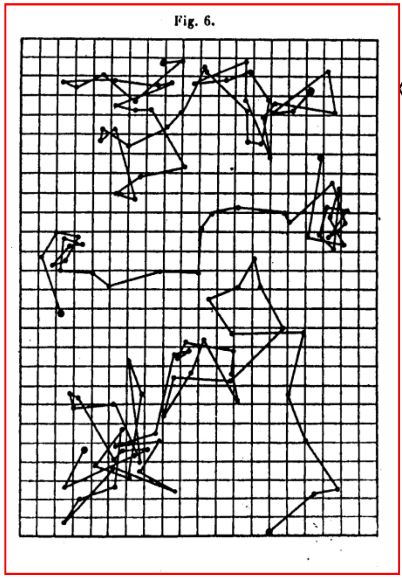
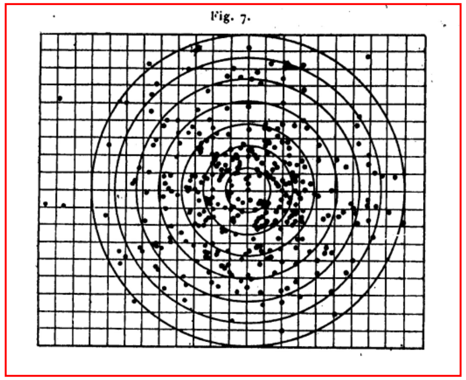
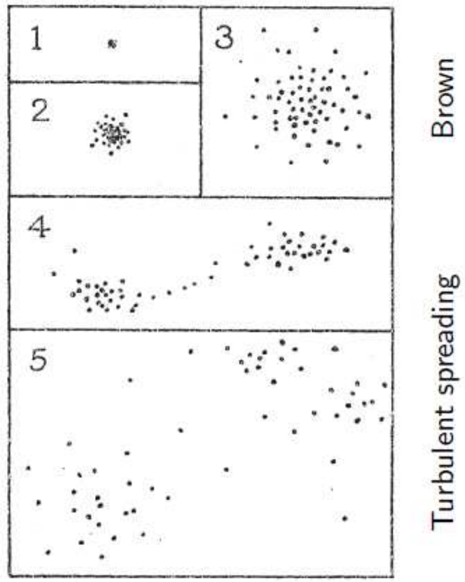




$$\frac{\partial}{\partial t} f(t,\vec{r}) = K\Delta f(t,\vec{r})$$

J.B. Perrin, 1909: Brownian motion of small grains of putty

$$\frac{\partial f}{\partial t} = \varepsilon \frac{\partial}{\partial l}\left(l^{4/3} \frac{\partial f}{\partial l} \right)$$



Figs. 1–5.

Richardson, 1926: dispersal of particles in the atmosphere vs dispersal of Brownian particles

Scale-invariant Markov random motions in inhomogeneous and non-stationary media

Aleksei Chechkin

Max Planck Institute for Physics of Complex Systems, Dresden, and Akhiezer Institute for Theoretical Physics NAS Ukraine, Kharkov

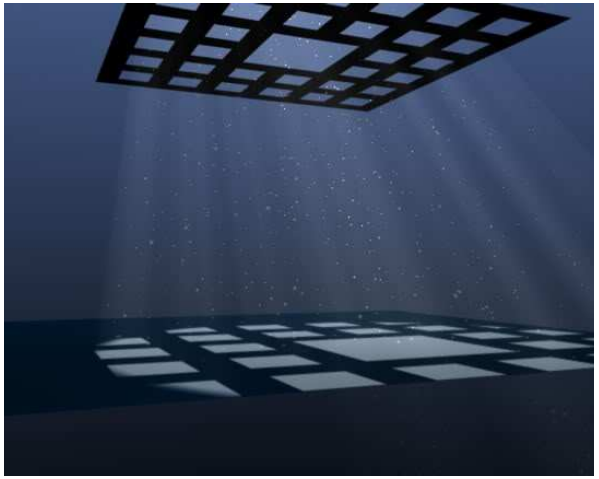
First (documented) observation of random walk

115] Глянь-но пильніше: як тільки осяйливе сонячне світло,
116] В дім зазирне, розчахнувши промінням півморок покою,
117] Повно тоді порошинок побачим: в освітленій смузі
118] Грають роями вони, пориваються в напрямках різних,
119] Наче в одвічній війні, нескінченні зав'язують битви,
120] Втоми не знаючи, цілі загони в борні заповзятій
121] То наче в купу збиваються, то розбігаються знову.

De Rerum Natura, Book II: “The Dance of Atoms”.

Titus Lucretius Carus

1st cent. BC



Reminder. Brownian motion: massive particle in a heat bath

- **Langevin SDE**

$$m\frac{dv}{dt} = -\gamma m v + \sqrt{2D_v}\,\xi(t) \quad , \quad \frac{dx}{dt} = v$$

Stokes' friction coefficient $\gamma = \frac{6\pi R}{m}\eta$

$$\langle \xi(t)\xi(t') \rangle = \delta(t-t')$$

$\xi(t)$: Gaussian

$$D_v = \gamma m k_B T = const$$

- **overdamped approximation**

$$\frac{dx}{dt} = \sqrt{2D_x}\,\xi(t) \qquad D_x = \frac{k_B T}{m\gamma}$$

$D_x \to D$

- **Diffusion equation**

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}, \quad -\infty \le x < \infty, \; f(x,0) = \delta(x)$$

$$G(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right) \;\Rightarrow\; \langle x^2 \rangle = 2Dt$$

Normal diffusion law

- **Wiener process: increments are (1) stationary, (2) Gaussian, (3) uncorrelated**

ANOMALOUS TRANSPORT / ANOMALOUS DYNAMICS

Anomalous diffusion law: the hallmark of anomalous transport

Normal diffusion

In particular, (ordinary) Brownian motion,
or Wiener process:

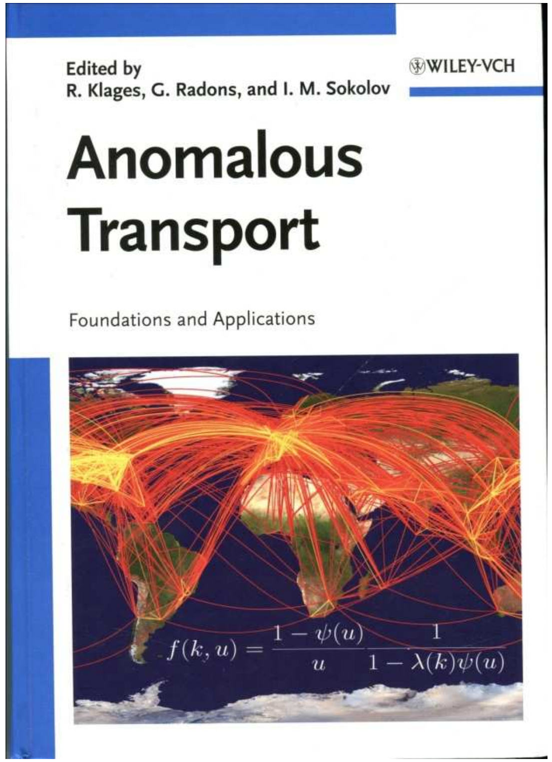
$$\left\langle \vec{R}^2(t) \right\rangle \propto t$$

Anomalous diffusion

$$\left\langle \vec{R}^2(t) \right\rangle \propto t^{\mu},\ \mu \neq 1$$

$\mu > 1$ superdiffusion (fast)

$\mu < 1$ subdiffusion (slow)



- **Anomalous is normal**
- **Happy families are all alike; every unhappy family is unhappy in its own way**
  L. Tolstoi , Anna Karenina (the very first sentence)

Different sources of anomaly

Long jumps → Lévy flights / Levy walks in space

Long temporal correlations → fractional motions

Long waiting times as arising from random potential models (*energetic disorder*) → Lévy flights in time

Geometrical constraints → Diffusion on fractal structures, e.g. on percolation clusters (*geometrical disorder*, or *labyrinthine environment*)

...

Inhomogeneous and/or non-stationary environment → Scaled Brownian motion, heterogeneous diffusion process

Generic types of variable diffusion processes

Heterogeneous diffusion process HDP x(t)

$$\frac{dx}{dt} = \sqrt{2D(x)}\,\xi(t)$$

Scaled Brownian motion SBM x(t)

$$\frac{dx}{dt} = \sqrt{2D(t)}\,\xi(t)$$

$\xi(t)$: white Gaussian noise, $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$

$$D(x) \propto |x|^{\alpha}$$

power law dependent

$$D(t) \propto t^{\alpha-1}$$

A. Fulinski (2013), Cherstvy, Ch, Metzler (2013)

Lim and Muniandy (2002)

Invariance under the scale transformation

$$t \to \lambda t\,, \quad x \to \lambda^{H} x$$

$$H = \frac{1}{2-\alpha}$$

$$\langle x^2 \rangle \propto t^{2H}$$

$$H = \frac{\alpha}{2}$$

- Doob's Theorem (J.L. Doob, Stochastic Processes, 1953): Markovianity
- Smoothing at x = 0, t = 0 $\quad D(x) \propto \left(x_0 + |x|\right)^{\alpha}$, $\quad D(t) \propto \left(\tau_0 + t\right)^{\alpha-1}$

Outline

I. INTRODUCTION AND MOTIVATION

- **Brownian motion in inhomogeneous medium, examples and motivation**
- **Brownian motion in non-stationary medium**
- **Turbulent diffusion and Richardson law**

II. HETEROGENEOUS DIFFUSION PROCESSES

- **Correlation properties: similar to FBM**
- **Time vs ensemble averages: similar but not identical to CTRW**

III. SCALED BROWNIAN MOTION

- **Time versus ensemble averages: “between” BM and CTRW**
- **Confined and aging SBM: universal aging depression and strong effect for a weak aging**
- **Ultraslow SBM and diffusion in granular gases**

IV. Σ, and what was not mentioned

In collaboration with

Anna Bodrova, Moscow / Potsdam / Berlin

Andrey Cherstvy, Potsdam

Jae – Hyung Jeon, Tampere / Potsdam / Seoul

Hadiseh Safdari, Tehran / Potsdam

Ralf Metzler, Potsdam

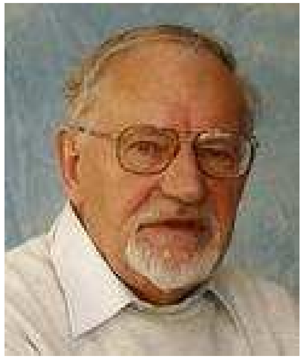
My personal motivation

Yurii Klimontovich, Moscow , USSR:

Anomalous diffusion in turbulent plasmas (Bohm diffusion)

Radu Balescu, Statistical Dynamics. Matter out of Equilibrium. ICL, 2000:

For a space-dependent diffusion coefficient the simple relation between diffusion coefficient and mean squared displacement breaks down and "strangely, it does not seem to be mentioned in the literature".



