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Effect of discreteness on a sine-Gordon three-soliton solution

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Abstract

The collision between a kink and a high amplitude breather in the Frenkel–Kontorova chain with a small degree of discreteness was studied numerically and the results were compared with an exact three-soliton solution to the sine-Gordon (SG) equation. It was found that there exists a narrow range of parameters of quasiparticles where the collision in the discrete system is strongly inelastic and that the inelastic collision occurs in the vicinity of a separatrix of the SG three-soliton solution. © 1998 Elsevier Science B.V.

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1. Introduction

In the last few decades the effects of discreteness on the properties of solitary waves have been widely studied [1–17]. It has been recognized that the continuum limit approximation cannot describe many effects important in condensed matter physics when the width of the soliton becomes of the order of lattice spacing.

The one-soliton solution (kink) has received much attention in most studies. For the static kink in the Frenkel–Kontorova model the exact solution has been obtained [1]. It has been shown that the height of the Peierls–Nabarro (PN) potential decreases rapidly with decreasing the lattice period [2]. The critical pinning velocity and pinning frequency have been derived [3] using the perturbation formalism [4]. The

change in the shape of a kink and the radiation power loss have been discussed [3]. With the use of the derived discretized Hamiltonian formalism the influence of static dressing on the kink pinning frequency and the PN potential have been studied [5,6]. It has been reported that a moving kink not only radiates phonons continuously but can also emit large bursts of phonon radiation [7]. The discreteness strongly affects the Brownian-like motion of the kink and the kink-pair nucleation process in the thermalized chains [8–12].

The effects of discreteness on the two-soliton solutions have also been studied [12–16]. It has been shown that the kink–kink and kink–antikink collisions in a discrete lattice result in an additional radiation of energy [12–14]. Using the PN barrier calculations as the base it has been shown that the breather lives extremely long [16]. For the kink–antikink collision in a perturbed SG equation the reflection window effect, first reported for the φ^4 -equation [9], has been

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studied [18]. Related work on resonant kink-impurity interaction has been carried out [19].

The equations of motion for the Frenkel–Kontorova model can be written in a form where the period of lattice spacing h gives a measure of discreteness and the width of corresponding static SG kink is equal to 1. The effects of discreteness are usually noticeable for $h \approx 1$ and they cannot be neglected for $h > 1$. Formally, the continuum approximation is valid for $h \ll 1$ but in fact it has been discovered by many authors that even for $h \leq 0.5$ there is a good correlation between a discrete system and its continuum analog. When one studies the effects of discreteness, the numerical calculations or examples are typically provided on the range $0.7 \leq h \leq 2$. For some special forms of the substrate potential the discreteness effects can be noticeable even for very wide solitons [13].

For the continuum equations perturbed by various Hamiltonian and/or dissipative terms the effects of nontrivial many-soliton collisions have been found [20]. For many-soliton collisions a new effect of discreteness appears: namely, for a small degree of discreteness there exists a narrow range of parameters of quasiparticles in which the collision is strongly inelastic [17,21].

The purpose of this Letter is to show that the kink-breather collision in the discrete SG system can be strongly inelastic even at a small h and it happens when the discrete system is close to a separatrix of the corresponding unperturbed SG equation.

2. Sine-Gordon three-soliton solution

We consider the Hamiltonian of the Frenkel–Kontorova model [22] in a dimensionless form,

$$H = \sum_n \left(\frac{p_n^2}{2} + \frac{1}{2h^2} (u_{n+1} - u_n)^2 + (1 - \cos u_n) \right), \quad (1)$$

where u_n is the displacement of the n th particle from an initial point with coordinate $x = nh$, p_n is the momentum of the particle with a unit mass, H/h is the density of energy and h is the only parameter of the system, which gives a measure of discreteness because the equations of motion obtained from the Hamiltonian (1),

$$\frac{d^2 u_n}{dt^2} - \frac{1}{h^2} (u_{n-1} - 2u_n + u_{n+1}) + \sin u_n = 0, \quad (2)$$

in the continuum limit ($h \rightarrow 0$) are reduced to the SG equation,

$$u_{tt} - u_{xx} + \sin u = 0. \quad (3)$$

A three-soliton solution to Eq. (3) which describes the collision between a kink and a breather can be obtained by the Bäcklund transformation (see, e.g., Ref. [23]). The solution has the form of a sum,

$$u = v + w. \quad (4)$$

Function v is the kink solution

$$v = 4 \arctan \exp B, \quad (5)$$

where $B = \delta_k (x - x_k - d_k t)$, $\delta_k^{-1} = \sqrt{1 - d_k^2}$ is the width of the kink, $0 \leq d_k < 1$ is the velocity of the kink, and x_k defines the position of the kink at the time $t = 0$.

