- susceptibility for arbitrary op M

$$\langle \tilde{M}(t)\rangle - \langle M(0)\rangle = \epsilon * \lambda(t) \quad , \quad \lambda(t) = \lambda_0 e^{-i\omega t}$$

$$\epsilon = \lim_{y\to 0^+} i\int_0^\infty dt\, e^{i\omega t - yt} \langle [\tilde{M}(t), \tilde{M}(0)]\rangle$$

$$= \lim_{y\to 0^+} i\int_0^\infty dt\, e^{i\omega t - yt} \sum_n \overbrace{\rho_n}^{\sim e^{-\beta H_0}} \Big(\underbrace{\langle n| e^{iH_0 t} M e^{-itH_0} M |n\rangle}{} - \text{h.c.}\Big)$$

$$= \sum_m \langle n| e^{itH_0} M e^{-itH_0} |m\rangle\langle m| M |n\rangle$$

$$= \sum_m e^{i(E_n - E_m)t} |\langle n| M |m\rangle|^2$$

$$= \lim_{y\to 0^+} i\int_0^\infty dt\, e^{-i\omega t - yt} \sum_{n,m} \rho_n \left(e^{-i(E_m - E_n)t} - e^{+i(E_m - E_n)t}\right) |M_{nm}|^2$$

$$= \lim_{y\to 0^+} i \sum_n \rho_n \sum_{m\neq n} |M_{nm}|^2 \left(\frac{1}{y - i(E_n - E_m + \omega)} - \frac{1}{y - i(E_m - E_n + \omega)}\right)$$

decompose: $\epsilon = \epsilon' + i\epsilon''$

$$\epsilon'' = \mathrm{Im}(\epsilon) = \sum_n \rho_n \sum_{m\neq n} |M_{nm}|^2 *$$

$$* \lim_{y\to 0^+} \left(\frac{y}{y^2 + (E_n - E_m + \omega)^2} - \frac{y}{y^2 + (E_m - E_n + \omega)^2}\right)$$

<u>recall</u>: $\lim_{y\to 0^+} \frac{y}{y^2 + x^2} = \pi\delta(x)$

$$\epsilon'' = \pi \sum_n \rho_n \sum_{m\neq n} |M_{nm}|^2 \Big(\delta(E_n - E_m + \omega) - \delta(E_m - E_n + \omega)\Big)$$

$$= \frac{1}{2}\Big(S(\omega) - S(-\omega)\Big) \quad (*)$$

def: spectral fn. of op. M

$$S(\omega) := 2\pi \sum_n f_n \sum_{m\neq n} |M_{nm}|^2 \, \delta(E_n - E_m + \omega)$$

fluctuation - dissipation relation:

in thermal equilibrium:

$$S(-\omega) = 2\pi \sum_{m\neq n} \frac{e^{-\beta E_n}}{Z} |M_{nm}|^2 \, \delta(E_m - E_n + \omega)$$

$m \leftrightarrow n$, dummy

$$= 2\pi \sum_{m\neq n} \frac{e^{-\beta E_m}}{Z} \underbrace{|M_{mn}|^2}_{=|M_{nm}|^2} \delta(E_n - E_m + \omega)$$

$$= 2\pi \sum_{m\neq n} \frac{e^{-\beta E_n}}{Z} |M_{nm}|^2 \, \underline{\delta(E_n - E_m + \omega)} \, e^{-\beta \overbrace{(E_m - E_n)}^{=\omega}}$$

$$= e^{-\beta\omega} S(+\omega)$$

$\overset{(*)}{\Rightarrow}$ $$\epsilon''(\omega) = \frac{1}{2} S(\omega)\left(1 - e^{-\beta\omega}\right)$$

$\Rightarrow$ $$\boxed{S(\omega) = \frac{2\,\epsilon''(\omega)}{1 - e^{-\beta\omega}}}$$

fluctuation - dissipation relation (FDR)

susceptibility: absorption from drive of $\langle M \rangle$

spectral fn: Fourier transform of fluctions
→ thermal
→ quantum

FDR: relates spectral fn to imaginary part of susceptibility

<u>back to Q</u> : how do we measure quantum geometry?