George Rowlands, Cambridge, 2007:

"Do you really believe in all that ???"

Ralf Metzler and Andrey Cherstvy, Potsdam, 2013: CTRW everywhere ???

Brownian motion in inhomogeneous medium

- Langevin SDE (tentative)

$$\frac{d\mathrm{v}}{dt} = -\gamma(x)\mathrm{v} + \sqrt{2D_{\mathrm{v}}(x)}\,\xi(t)$$

$$D_{\mathrm{v}} = \gamma(x) m k_B T(x)$$

- overdamped

$$\frac{dx}{dt} = \sqrt{2D_x(x)}\,\xi(t)$$

$$D_x = \frac{k_B T(x)}{m\gamma(x)}$$

Famous problem: Ito, Stratonovich, or Haenggi - Klimontovich ?

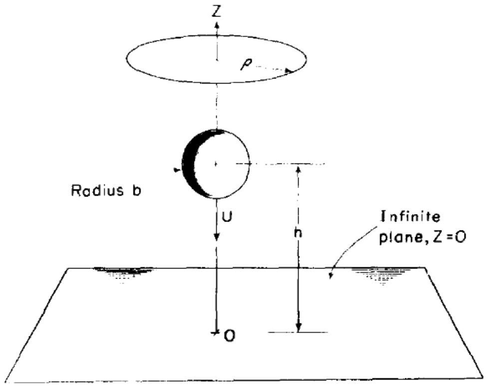
Problem: estimation of space-dependent diffusion coefficient from experimental data (Hoffmann, *Bernoulli*, 1999)

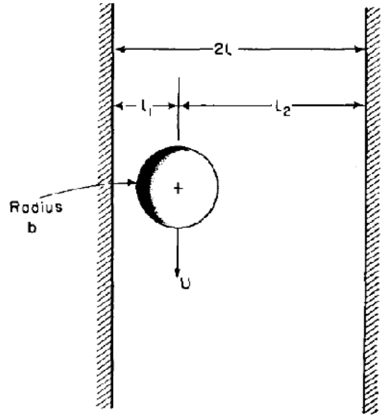
Brownian particle diffusing near a wall

Lorentz (1907): $\gamma(x) \sim \gamma_{Stokes}\left(1+\frac{9}{8}\frac{R}{h}\right), h >> R$

Reflects long-range nature of hydrodynamic interactions

Brener (1961)





State-dependent diffusion in soft-matter systems

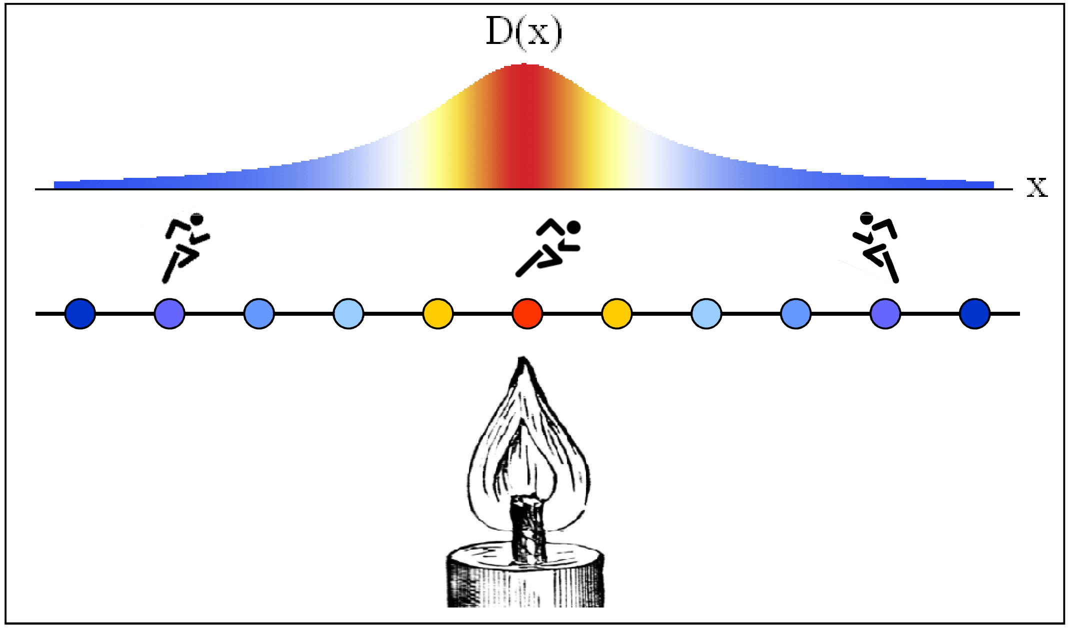
- Transport of macromolecules in a spatially inhomogeneous medium, such as a polymer gel and a porous medium (Viramontes-Gamboa et al., 1995, Tong et al., 1997)
- Brownian motion of colloidal particles between two nearly parallel walls (Lancon et al, 2001)
- Affects interpretation of single-molecule force-extension experiments (Neto et al., 2005, Goshen et al., 2005)
- …and verification of FT in a colloidal suspension near a wall (Blickle et al., 2006)

Position- and time-dependent temperature profile in a heated/cooled system

D = D(T(**r**,t))

Fourier - Kirchhoff equation

$$\frac{\partial T}{\partial t} = \Lambda \frac{\partial^2 T}{\partial \mathbf{r}^2} + S(\mathbf{r}, t)$$



T(**r**,t) ↔ S(**r**,t): in principle, any required T(**r**,t) can be assumed (Fulinski 2013)

Stochastic climate theory and modeling

Franzke et al., 2014

- State vector **z** splitted into slow **x** and fast **y** components (scale separation in space and time)

- Functional form of reduced climate models

$$\frac{d\mathbf{x}}{dt} = F + L\mathbf{x} + B(\mathbf{x},\mathbf{x}) + M(\mathbf{x},\mathbf{x},\mathbf{x}) + \sigma_A \xi_A + \sigma_M(\mathbf{x})\xi_M$$

The magnitude of fluctuations is dependent on the state of the system

- Intuitively: on a windless day, the fluctuations are very small, whereas on a windy day not only the mean wind strong, but also the fluctuations around this mean are large

- State-dependent noise is important for deviations from Gaussianity and thus extremes

Stochastic climate theory and modeling (continued)

- **CAM noise model: univariate version as a physically plausible null hypothesis for non-Gaussian climate variability**

$$\frac{dx}{dt} = -\lambda x + \xi_A + x\xi_M$$

Sura, 2013

- Linear model with state-dependent noise

- Produce power-law tails

- Capture some (experimentally observed) properties of SST and SSH dynamics

More examples of space dependent diffusivities:

- Atmospheric diffusion of substances released from the infinite line source, $D(z) \propto z^{\alpha}$, $\alpha \leq 1$ (Koch, 1988)

- Transport of radionuclides in strongly inhomogeneous geological formations, $D(r) \propto \ln^{1/2}(r)$ (Goloviznin et al., 2010)

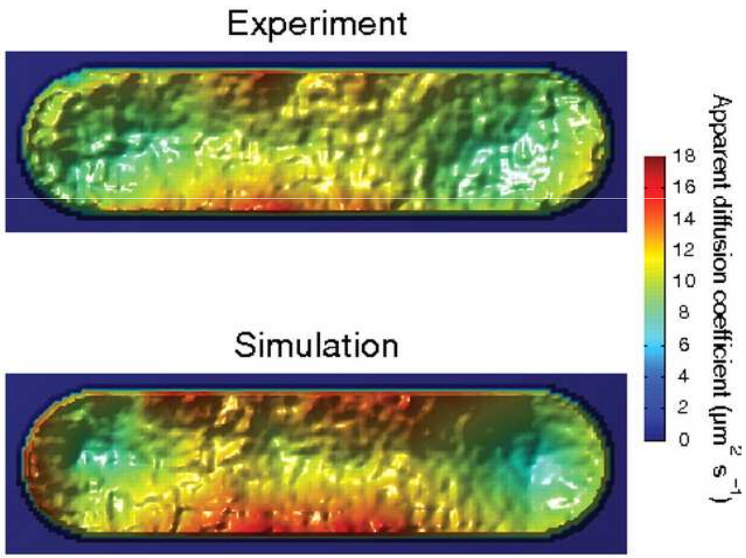
- Radiation-induced diffusion, $D(x) \propto \exp(-k\,x)$ (Kesarev et al., 2008)

...

Biomotivation. **Physical causes of heterogeneous D(x):**

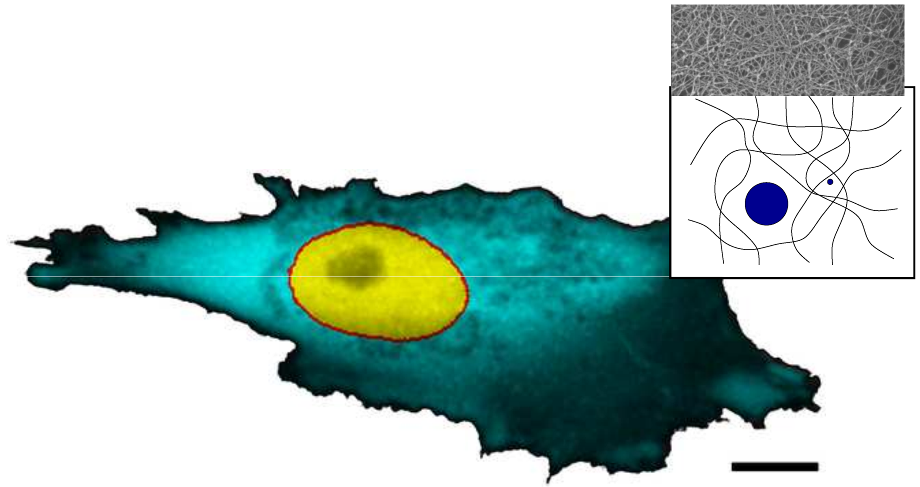
Varying obstacle density, cyto-skeletal network, cytoplasm viscosity, porosity etc.