The second part of the solution (4) is

$$w = 4 \arctan(\eta X / \omega Y), \quad (6)$$

with

$$X = 2\omega(\sinh D - \cos C \sinh B) + 2\delta_k \delta_b (d_k - d_b) \sin C \cosh B, \quad (7)$$

$$Y = 2\eta(\cos C + \sinh D \sinh B) - 2\delta_k \delta_b (1 - d_k d_b) \cosh D \cosh B, \quad (8)$$

where $C = -\omega \delta_b (t - d_b (x - x_b)) + 2\pi m$ with an integer m , $D = \eta \delta_b (x - x_b - d_b t)$, $\delta_b^{-1} = \sqrt{1 - d_b^2}$, $0 \leq d_b < 1$ is the velocity of the breather, $\eta = \sqrt{1 - \omega^2}$, the frequency of the breather $0 \leq \omega < 1$ defines the amplitude or the rest energy of the breather (see Eq. (10) at $d_b = 0$), and x_b defines the position of the breather at the time $t = 0$. Note that the center of kink in the solution (4) is not located at x_k ; and that the center of breather is not at x_b .

In the continuum limit the velocity of the breather d_b , its wavelength λ and period T are related to each other by the following expressions,

$$|d_b| = \lambda/T, \quad \lambda = 2\pi \delta_b |d_b| / \omega, \quad T = 2\pi \delta_b / \omega. \quad (9)$$

The amplitude A and the energy E_b of the breather are

$$A = 4 \arctan(\eta/\omega), \quad E_b = 16\eta\delta_b. \quad (10)$$

The energy of the kink is $E_k = 8\delta_k$ and the energy of the solution (4) is $E = E_k + E_b$.

The coordinate and the time of collision of kink and breather are

$$x_c = \frac{d_k x_b - d_b x_k}{d_k - d_b}, \quad t_c = \frac{x_b - x_k}{d_k - d_b}. \quad (11)$$

Eq. (4) predicts the purely elastic collision between kink and breather when they recover their initial shapes and there is no energy and momentum exchange.

3. Conditions of inelastic collision

In order to study the collision of a kink and a breather in the Frenkel–Kontorova chain the equations of motion (2) were integrated numerically. Eq. (4) after the substitution $x \rightarrow nh$ was used to define the initial conditions. For the discreteness parameter we took the values $h \leq 0.1$. It was found that for such a small h and for not very small ω the continuum solution (4) describes the behavior of the discrete system with a very high accuracy while kink and breather do not interact. When the quasiparticles collide the situation becomes more complicated. For the most part the above statement can be expanded to the colliding quasiparticles, but in the present paper we show that for some special choice of parameters the inelasticity of collision drastically increases.

It has been shown that for nonlinear Hamiltonian system a high-frequency, small-amplitude perturbation produces a small perturbation of the invariant curves situated far from separatrices but for the invariant curves which are in the vicinity of a separatrix the influence of the perturbation results in qualitatively new dynamics, namely, in stochastic instability [24]. The stochastic instability appears inside a layer in the vicinity of a separatrix and the width of the layer decreases exponentially as the amplitude of perturbation decreases [24]. This means that in the vicinity of a separatrix there always exists a stochastic region, no matter how small the amplitude of the perturbation.

Let us find the conditions when the kink–breather system is close to a separatrix. Let u and u^* be the two

different particular solutions to Eq. (3) of the form (4). The solutions differ by the opposite signs of the velocities of quasiparticles,

$$d_k = -d_k^*, \quad d_b = -d_b^*, \quad (12)$$

and the positions of quasiparticles at the time $t = t_c$ in the solution u^* ,

$$\begin{aligned} x_k^* &= \frac{x_k(d_k + d_b) - 2d_k x_b}{d_b - d_k}, \\ x_b^* &= \frac{2d_b x_k - x_b(d_k + d_b)}{d_b - d_k}, \end{aligned} \quad (13)$$

are chosen in such a way that $x_c = x_c^*$ and $t_c = t_c^*$. The magnitudes of all other parameters in u and u^* are the same.