$$H = H_0 + \lambda V \,, \quad V = \partial_\lambda H \quad ; \quad \partial_\alpha = \partial/\partial\lambda_\alpha$$

- geometric tensor $\chi_{\alpha\beta}$ is related to Kubo suscept. of $\partial_\alpha H$:

$$\chi_{\alpha\beta} = \sum_{n\neq 0} \langle 0|A_\alpha|n\rangle\langle n|A_\beta|0\rangle = \sum_{n\neq 0} \frac{\langle 0|\partial_\alpha H|n\rangle\langle n|\partial_\beta H|0\rangle}{(E_n-E_0)^2}$$

1) $$\frac{1}{(E_n-E_0)^2} = \int_{-\infty}^{\infty} d\omega \, \frac{1}{\omega^2}\, \delta(E_n-E_0-\omega)$$

$$= \lim_{\gamma\to 0^+} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-i(E_n-E_0-\omega)t}\, e^{-\gamma|t|}$$

2) $$\langle 0|e^{iE_0 t}\partial_\alpha H e^{-iE_n t}|n\rangle = \langle 0|\widetilde{\partial_\alpha H}(t)|n\rangle$$ int. picture repr. of $\partial_\alpha H$

$$\chi_{\alpha\beta} = \int \frac{d\omega}{\omega^2} \lim_{\gamma\to 0^+} \int \frac{dt}{2\pi} e^{-\gamma|t|} \sum_{n\neq 0} \langle 0|\widetilde{\partial_\alpha H}(t)|n\rangle\langle n|\widetilde{\partial_\beta H}(0)|0\rangle$$

$$= \int \frac{d\omega}{\omega^2} \lim_{\gamma\to 0^+} \int \frac{dt}{2\pi} e^{-\gamma|t|} \times$$

$$\times \underbrace{\sum_{n=0}^{\infty} \langle 0|\widetilde{\partial_\alpha H}(t)|n\rangle\langle n|\widetilde{\partial_\beta H}(0)|0\rangle - \langle 0|\widetilde{\partial_\alpha H}|0\rangle\langle 0|\widetilde{\partial_\beta H}|0\rangle}_{= \langle 0|\widetilde{\partial_\alpha H}(t)\widetilde{\partial_\beta H}(0)|0\rangle_c}$$

$$= \frac{1}{2\pi} \int \frac{d\omega}{\omega^2} \underbrace{\lim_{\gamma\to 0^+} \int dt\, e^{-\gamma|t|} e^{i\omega t} \langle 0|\widetilde{\partial_\alpha H}(t)\widetilde{\partial_\beta H}(0)|0\rangle_c}_{=: S_{\alpha\beta}(\omega) \text{ spectral fn of } \partial_\alpha H}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{S_{\alpha\beta}(\omega)}{\omega^2} = \int_{0}^{\infty} \frac{d\omega}{2\pi} \frac{S_{\alpha\beta}(\omega) + S_{\alpha\beta}(-\omega)}{\omega^2}$$

from FDR at $\beta \to \infty$ (0-temperature)
thermal expect. $\to$ GS expect. value

$$\beta \to \infty \Rightarrow S_{\alpha\beta}(\omega) \overset{FDR}{=} \begin{cases} 2\,\epsilon''_{\alpha\beta}(\omega) & , \ \omega > 0 \\ 0 & , \ \omega < 0 \end{cases}$$

$$\Rightarrow \chi_{\alpha\beta}(\omega) = \int_0^\infty \frac{d\omega}{2\pi} \frac{S_{\alpha\beta}(\omega)}{\omega^2} \quad \text{geom. tensor}$$

Fubini - Study metric:

$$g_{\alpha\beta} = \int_0^\infty \frac{d\omega}{2\pi} \frac{\epsilon''_{\alpha\beta}(\omega) + \epsilon''_{\beta\alpha}(\omega)}{\omega^2}$$

can be measured directly from the symmetric part of susceptibility $\epsilon''(\omega)$ which defines the relation b/w fluctuations & energy absorption ($M = \partial_\alpha H$)