Geometry-mediated position-dependent diffusivity in *E.Coli*



J. Elf et al., Proc. Natl. Acad. Sci. USA (2011)

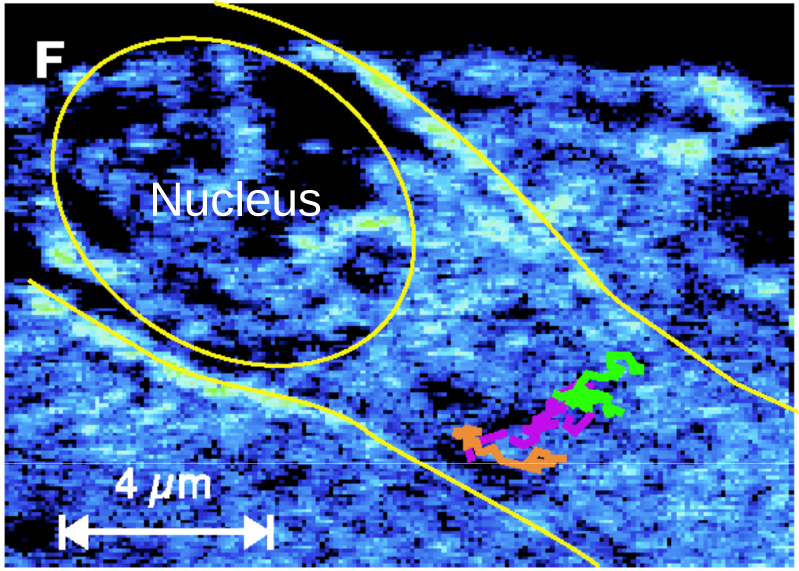
Heterogeneous diffusivity of mammalian cell cytoplasm

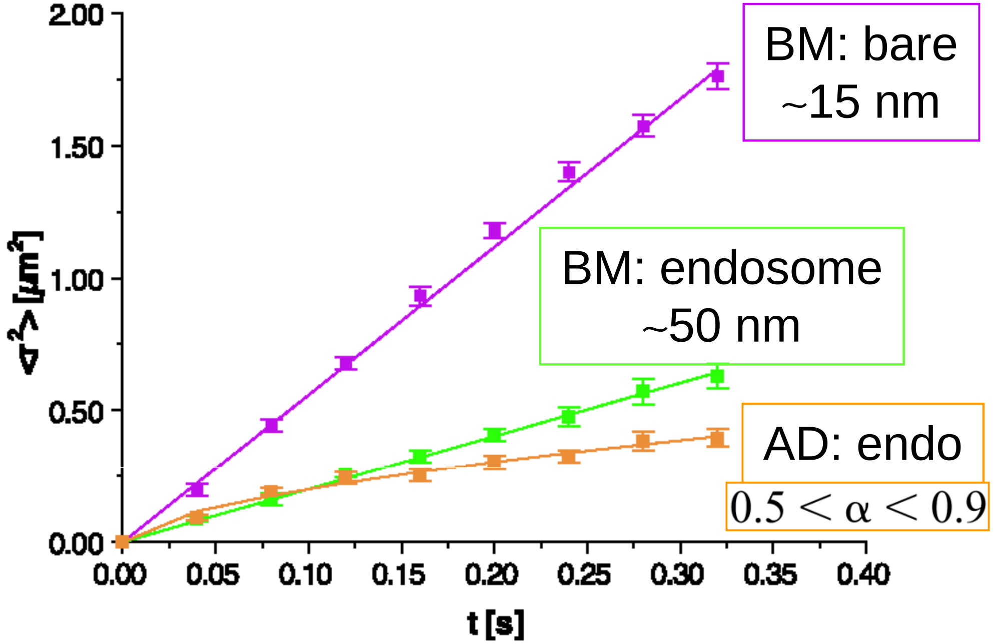


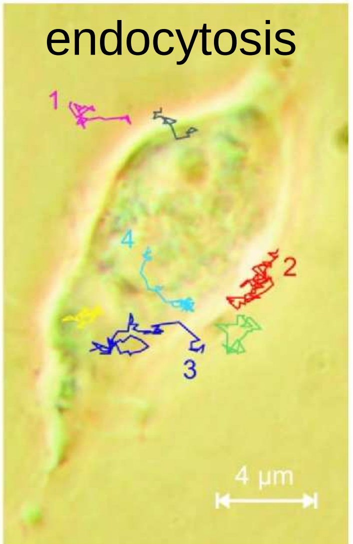
FRAP color intensity is related to effective porosity
26 kDa EYFP (~4 nm) diffusion
Substrate-adhered “fried-egg” like NLFK and HeLa cells

T. Kuehn et al., PLoS One (2011)

Infection pathways of AAV viruses







BM: 53 viruses
AD: 51 viruses
BM+drift: 9 viruses

ND or BM: Azimuthal Traces

AD: Radial Traces

G. Seisenberger et al., Science **294**, 1929 (2001)

Brownian motion in non-stationary medium

- ♦ Langevin SDE (tentative)

$$\frac{d\mathrm{v}}{dt} = -\gamma(t)\mathrm{v} + \sqrt{2D_{\mathrm{v}}(t)}\,\xi(t)$$

$$D_{\mathrm{v}} = \gamma(t)\frac{k_B T(t)}{m}$$

- ♦ overdamped

$$\frac{dx}{dt} = \sqrt{2D_x(t)}\,\xi(t)$$

$$D_x = \frac{k_B T(t)}{m\gamma(t)}$$

- Langevin equation for a particle in a heat bath with time-dependent temperature (Bray and Casado, 1990)

Caution: "Time-dependent diffusion coefficient for anomalous diffusion"

$$D_a(t) = \lim_{t\to\infty}\frac{\langle x^2(t)\rangle}{t}$$

Can be introduced for any anomalous diffusion process

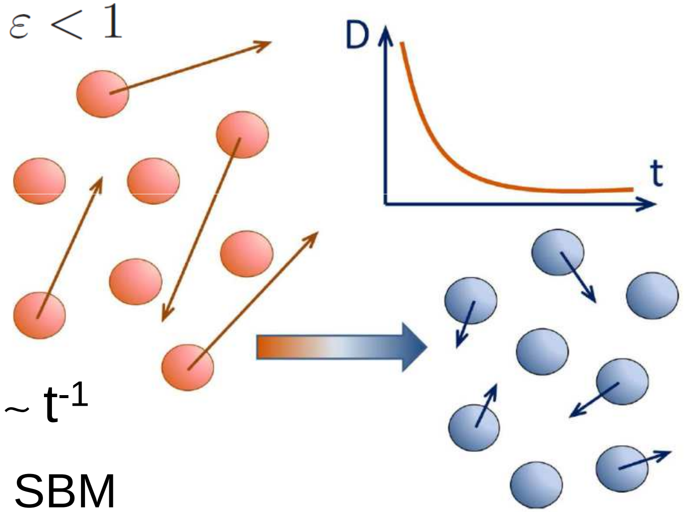
$$\langle x^2(t)\rangle = \int_0^t D(t')dt' \neq \quad D_a(t)t$$

But: "almost" the same for D(t)$\propto$ t$^{\alpha-1}$

- ♦ Magnetic resonance imaging – measured water diffusion in muscles and in brain, D(t) ~ D$_\infty$ + const·t$^{-\theta}$, $\theta > 0$ (Novikov et al, 2013)

Granular gases and Ultraslow SBM

- Granular particles collide inelastically and lose a fraction of their kinetic energy during collisions which transforms into heat stored in internal degrees of freedom
- No external forces, the gas evolves freely and gradually cools down
- The first stage of its evolution, the granular gas is in the **homogeneous cooling state** characterized by uniform density and absence of macroscopic fluxes
- Constant restitution coefficient $\varepsilon = |v'_{12}/v_{12}|$ $\quad 0 < \varepsilon < 1$
- Haff's law $T(t) = T_0/\left(1 + t/\tau_0\right)^2$
- Self-diffusion coefficient



$$D(t) = 3T(t)\tau_c(t)/(2m) = D_0/(1 + t/\tau_0) \sim t^{-1}$$

as ultraslow SBM

MSD: from ballistic motion $\quad$ to ultraslow diffusion

$$\langle R^2(t)\rangle \sim t^2 \quad \Rightarrow \quad \langle R^2(t)\rangle \sim \ln(t)$$

Expected ultraslow SBM behavior

Geophysical and environmental processes

- Occur under the influence of external time-dependent and random forcing
- Stochastic hydrology: snowmelt dynamics

R.L. Bras, Hydrology MA, 1990

A. Molini et al, 2011

The amount of fresh water potentially available from both snow accumulation and rainfall during the melting season in mountainous regions

$$\frac{dh}{dt} = -qkt^{\alpha} + \sqrt{2kt^{\alpha}}\,\xi(t)$$

$\alpha \approx 0.25$

Power-law time dependent drift directed towards the total depletion of snow mantle

Power-law diffusion

Positive excursions: precipitation events

Negative excursions: pure melting periods

- Variablity of the process is expected to increase proceeding into the warm season

Nonstationary increments, scaling distributions, and variable diffusion processes in financial markets

Kevin E. Bassler*†, Joseph L. McCauley*‡, and Gemunu H. Gunaratne*§¶

*Department of Physics and †Texas Center for Superconductivity, University of Houston, Houston, TX 77204; ‡J. E. Cairnes Graduate School of Business and Public Policy, National University of Ireland, Galway, Ireland; and §Institute of Fundamental Studies, Kandy, Sri Lanka

Communicated by Mitchell J. Feigenbaum, The Rockefeller Unive

Intraday fluctuations in the Euro-Dollar exchange rate: 1-min-interval tick data 1999-2004