One can see that if

$$\omega(x_b - x_k) = 2\pi m \delta_b (d_k - d_b), \quad (14)$$

where m is an integer, then at the time $t = t_c$,

$$u(x, t_c) \equiv u^*(x, t_c), \quad (15)$$

and for the velocities one has $u_t(x, t_c) = -u_t^*(x, t_c)$. Under the additional conditions

$$d_k \rightarrow 0, \quad d_b \rightarrow 0, \quad (16)$$

one has $u_t(x, t_c) \rightarrow 0$ and $u_t^*(x, t_c) \rightarrow 0$ for any x , or

$$u_t(x, t_c) \rightarrow u_t^*(x, t_c) \quad (17)$$

for any x .

This means that if the conditions (14) and (16) are nearly fulfilled then at a certain time $t = t_c$ two qualitatively different solutions u , u^* as well as their derivatives with respect to time u_t , u_t^* are close to each other. The solutions coincide at $t = t_c$ under the conditions $d_k = 0$, $d_b = 0$ and Eq. (14). This limiting particular solution plays a role of a separatrix.

Physically, Eq. (14) gives the phase of a strongly inelastic kink–breather collision. An integer m appears due to the periodicity of the breather motion.

Turning back to the discrete equation (2) one can say that it corresponds to the SG equation perturbed by the discretization. In the discrete system with small h , the PN potential plays the role of a high-frequency, small-amplitude perturbation of the motion of rather wide quasiparticles. The influence of this perturbation is small for separating a kink and a breather and it is

small in most cases for colliding quasiparticles as well. A completely different type of situation occurs when both conditions (14) and (16) are nearly satisfied. In that instance even for a small h one can expect the considerable difference between pictures of collision in the continuum system (3) and in the discrete system (2) because the motion occurs close to the separatrix.

4. Numerical results

As one would expect the study of the dynamics inside the chaotic layer is very sensitive to numerical scheme. To avoid this fundamental difficulty we did not approach the closest vicinity of the separatrix. The Störmer method of order six [25] with a time step size $\Delta t = 0.005$ was used. The numerical data reported in this Letter did not vary noticeably with further decreasing of Δt .

The kink-breather collision in the chain equation (2) was studied for different breather initial positions x_b when other parameters were fixed. This means that we changed the phase of collision.

For the kink we put $x_k = 0$, $d_k = 0$ and for the breather $x_b > 0$, $d_b < 0$. Then the condition (14) can be written through the breather wavelength λ as

$$x_b/\lambda = m, \quad (18)$$

where m is a positive integer.

Before the collision the kink is at rest, therefore the velocity of the kink after the collision d_k^* is a suitable measure of inelasticity of the kink-breather interaction. The velocity d_k^* was determined when the kink was far from the breather after the collision. To define the position of the kink in the discrete system, a linear interpolation between the nodes of the grid was employed.

In Fig. 1, d_k^* is presented as a function of dimensionless initial distance between the kink and the breather $\xi = x_b/\lambda$ at $\omega = 0.1$, $d_b = -0.2$, $h = 0.1$. From the periodicity of the breather motion we only need to consider $1/2 \leq \xi \leq 3/2$. Different ξ means a different phase of the kink-breather collision. One can see that the inelasticity of collision strongly depends on the phase of collision. In the vicinity of $\xi = 1$ the inelasticity of collision drastically increases. The maximum value of $|d_k^*|$ is equal to 0.061 but $|d_k^*|$ does not exceed 10^{-4} out of the range $0.9 < \xi < 1.1$. The left inset

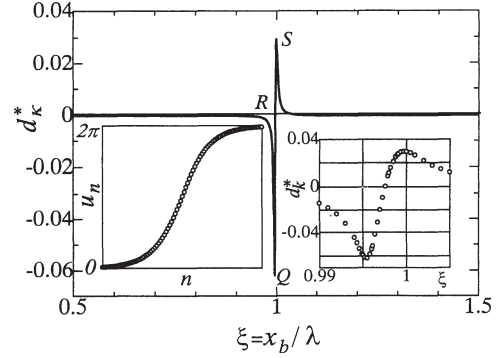


Fig. 1. Velocity of the kink after the collision d_k^* as a function of dimensionless initial distance ξ between the kink and the breather at $h = 0.1$, $\omega = 0.1$, $d_b = -0.2$, $d_k = 0$, $x_k = 0$. The left inset shows the discrete static kink at $h = 0.1$. The right inset shows the numerical data in the vicinity of $\xi = 1$.

shows the discrete static kink at $h = 0.1$. The right inset shows the numerical data in the vicinity of $\xi = 1$. In the example being considered the condition (16) is nearly fulfilled and at $\xi = 1$ the condition (14) is satisfied exactly.