<u>at finite temperature</u>:

$$\chi_{\alpha\beta} = \int_0^\infty \frac{d\omega}{2\pi} \frac{S_{\alpha\beta}(\omega) + e^{-\beta\omega} S_{\alpha\beta}(\omega)}{\omega^2} = \int_0^\infty \frac{d\omega}{2\pi} \frac{1+e^{-\beta\omega}}{\omega^2} S_{\alpha\beta}(\omega)$$

$$\overset{FDR}{=} \int_0^\infty \frac{d\omega}{2\pi} \frac{1+e^{-\beta\omega}}{\omega^2} \frac{2\,\epsilon''(\omega)}{1-e^{-\beta\omega}} = \int_0^\infty \frac{d\omega}{\pi} \frac{1}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) \epsilon''_{\alpha\beta}(\omega)$$

$$\Rightarrow g_{\alpha\beta} = \frac{1}{2}\left(\chi_{\alpha\beta} + \chi_{\beta\alpha}\right) = \int_0^\infty \frac{d\omega}{2\pi} \frac{1}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right)\left(\epsilon''_{\alpha\beta}(\omega) + \epsilon''_{\beta\alpha}(\omega)\right)$$

measureable in experiments

- <u>metric tensor</u> important in quantum info. theory known as Fisher information or fidelity susceptibility
  - useful to distinguish quantum states
  - useful for parameter estimation: metric tensor sets a fundamental bound on the ability to determine unknown parameters of a system $\to$ <u>Cramér-Rao bound</u>

- can we also measure the Berry curvature $F_{\alpha\beta}$?

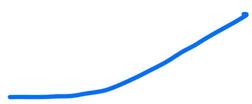
<u>recall</u>: in co-moving frame

$$\tilde{H} = \underbrace{U^+ H U}_{\text{diag.}} - \underbrace{\dot{\lambda}_\alpha \underbrace{\tilde{A}_\alpha}_{\substack{\text{creates all}\\ \text{excitations}}}}_{\dot{\lambda}\to 0,\text{ adiabatic limit} \to 0}$$

$\Rightarrow$ gauge pot. A_α defines both the non-adiabatic response & quantum geometry

- consider a system in GS $|0\rangle$ of $H(\lambda_0)$,
  $\lambda_0 = \lambda(t=0)$

  <u>assume</u>: $\dot{\lambda}(t)$ is a smooth fn of time & $\dot{\lambda}(0) = 0$

  $\lambda(t) =$ 

  $\Rightarrow$ to leading order in $|\dot{\lambda}|$, system follows the GS of $\tilde{H}$ adiabatically

$\rightarrow$ apply 1st order pert. theory in $\dot{\lambda}$ in co-moving frame

$\Rightarrow$ transition amplitude to excited state $|n\rangle$:

$$a_n \approx \dot{\lambda}_\alpha \frac{\langle n | A_\alpha | 0\rangle}{E_n - E_0} \qquad (*)$$

- describe observables as a generalized force operators conjugate to some coupling λ_β:

$$\mathcal{M}_\beta := -\partial_\beta H$$

$\rightarrow$ matrix elements of $\mathcal{M}_\beta$ appear in geom. tensor

<u>def</u>: generalized force: $M_\beta := \langle 0 | \mathcal{M}_\beta | 0 \rangle$

example:

- generalized force:		observable:
magnetization	$\leftrightarrow$	magnetic field
el. current	$\leftrightarrow$	vector potential
$\vdots$		$\vdots$

- any observable $\mathcal{O}$ can be represented as a generalized force operator by adding a source term $-\lambda\mathcal{O}$ to Hamiltonian