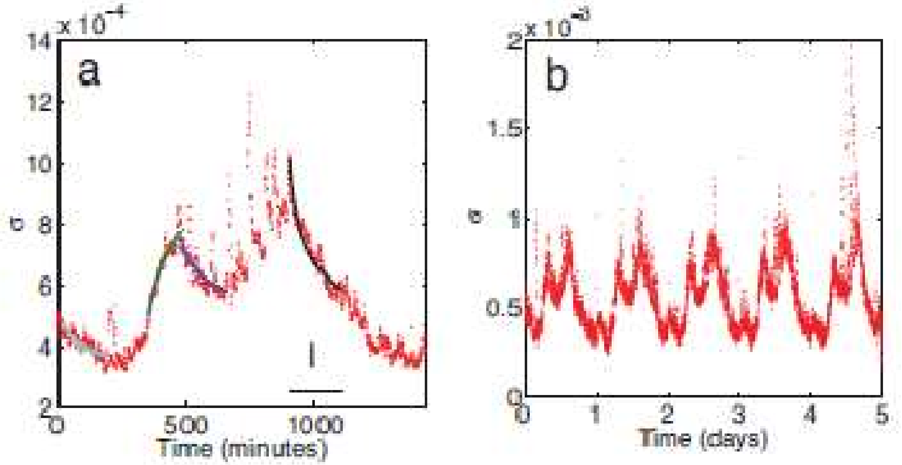


Fig. 1. Intraday increments in the Euro–Dollar exchange are nonstationary. (*a*) Standard deviation $\sigma(t) = \sqrt{\langle x(t)^2 \rangle}$ of the increments of the daily Euro–Dollar exchange as a function of the time of day (in Greenwich Mean Time). Our statistical analysis assumes that $x(t)$ follows the same stochastic process each trading day. The average indicated by the brackets ⟨.⟩ is taken over the ~1,500 trading days between 1999 and 2004, and the standard error at each point is typically 3%. Note that, if the stochastic dynamics had stationary increments, $\sigma(t)$ would be constant. Instead, it varies by more than a factor of 3 during the day, thus showing explicitly that the exchange rate has nonstationary increments. Notice also that $\sigma(t)$ scales in time during several intervals, four of which are highlighted by colored lines that are power-law fits. Our analysis focuses on the interval I shown by the horizontal solid line. (*b*) Weekly behavior of $\sigma(t)$ for the same data. Observe that it exhibits an approximate daily periodicity, thereby justifying our assumption of the daily repeatability of the stochastic process underlying the Euro–Dollar exchange rate.

$$D(x,t) \approx t^{2H-1}\mathcal{D}(u), \quad u = \frac{x}{\tau^H}$$

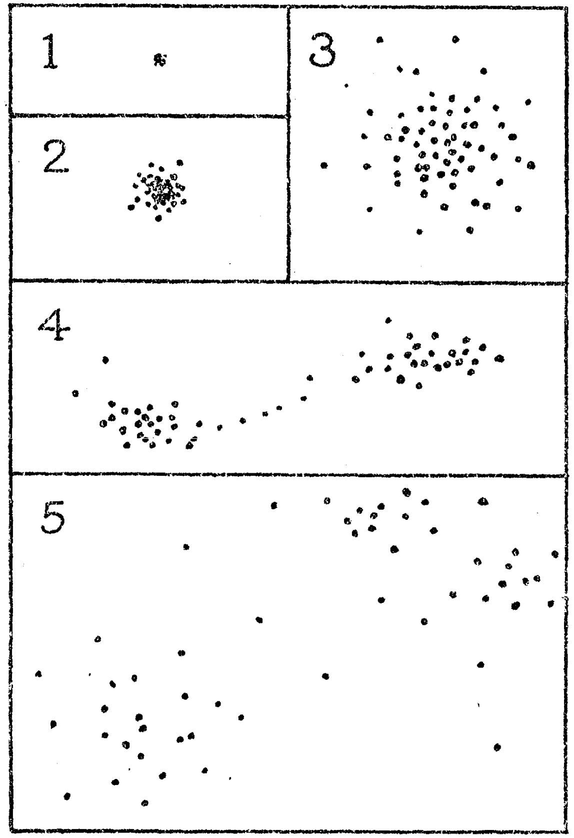
$H \approx 0.35$

Variable diffusion processes exhibit “clustering of volatility”

Richardson (relative) diffusion in turbulence

$$\left\langle \vec{l}^{\,2}(t) \right\rangle \propto t^3$$

The failure of the dispersal of a point-charge to serve as a mathematical element, from which the dispersal of an extended system may be built up, appears to be intimately connected with the fact that in the atmosphere the dispersal goes on in patches. That is to say, a small dense cluster of marked molecules, represented by the dot in fig. 1 which, by molecular diffusion alone, would spread through the successive spherical clusters shown in figs. 2 and 3, actually seldom passes through the large spherical stage 3, because it is first sheared into two detached clusters as suggested in fig. 4. These are carried far from one another, and are likely to be again torn into smaller pieces as in fig. 5. Meanwhile each of the torn parts is gradually spreading by molecular diffusion. These diagrams are, of course, merely illustrative fictions.



FIGS. 1–5.

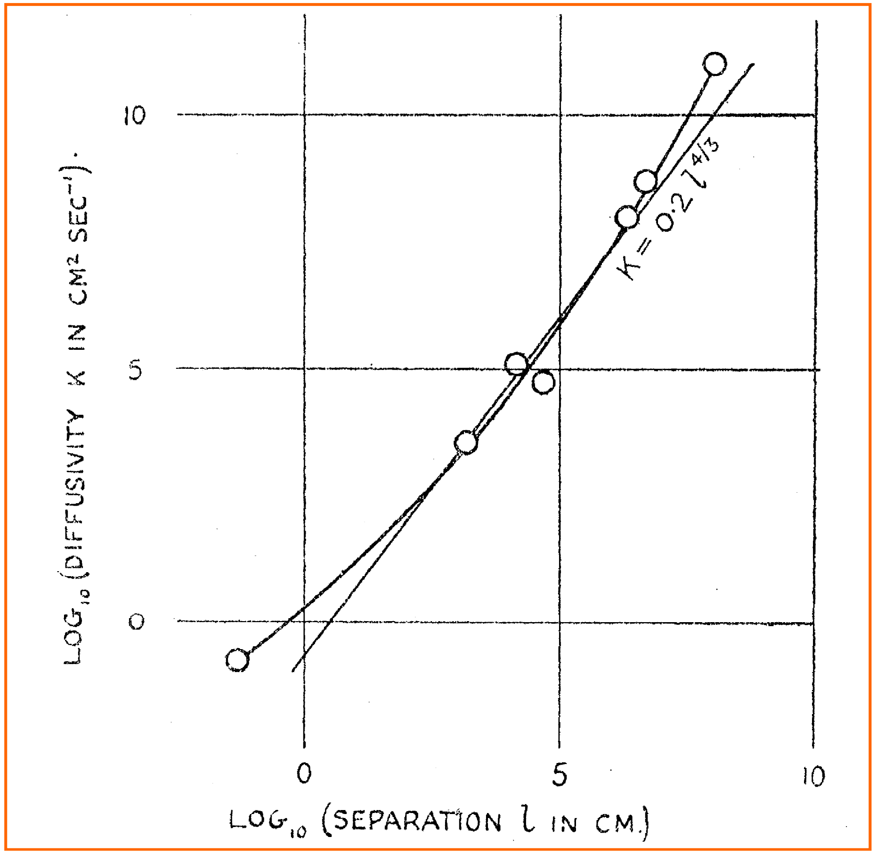
Lewis F. Richardson

Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, Vol. 110, No. 756 (Apr. 1, 1926), 709-737.

Classical example of superdiffusion

Richardson law:

$$\left\langle \vec{l}^{\,2}(t) \right\rangle \propto t^3$$

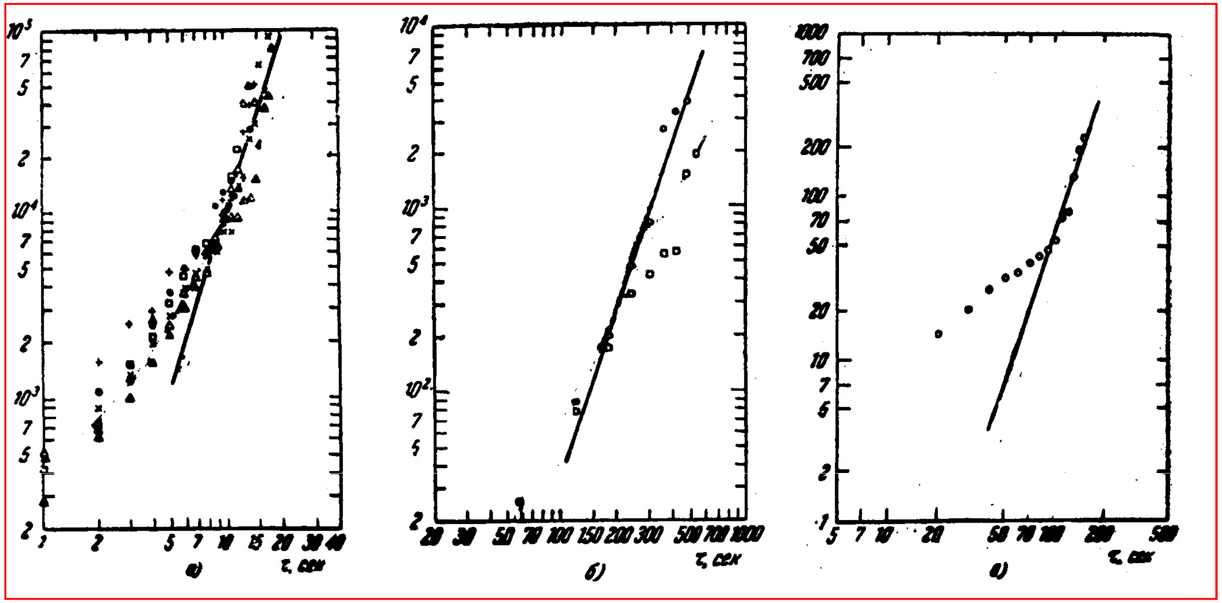


$$\frac{\partial f}{\partial t} = \varepsilon \frac{\partial}{\partial l}\left(l^{4/3} \frac{\partial f}{\partial l} \right)$$