At $h \leq 0.1$ any noticeable radiation emitted during the inelastic collision was not observed. The increase of the kink energy was practically equal to the decrease of the breather energy.

Referring to Fig. 1 one can define the width of the range of inelastic collision as $W = \xi_S - \xi_Q$, the center of the range as ξ_R and the greatest possible inelasticity of collision at given h , ω , d_b as $I = d_k^*(\xi_S) - d_k^*(\xi_Q)$, where ξ_Q , ξ_R , ξ_S are abscissas of the minimum, zero and maximum values of the curve $d_k^*(\xi)$, respectively.

In Fig. 2 the influence of parameters ω , d_b , h on the W , ξ_R , I is presented. In Fig. 2a the ω -dependences of W and I are shown at $h = 0.1$, $d_b = -0.2$. The d_b -dependences of W and I are shown in Fig. 2b at $h = 0.1$, $\omega = 0.2$. Finally, the h -dependences of W and I are shown in Fig. 2c and the h -dependence of ξ_R is shown in Fig. 2d at $d_b = -0.2$, $\omega = 0.15$.

It can be seen from Fig. 2a–c that the greatest possible inelasticity of collision I increases with decreasing ω and with decreasing $|d_b|$. Magnitude of I decreases with the decrease in h . Increase of I with decreasing $|d_b|$ results from the approaching to the separatrix (see Eq. (16)). Decrease of I with decreasing h is caused by the decreasing of the perturbation amplitude resulting in a narrowing of the chaotic layer. The width of the inelastic interaction region W increases

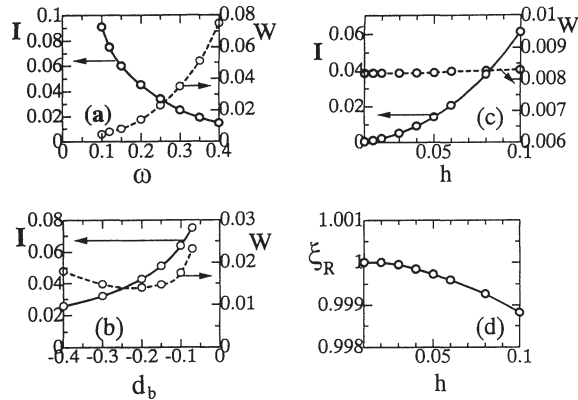


Fig. 2. (a) The ω -dependences of the width of inelastic collision range W and the greatest possible inelasticity of collision I at $h = 0.1$, $d_b = -0.2$. (b) d_b -dependences of W and I at $h = 0.1$, $\omega = 0.2$; (c) h -dependences of W and I at $d_b = -0.2$, $\omega = 0.15$; and (d) h -dependence of the center of inelastic collision range ξ_R at $d_b = -0.2$, $\omega = 0.15$. Numerical data are shown by the circles. Curves are guides to the eye.

with ω , does not depend on h and is a nonmonotone function of d_b .

As Fig. 2d suggests, at small h the center of the inelastic interaction region ξ_R approaches 1, which is in agreement with Eq. (14).

Curves in Fig. 2a terminate at $\omega \approx 0.1$ and in Fig. 2b at $d_b \approx -0.07$. In the regions where the curves are undefined the picture of collision changes qualitatively, namely, the reflection of kink and breather or the breakup of the breather into a kink–antikink pair takes place. Both these types of inelastic kink–breather collisions have been observed for the SG equation with other perturbations [20].

The resonant kink–antikink reflection in the modified SG model [18] occurs through the two bounce interaction, therefore the change from the regime of passing through each other to the regime of reflection causes a step-wise increase in the time of collision. In our numerical experiments, as the regime of interaction changes, the time of kink–breather collision changes monotonically. This is one of the reasons why we consider that the effect being discussed here has another nature than the resonant interaction effect [18].

5. Conclusion

A new effect of discreteness on a SG three-soliton solution which can be prominent even for a small degree of discreteness was studied.

The physical meaning of the effect is that if one considers the parameters of quasiparticles as random variables, then the probability of strong inelasticity of their collision can be introduced. This probability decreases with decreasing h but differs from zero even for very small h .

In the present Letter the discrete SG system was treated but the effect under consideration has been observed in another nearly integrable chain [21], therefore it appears as a common effect of discreteness on many-soliton solutions.

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