- leading order non-adiabatic response

$$M_\beta = \langle \mathcal{M}_\beta \rangle = \underbrace{\langle 0 | \mathcal{M}_\beta | 0 \rangle}_{= M_\beta^{(0)}} - \sum_{n\neq 0} a_n^* \langle n | \partial_\beta H | 0 \rangle + \text{h.c.}$$

gen. force in inst. GS

$$\overset{(*)}{\approx} M_\beta^{(0)} + i\, \dot\lambda_\alpha \sum_{n\neq 0} \frac{\langle 0 | \partial_\alpha H | n \rangle \langle n | \partial_\beta H | 0 \rangle}{(E_n - E_0)^2} - (\alpha \leftrightarrow \beta)$$

$$= M_\beta^{(0)} + i\, F_{\beta\alpha} \dot\lambda_\alpha$$

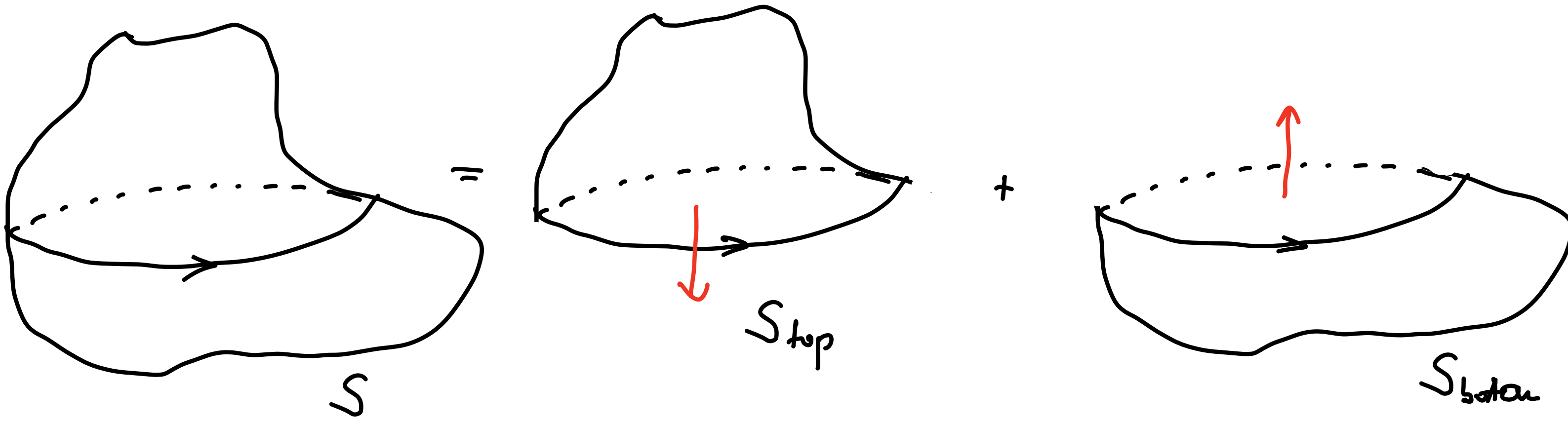
=> leading-order non-adiabatic correction to generalized force comes from product of Berry curvature & rate of change of $\dot{\vec\lambda}$

analogy:

	EM		QM
i)	vector pot. $\vec A(\vec r)$	$\leftrightarrow$	Berry connection $\vec A(\vec\lambda) = \langle 0(\vec\lambda) | i \vec\nabla_\lambda | 0(\vec\lambda)\rangle$
ii)	AB phase $\varphi_{AB} = \oint \vec A \cdot d\vec r$	$\leftrightarrow$	Berry phase $\gamma = \oint \vec A(\vec\lambda) \cdot d\vec\lambda$
iii)	magnetic field / EM field $F_{ab} = \partial_a A_b - \partial_b A_a = \varepsilon_{abc} B_c$	$\leftrightarrow$	Berry curvature $F_{\mu\nu}(\vec\lambda) = \partial_\mu A_\nu - \partial_\nu A_\mu$

- since Berry curvature behaves like a magn. field in parameter space, it follows that Kubo correction (i.e. linear non-adiabatic response) is the corresponding Lorentz force (or Coriolis force)