L.F. Richardson, Proc. Royal Soc. London, 1926

Classical example of superdiffusion

$$\left\langle \vec{l}^{\,2}(t) \right\rangle \propto t^3$$



a) Frenkiel and Katz (1956),
b) Seneca (1955),
c) Kellogg (1956)

$\left\langle \vec{l}^{\,2}(t) \right\rangle$ **versus t (Gifford, 1957)**

Gave inspiration to:

Monin (1955, 1956) , **Tchen** (1959) :
Space – fractional diffusion equation

$$\frac{\partial}{\partial t} p(\vec{l},t) = -D(-\Delta)^{1/3} p(\vec{l},t)$$

Klafter, Blumen, Shlesinger (1987):
Lévy walks

$$f(k,s) = \frac{1-\psi(s)}{u} \frac{1}{1-\psi(k,s)}$$

$$\psi(r,t) = Cr^{-\mu}\delta\left(r - t^{\nu}\right)$$

How to explain the Richardson diffusion ?

$$\left\langle \vec{l}^{\,2}(t) \right\rangle \propto t^3$$

In 3D: $\frac{\partial}{\partial t} f(\vec{l},t) + div \vec{J} = 0$ + $\vec{J}(\vec{l},t) = -D(l,t)\, grad\, f(\vec{l},t)$

$$\frac{\partial}{\partial t} f(\vec{l},t) = \frac{\partial}{\partial l_j} D(l,t) \frac{\partial}{\partial l_j} f(\vec{l},t) \quad , \quad l = \left| \vec{l} \right|$$

Constructed by analogy with diffusion equation, can not be derived or strictly justified

Richardson (1926) : $D(l,t) \equiv D(l) \propto l^{4/3}$

Batchelor (1952) : $D(l,t) \equiv D(t) \propto t^2$

Hentschel and Proccacia (1984) $D(l,t) \propto t^a l^b \quad , \quad 2a + 3b = 4$

$$D(l,t) \propto t\, l^{2/3}$$

$$\left\langle \vec{l}^{\,2}(t) \right\rangle \propto t^3$$

Can not be fixed on dimensional grounds alone

BUT: The distribution functions are different, and experiment might be used to distinguish between the choices

Part II. Inhomogeneous medium: Heterogeneous diffusion process (HDP)

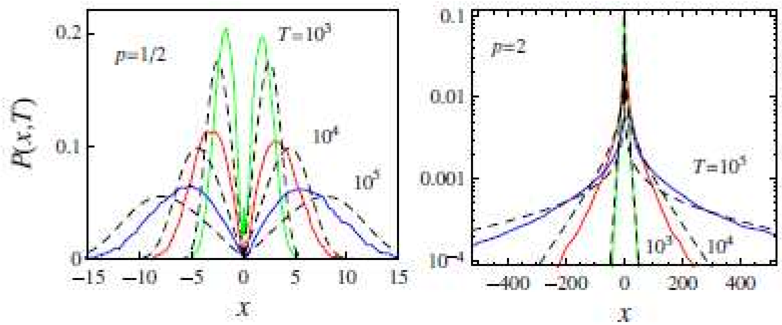
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2D(x)}\zeta(t) \qquad\qquad D(x) = D_0|x|^\alpha$$

Stratonovich interpretation: $$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\sqrt{D(x)}\frac{\partial}{\partial x}\left(\sqrt{D(x)}P(x,t)\right)\right]$$

$$y(x) = \int^x \frac{\mathrm{d}x'}{\sqrt{2D(x')}}$$

y(t) the Wiener process

$$P(x,t) = \frac{|x|^{1/p-1}}{\sqrt{4\pi D_0 t}}\exp\left(-\frac{|x|^{2/p}}{(2/p)^2 D_0 t}\right)$$



Ensemble averaged MSD (EMSD)

$$\langle x^2(t)\rangle = \frac{\Gamma(p+1/2)}{\pi^{1/2}}\left(\frac{2}{p}\right)^{2p}(D_0 t)^p \qquad p = 2/(2-\alpha)$$

$\alpha<0$ subdiffusion
$\alpha>0$ superdiffusion

$\alpha < 2$

Correlation properties of HDP

$$\langle x(t_1)x(t_2)\rangle = \frac{2^{p+1}}{\sqrt{\pi}} \frac{\Gamma(p+1)\Gamma(\frac{p}{2}+1)}{p^{2p}\Gamma(\frac{p}{2}+\frac{1}{2})}[D_0 t_1]^{(p+1)/2}$$

$$\times [D_0(t_2-t_1)]^{(p-1)/2} {}_2F_1\left(\frac{1-p}{2}, \frac{p}{2}+1; \frac{3}{2}; \frac{-t_1}{t_2-t_1}\right)$$

$p = 2/(2-\alpha)$

$\alpha = 0$:
Brownian limit

Reminder: consecutive increments of FBM $\left\langle [B_H(t)-B_H(t-\tau)][B_H(t+\tau)-B_H(t)]\right\rangle = \left(2^{2H-1}-1\right)\tau^{2H} \Rightarrow$

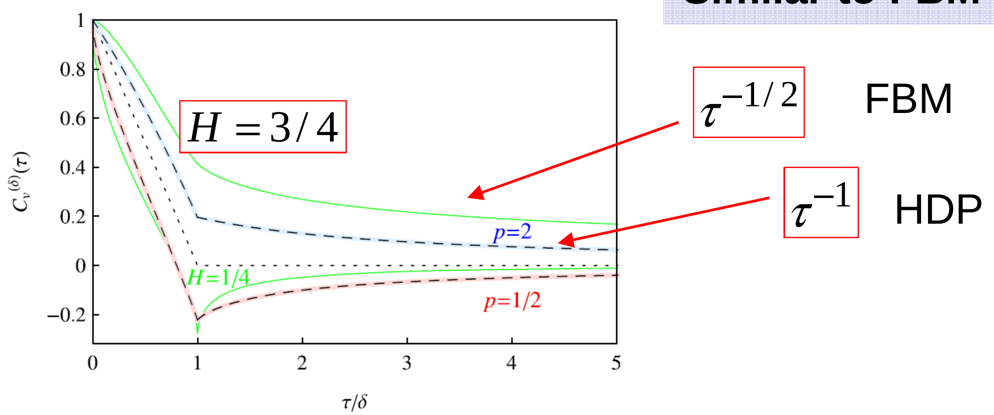
$\Rightarrow$ < 0, 0<H<1/2 (sub) ; > 0, 1/2<H<1 (super)

Consequitive increments of HDP $\tau << t$

$$\langle [x(t)-x(t-\tau)][x(t+\tau)-x(t)]\rangle \sim \begin{cases} -\tau\sqrt{D_0/t}, & p=\frac{1}{2}, \\ (D_0\tau)^2, & p=2. \end{cases}$$

antipersistence ($p=\frac{1}{2}$)
persistence ($p=2$)
Similar to FBM

$$C_v^{(\delta)}(\tau) = \frac{\langle [x(\tau+\delta)-x(\tau)][x(\delta)-x(0)]\rangle}{\sqrt{\langle [x(\tau+\delta)-x(\tau)]^2}\sqrt{\langle [x(\delta)-x(0)]^2\rangle}}.$$



Cherstvy, Ch, Metzler, NJP 2013

Nonergodic behavior of HDP

Time-ensemble averaged MSD (TEMSD)

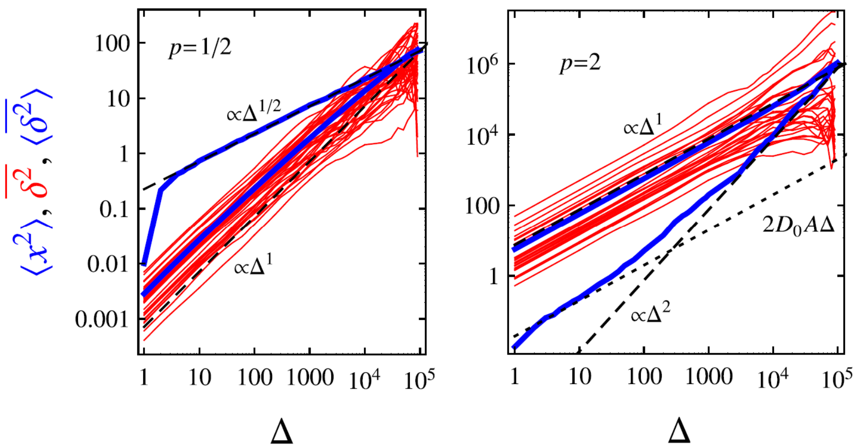
$$\left\langle \overline{\delta^2(\Delta)} \right\rangle = \frac{1}{T-\Delta}\int_0^{T-\Delta} \left\langle [x(t+\Delta)-x(t)]^2 \right\rangle \mathrm{d}t$$

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle = \frac{\Gamma(p+1/2)}{\pi^{1/2}} \left(\frac{2}{p}\right)^{2p} D_0^p \frac{\Delta}{T^{1-p}} = 2D_{eff}\Delta \quad , \quad D_{eff} \simeq T^{p-1}$$

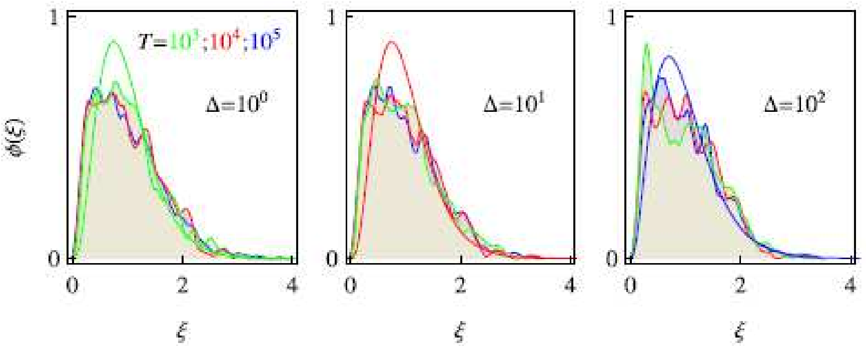
Relation between TEMSD and EMSD: CTRW – like behavior

