Checking the algebra of the phase-space point operators. CAH, 16.i.24.

```
Clear["Global`*"];
```

Set up the definitions.

In[∘]:=
```
ω = Exp[2 I π / 3];
X = {{0, 0, 1}, {1, 0, 0}, {0, 1, 0}};
Z = {{1, 0, 0}, {0, ω, 0}, {0, 0, ω^2}};
T[a_, ap_] := ω^(-2 a ap) MatrixPower[Z, a].MatrixPower[X, ap];
```

Check the various identities that these should obey.

In[∘]:=
```
ω^3
MatrixPower[X, 3] // MatrixForm
MatrixPower[Z, 3] // MatrixForm
ConjugateTranspose[X].X // MatrixForm
ConjugateTranspose[Z].Z // MatrixForm
```

Out[∘]= 1

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Look at some examples of Pauli string operators for a single site.

In[∘]:=
```
Table[T[i, j] // MatrixForm, {i, 0, 2}, {j, 0, 2}]
```

Out[∘]=

$$\left\{\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2 i \pi}{3}} & 0 \\ 0 & 0 & e^{-\frac{2 i \pi}{3}} \end{pmatrix}, \begin{pmatrix} 0 & 0 & e^{\frac{2 i \pi}{3}} \\ e^{-\frac{2 i \pi}{3}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{-\frac{2 i \pi}{3}} & 0 \\ 0 & 0 & 1 \\ e^{\frac{2 i \pi}{3}} & 0 & 0 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\frac{2 i \pi}{3}} & 0 \\ 0 & 0 & e^{\frac{2 i \pi}{3}} \end{pmatrix}, \begin{pmatrix} 0 & 0 & e^{-\frac{2 i \pi}{3}} \\ e^{\frac{2 i \pi}{3}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{\frac{2 i \pi}{3}} & 0 \\ 0 & 0 & 1 \\ e^{-\frac{2 i \pi}{3}} & 0 & 0 \end{pmatrix}\right\}\right\}$$

Define the sum of Pauli strings and check its form.

In[∘]:=
```
S = Simplify[Sum[T[i, j], {i, 0, 2}, {j, 0, 2}]];
```

In[∘]:=
```
S // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$

Define the phase space point operators and check their form.

In[∘]:=
```
A[b_, bp_] := Simplify[(1/3) T[b, bp].S.ConjugateTranspose[T[b, bp]]];
```

In[∘]:=
```
Table[A[i, j] // MatrixForm, {i, 0, 2}, {j, 0, 2}]
```

Out[∘]=

$$\left\{\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -(-1)^{1/3} \\ 0 & (-1)^{2/3} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & (-1)^{2/3} \\ 0 & 1 & 0 \\ -(-1)^{1/3} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -(-1)^{1/3} & 0 \\ (-1)^{2/3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & (-1)^{2/3} \\ 0 & -(-1)^{1/3} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -(-1)^{1/3} \\ 0 & 1 & 0 \\ (-1)^{2/3} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & (-1)^{2/3} & 0 \\ -(-1)^{1/3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\right\}$$

Check the trace identity.

In[∘]:=
```
Table[FullSimplify[Tr[A[IntegerDigits[i, 3, 2][[1]], IntegerDigits[i, 3, 2][[2]]].A[
    IntegerDigits[j, 3, 2][[1]], IntegerDigits[j, 3, 2][[2]]]]], {i, 0, 8}, {j, 0, 8}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$