→ on a closed GS parameter manifold, this "Lorentz" force leads to a quantized response:



arbitrary closed parameter manifold

· consider Berry phases / Berry fluxes

$$\gamma_{top} = \int_{S_{top}} F_{\alpha\beta}\, d\lambda_\alpha \wedge d\lambda_\beta \quad ; \quad \gamma_{bottom} = -\int_{S_{bottom}} F_{\alpha\beta}\, d\lambda_\alpha \wedge d\lambda_\beta$$

opposite orientation

recall: γ is the physical phase acquired by wave fn. during adiabatic motion in parameter space

since wave fn is unique: $\gamma_{top} = \gamma_{bottom} \mod 2\pi$

$$\Rightarrow \gamma_{top} - \gamma_{bottom} = 2\pi n = \oint_S F_{\alpha\beta}\, d\lambda_\alpha \wedge d\lambda_\beta$$

$$C_1 := \frac{1}{2\pi} \oint_S F_{\alpha\beta}\, d\lambda_\alpha \wedge d\lambda_\beta \in \mathbb{N} \text{ integer}$$

1st chern number of closed surface S

→ example of topological invariant

- intuitive understanding using analogy to EM

EM: flux thru closed surface $\propto$ charge inside

Gauss' law of magnetism:

$C_1 \propto q_m$ effective "magnetic" charge

topological charge

Dirac: if magnetic monopoles exist, then magnetic charge must be quantized

$\Leftrightarrow$ quantization of Chern number

- in parameter space: isolated degenaries can lead to source of Berry curvature & give rise to $C_1 \neq 0$ as topo inv.

<u>Example</u>: 2LS $\quad \vec{\lambda} = (\theta, \varphi)$

$$H = \vec{h}(\theta,\varphi)\cdot\vec{\sigma} \quad , \quad \vec{h} = h\begin{pmatrix}\cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ \sin\theta\end{pmatrix}$$

$$\Rightarrow |0\rangle = \begin{pmatrix}\cos\theta/2 \\ e^{i\varphi}\sin\theta/2\end{pmatrix}$$

$$\Rightarrow F_{\varphi\theta} = \frac{1}{2}\sin\theta \qquad \text{Berry curvature}$$

Chern number:

$$C_1 = \frac{1}{2\pi}\oint_{S^2} F_{\alpha\beta}\, d\lambda_\alpha \wedge d\lambda_\beta = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, F_{\varphi\theta}$$

$$= \frac{1}{2}\int_0^{\pi} d\theta \sin\theta = 1$$

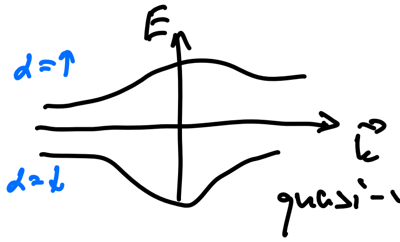
$\Rightarrow$ GS wave fn (Bloch vector) wraps exactly <u>once</u> if we change $\vec{h}$ over the entire solid angle

$\rightarrow$ Chern is sourced by isolated degeneracy in spectrum of H at $h=0$ (origin of Bloch sphere)

HW: for arbitrary spin-S : $c_1 = 2S$

Example 2: topological insulators

take 2D free fermion system w/ 2 energy bands



$|u_\alpha(\vec{k})\rangle$: Bloch waveft.

parameter space : Brillouin zone

$\vec{\lambda} = (k_x, k_y) = \vec{k}$

Berry connection: $A^\alpha_{k_j} = i \langle u_\alpha(\vec{k}) | \partial_{k_j} u_\alpha(\vec{k}) \rangle$

-ll- curvature: $F^\alpha_{k_x k_y} = \partial_{k_x} A^\alpha_{k_y} - \partial_{k_y} A^\alpha_{k_x}$

only non-vanishing component

Chern number: $C_1^\alpha = \oint_{BZ} dk_x \, dk_y \; F^\alpha_{k_x k_y}$

HW: show $\sum_\alpha C_1^\alpha = 0$