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle = (\Delta/T)^{1-p} \langle x^2(\Delta) \rangle$$

Amplitude scatter PDF $\phi(\xi = \overline{\delta^2}/\langle\overline{\delta^2}\rangle)$



Cherstvy, Ch, Metzler, NJP 2013, PCCP 2014



BUT: $\phi(0) = 0$ contrast to CTRW

Ergodicity Breaking: EB ≠ 0

$$EB = \frac{\left(\left\langle (\overline{\delta^2})^2 \right\rangle - \left\langle \overline{\delta^2} \right\rangle^2\right)}{\left\langle \overline{\delta^2} \right\rangle^2}$$

(Rytov et al., 1966, Barkai et al., 2008)

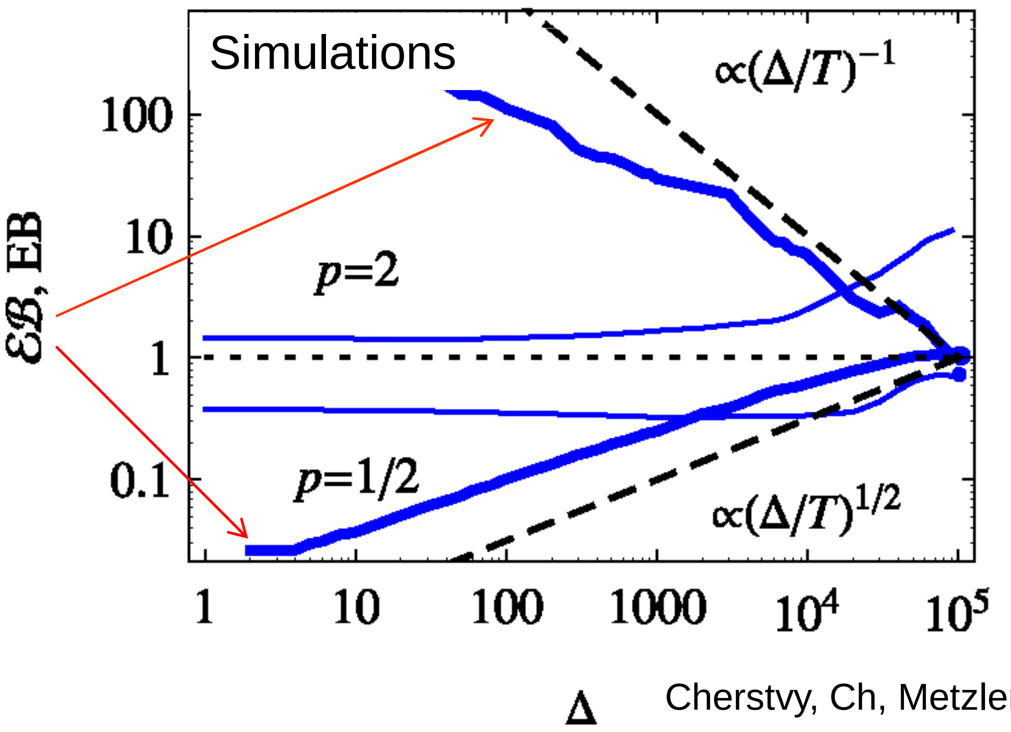
CTRW: (He, Burov, Metzler, Barkai, 2008)

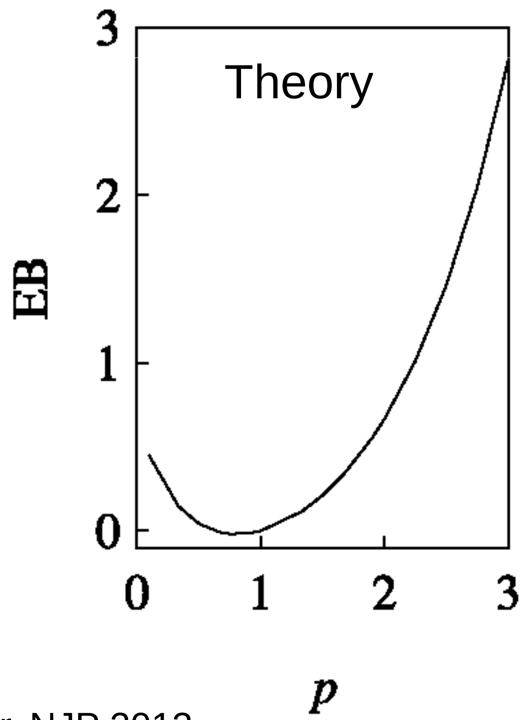
$$\lim_{T/\Delta\to\infty} EB = \frac{2\Gamma^2(1+p)}{\Gamma(1+2p)} - 1 \neq 0$$

HDP:

$$\lim_{T/\Delta\to\infty} EB = \frac{2}{3}, \quad p = 2$$

similar to CTRW





Cherstvy, Ch, Metzler, NJP 2013

II. Nonstationary medium: Scaled Brownian Motion (SBM)

$$\dot{x}(t) = \sqrt{2D(t)}\ \xi(t)$$

Lim & Muniandi (2002)

$$D(t) = \alpha K_\alpha t^{\alpha-1}$$

$$\langle x^2(t)\rangle \simeq 2K_\alpha t^\alpha$$

α > 1 superdiffusion
α < 1 subdiffusion

$$P(x,t) = \frac{1}{\sqrt{4\pi K_\alpha t}} \exp\left(-\frac{x^2}{4K_\alpha t}\right)$$

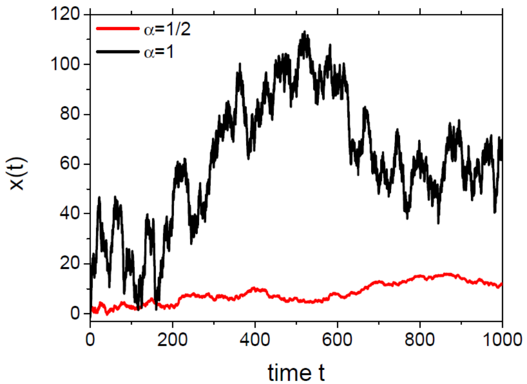
ACF , t < s $\quad \langle x(t)x(s)\rangle = \langle x(t)^2\rangle$

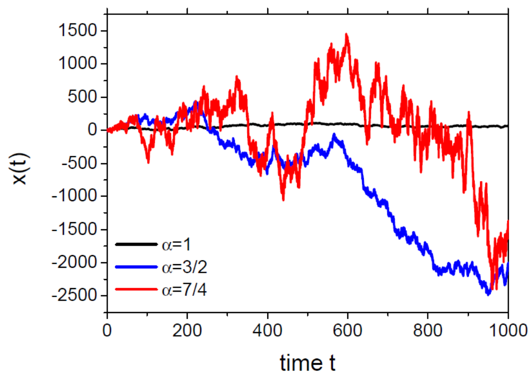
both similar to ordinary BM

Independent increments, $t_3 > t_2 > t_1$ $\quad \langle [x(t_2) - x(t_1)][x(t_3) - x(t_2)]\rangle = 0$

α = 1 (black)

α = 1/2 (red)





α = 1 (black)

α = 7/4 (blue)

α = 3/2 (red)

Nonergodic behavior of SBM

$$\overline{\delta^2(\Delta)} = \frac{1}{T-\Delta}\int_0^{T-\Delta}\left(x(t'+\Delta)-x(t')\right)^2 dt'$$

T : trajectory length

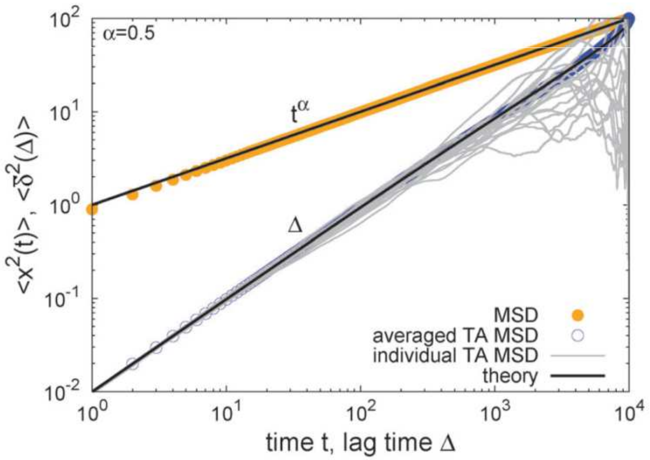
$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \sim 2K_\alpha \frac{\Delta}{T^{1-\alpha}} = 2D_{eff}\,\Delta$$

CTRW-like behavior

$$D_{eff} \sim T^{\alpha-1}$$

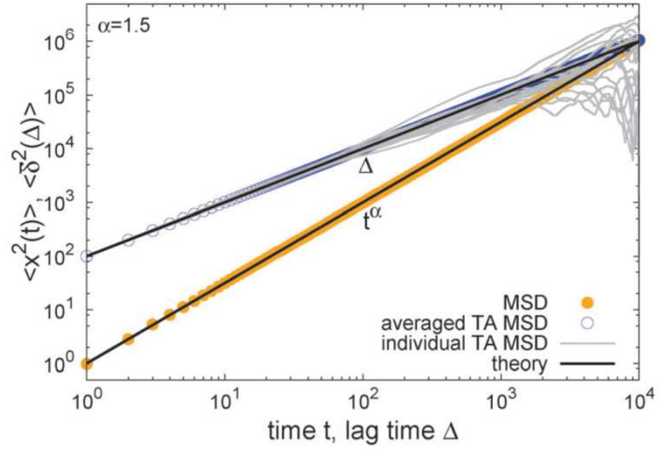
increases, α > 1 (superdif)
decreases, α < 1 (subdif)

$\left\langle \overline{\delta^2(\Delta)} \right\rangle = (\Delta/T)^{1-p}\langle x^2(\Delta)\rangle$ similar to CTRW and HDP



α = 1/2

α = 3/2

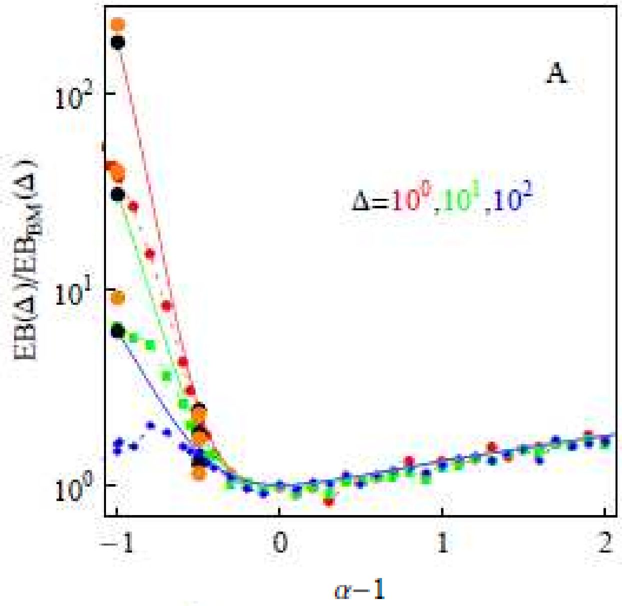


(Jeon, Ch, Metzler, PCCP Com 2014)

BUT: $\lim_{T\to\infty} EB(\Delta) = 0$ contrast to CTRW and HDP…

… and similar to BM $EB(\Delta) \sim \frac{4}{3}\frac{\Delta}{T} \underset{t\to\infty}{\to} 0$ (Thiel and Sokolov, 2014)

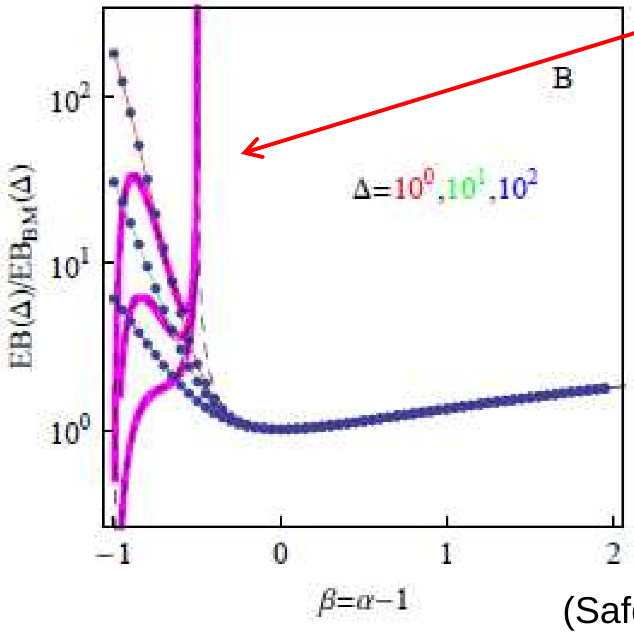
Quantifying non-ergodicity of SBM : EB at small, but finite Δ / T



Full analytical solution and Langevin simulations, Δ / T << 1

$$EB(\Delta) \sim \begin{cases} \frac{4}{3}\frac{\alpha^2}{2\alpha-1}\frac{\Delta}{T}, & \alpha > 1/2 \\ 4C(\alpha)\left(\frac{\Delta}{T}\right)^{2\alpha}, & \alpha < 1/2 \end{cases}$$

$C(\alpha=1/2) = \infty$, $C(0) = 0$



Spurious discontinuity of EB at $\alpha = 1/2$

$$EB_{\alpha=1/2}(\Delta) \sim \frac{\Delta}{3T}\left[\log\left(T/\Delta\right) + 2\log 2 - 5/6\right]$$

$$EB_{\alpha=0}(\Delta) \sim \frac{4\left(\pi^2/6-1\right)}{\left(\log\left[T/\Delta\right]+1\right)^2}$$

(Bodrova, Ch, Cherstvy, Metzler, NJP 2015)

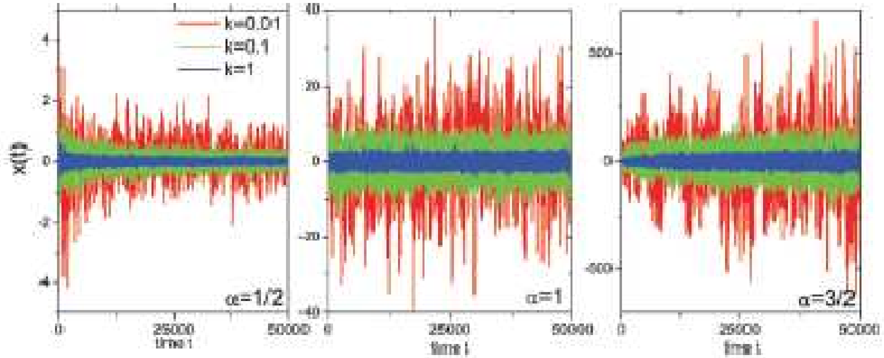
(Safdari, Cherstvy, Ch, Thiel, Sokolov, Metzler, JPA 2015)

Confined SBM

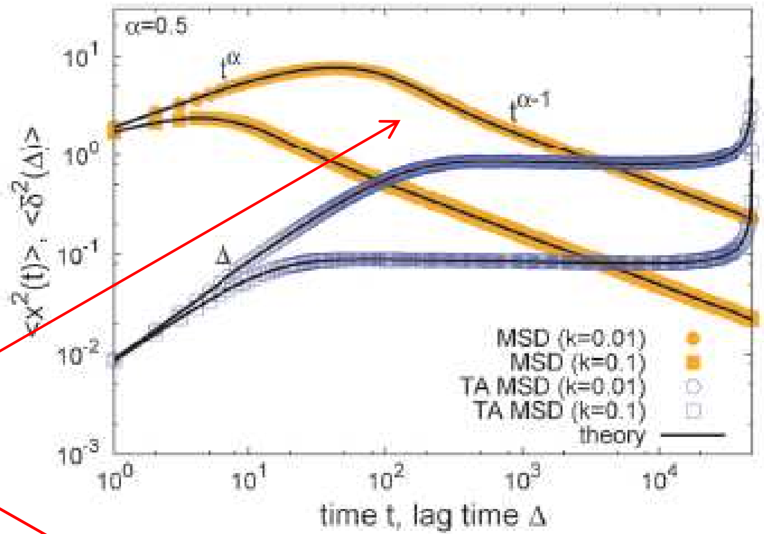
$$\frac{\partial}{\partial t}P(x,t)=\frac{\partial}{\partial x}\left(kx+D(t)\frac{\partial}{\partial x}\right)P(x,t)$$

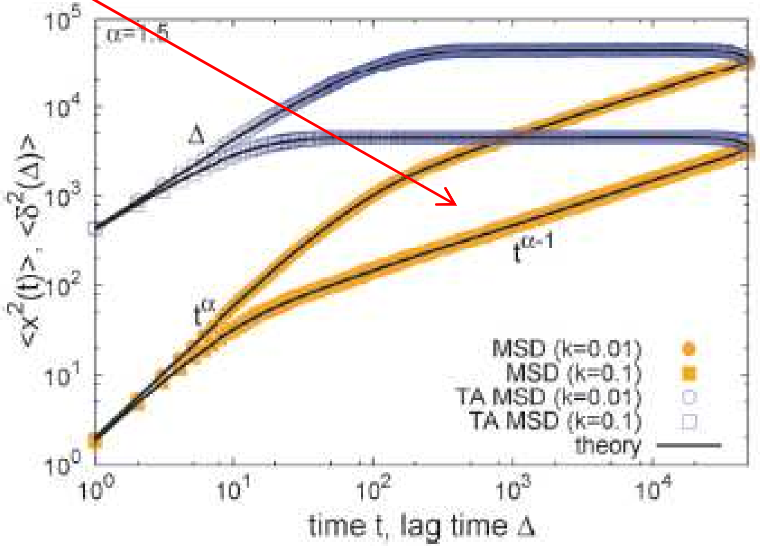
$$\left\langle x^2(t)\right\rangle \sim \frac{\alpha K_\alpha}{k}t^{\alpha-1} \quad , \quad t>>1/k$$

- EMSD: After the free anomalous diffusion behavior at short times we observe a turnover to a power-law behavior with negative or positive scaling exponent



subdiff normal superdiff





- TEMSD: plateau at $\Delta >> 1/k$

Aging SBM: initiated at t = 0 and measured from t_a

$$\overline{\delta_a^{\,2}(\Delta)} = \frac{1}{T-\Delta}\int_{t_a}^{T+t_a-\Delta}\left(x(t'+\Delta)-x(t')\right)^2 dt'$$

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \sim 2K_\alpha \frac{\Delta}{T^{1-\alpha}}$$

- **Universal aging depression**

$$\boxed{\left\langle \overline{\delta_a^{\,2}(\Delta)} \right\rangle \sim \Lambda_\alpha\left(t_a/T\right)\left\langle \overline{\delta^2(\Delta)} \right\rangle \quad , \quad t, t_a >> \Delta}$$

Identical to the aged subdiffusive CTRW and HDP !

Depression factor $\quad \Lambda_\alpha\left(z\right) = \left(1+z\right)^\alpha - z^\alpha$

Limit of strong aging ta >> t $\quad \left\langle \overline{\delta_a^{\,2}(\Delta)} \right\rangle \sim 2\alpha K_\alpha t_a^{\alpha-1}\Delta$

- **Apparent restoration of ergodicity in the strong aging limit**

$$\left\langle \overline{\delta_a^{\,2}(\Delta)} \right\rangle = \left\langle x^2(\Delta) \right\rangle_a$$

Subdiff CTRW, HDP, SBM

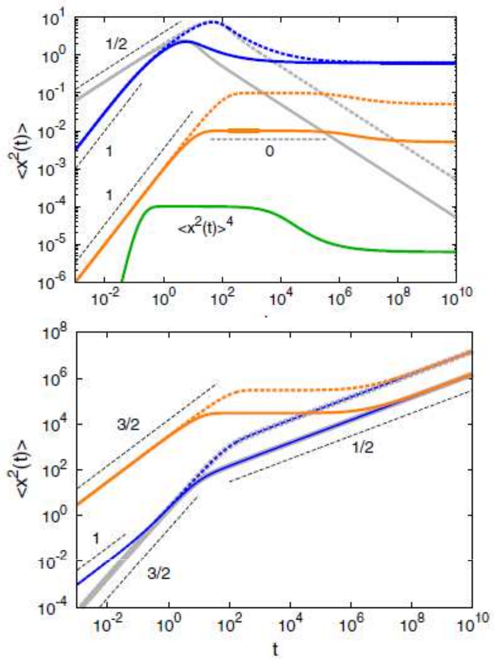
Cherstvy, Ch, Metzler, JPA 2014; Safdari, Ch, Jafari, Metzler, PRE 2015

Aging confined SBM

- **Strong effect of a weak aging** t >> 1/k, t_a << 1/k

$$\left\langle x^2(t)\right\rangle_a \sim \frac{\alpha K_\alpha}{k} t^{\alpha-1} + 2K_\alpha t_a^\alpha$$

0 for subdiff



The leading long-time behavior
for 0 < α < 1 is the plateau

$$\left\langle x^2(t)\right\rangle_a \sim 2K_\alpha t_a^\alpha \quad , \quad t >> 1/k$$

-> Even for very weak aging the EMSD becomes ta dependent

-> Stems from the initial free motion during the aging period

-> Superdiffusion: the leading order term shows the growth

$$\left\langle x^2(t)\right\rangle_a \sim \alpha K_\alpha k^{-1} t^{\alpha-1}$$

- **Universal depression again**

$$\overline{\left\langle \delta_a{}^2(\Delta)\right\rangle} \sim \Lambda_\alpha\left(t_a/T\right)\overline{\left\langle \delta^2(\Delta)\right\rangle} \quad , \quad t, t_a >> \Delta >> 1/k \quad or \quad \Delta << 1/k$$

$$\Lambda_\alpha(z) = (1+z)^\alpha - z^\alpha$$

Ultraslow SBM and Granular Gases

- **Ultraslow scaled Brownian motion: "degenerate" SBM , α = 0**

$$dx(t)/dt = \sqrt{2D(t)}\xi(t) \qquad D(t) = \frac{D_0}{1+t/\tau_0}$$

EMSD $$\langle x^2(t)\rangle = 2\int_0^t dt' \int_0^t dt'' \sqrt{D(t')D(t'')}\,\langle \xi(t')\xi(t'')\rangle = 2D_0\tau_0 \log\left(1+t/\tau_0\right)$$ ~ log t

One more representative of the family of ultraslow random processes

- **Granular gas in homogeneous cooling state**

Haff's law

θ: temperature

$$\theta(t) = \frac{\theta_0}{(1+t/\tau_0)^2}$$

Self-diffusion coefficient $$D(t) = \frac{3\theta(t)\tau_c(t)}{m} = \frac{D_0}{(1+t/\tau_0)} \propto t^{-1}$$ as ultraslow SBM

MSD: from ballistic motion $$\langle R^2(t)\rangle \sim t^2$$

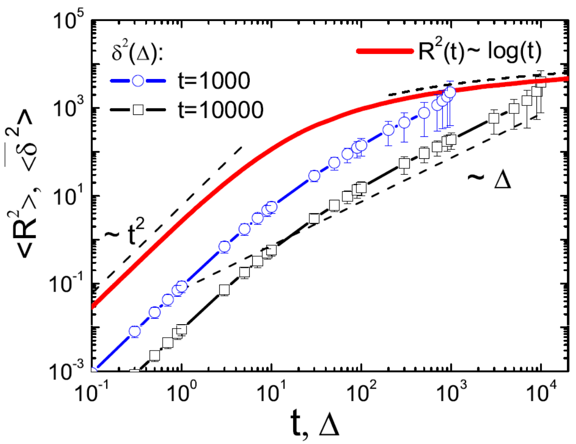
to ultraslow diffusion $$\langle R^2(t)\rangle \sim \ln(t)$$

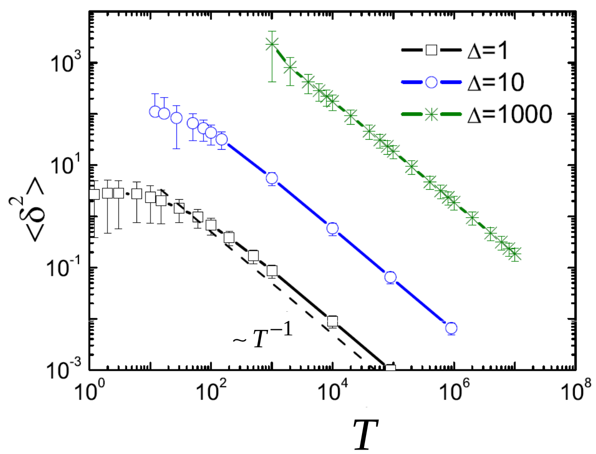
Expected ultraslow SBM behavior

Granular gases versus Ultraslow SBM

TEA MSD: from $\langle\overline{\delta^2(\Delta)}\rangle \sim \Delta^2$ to $\left\langle \overline{\delta^2(\Delta)} \right\rangle \simeq \frac{\Delta}{T}$ ≠ $\left\langle \overline{\delta^2(\Delta)} \right\rangle \simeq \frac{\Delta}{T} \ln\frac{T}{\Delta}$ ⇦ TEA MSD for ultraslow SBM

Scaling persuasively confirmed in MD simulations





$$\left\langle \overline{\delta^2(\Delta)} \right\rangle = \frac{1}{T-\Delta}\int_0^{T-\Delta} \left\langle \left(\mathbf{R}(t'+\Delta)-\mathbf{R}(t')\right)^2 \right\rangle dt' \quad = \left\langle \overline{\delta_0^2(\Delta)} \right\rangle + \Xi(\Delta)$$

SBM

beyond SBM approximation

$$\left\langle \overline{\delta_0^2(\Delta)} \right\rangle \simeq \frac{6D_0\tau_0\Delta}{T}\left(\ln\left(\frac{T}{\Delta}\right)+1\right)$$

log-term is cancelled out

$\gamma = 0.577\ldots$

Euler's constant

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \simeq 6D_0\tau_0 C(\beta)\frac{\Delta}{T} \qquad C(\beta) = \gamma + \frac{1}{\beta} + \psi(\beta) \qquad \beta = \tau_0/\tau_v$$

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz}$$

Digamma function

Beyond the overdamped SBM

$$dx(t)/dt = \sqrt{2D(t)}\xi(t) \Rightarrow dv/dt + v/\tau_v(t) = \sqrt{2D(t)}/\tau_v(t)\xi(t)$$

$$D(t) = \frac{D_0}{1+t/\tau_0} \qquad T(t) = T(0)/\left(1+t/\tau_0\right)^2 \qquad \tau_v^{-1} = \tau_v^{-1}(0)\sqrt{T(t)/T_0}.$$

$$\langle v(t_1)v(t_2)\rangle = \frac{T(0)\tau_0}{m\tau_v(0)(\beta-1)}\frac{(1+t_1/\tau_0)^{\beta-2}}{(1+t_2/\tau_0)^{\beta}}.$$

$$\overline{\delta^2(\Delta)} = \frac{1}{T-\Delta}\int_0^{T-\Delta}\left(x(t'+\Delta)-x(t')\right)^2 dt'$$

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \simeq 6D_0\tau_0 C(\beta)\frac{\Delta}{T}$$

As in granular gas, but in contrast to overdamped SBM:

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \simeq \frac{\Delta}{T}\ln\frac{T}{\Delta}$$

Summary: Markovian scale-invariant motions in inhomogeneous and non-stationary media

	HDP $D(x) \sim |x|^{\alpha}$, $\alpha < 2$	SBM $D(t) \sim t^{\alpha-1}$, $\alpha \geq 0$
♦ **MSD** $\langle x^2(t) \rangle$	$\sim t^p$, $p = 2/(2 - \alpha)$	$\sim t^{\alpha}$, $\alpha \neq 0$ $\sim \log t$, $\alpha = 0$
♦ **Correlations**	Power-law correlations, persistent for p > 1 antipersistent p < 1	uncorrelated increments
♦ **TE MSD** $\left\langle \overline{\delta^2(\Delta)} \right\rangle$ T : trajectory length	$\sim \Delta / T^{1-\alpha}$ (sim.CTRW)	$\sim \Delta / T^{1-\alpha}$ $\alpha \neq 0$ (sim.CTRW) $\sim (\Delta / T)\ln(T / \Delta)$, $\alpha = 0$
♦ **EB parameter**	$\lim_{T/\Delta \to \infty} EB \neq 0$ (sim.CTRW)	$\lim_{T/\Delta \to \infty} EB = 0$ (sim. BM, FBM ≠ sim.CTRW)

♦ **Aging behavior**

Similar to CTRW

$$\left\langle \overline{\delta_a{}^2(\Delta)} \right\rangle \sim \Lambda_\alpha(t_a / T) \left\langle \overline{\delta^2(\Delta)} \right\rangle \quad , \quad t, t_a >> \Delta$$

$$\Lambda_\alpha(z) = (1+z)^\alpha - z^\alpha$$

Not mentioned here :

- ♥ **More properies of confined SBM and HDP**

  (Jeon, Ch, Metzler, PCCP Com 2014; Cherstvy, Ch, Metzler, JPA 2014)

- ♥ **Aging HDP and SBM**

  (Cherstvy, Ch, Metzler, JPA 2014; Safdari, Ch, Jafari, Metzler, PRE 2015)

- ♥ **HDP in 2D**

  (Cherstvy, Ch, Metzler, Soft Matter 2014)

- ♥ **First passage problem for HDP and SBM**

  (Cherstvy, Ch, Metzler, Soft Matter 2014; Safdari, Ch, Jafari, Metzler, PRE 2015; Cherstvy, Ch, Metzler, in preparation)

- ♥ **Viscoelastic granular gas: more similarity to SBM with $\alpha = 1/6$**

  (Bodrova, Ch, Cherstvy, Metzler, PCCP 2015)

- ♥ **Beyond the overdamped SBM: weakly damped SBM and USBM**

  (Bodrova, Ch, Cherstvy, Metzler, NJP 2015; PCCP 2015; in preparation)

- Anomalous is normal
- Happy families are all alike; every unhappy family is unhappy in its own way

Theory is far from the end, fortunately…