3. Magic and mana.

"Magic": the "non-Cliffordness" of a quantum state, i.e. how many non-Clifford gates its preparation circuit would need.

Magic $\neq$ entanglement, since Clifford circuits <u>can</u> generate entanglement.

Theorem: universal quantum computation can be achieved with only Clifford gates plus <u>one</u> non-Clifford ("magic") gate, e.g. $T: |0\rangle \to |0\rangle, |1\rangle \to e^{i\pi/4}|1\rangle$.

Generalise to qutrits — structure is actually simpler in this case:

$$Z = |0\rangle\langle 0| + \omega|1\rangle\langle 1| + \omega^2|2\rangle\langle 2| \qquad \omega = e^{2i\pi/3}$$

$$X = |1\rangle\langle 0| + |2\rangle\langle 1| + |0\rangle\langle 2|.$$

$$T_{ab} \equiv \omega^{-2ab} Z^a X^b$$

'Pauli' strings:

$$T_{\underline{a}} = T_{a_1 b_1} T_{a_2 b_2} \cdots T_{a_N b_N}. \quad \text{(outer product: } N \text{ qutrits)}.$$

Clifford group: $C = \{ U : U T_{\underline{a}} U^\dagger = e^{i\phi} T_{\underline{b}} \}$, i.e. Clifford-group unitaries map Pauli strings to themselves^{J} (up to a phase). Generators of $\mathcal{C}$:

$$K = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \omega \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \qquad S = \sum_{ij} |i;i\oplus j\rangle\langle ij|$$

phase gate   qutrit Hadamard gate   sum gate

Explicitly,

$$\begin{aligned} S = {} & |00\rangle\langle 00| + |01\rangle\langle 01| + |02\rangle\langle 02| \\ & + |11\rangle\langle 10| + |12\rangle\langle 11| + |10\rangle\langle 12| \\ & + |22\rangle\langle 20| + |20\rangle\langle 21| + |21\rangle\langle 22|, \end{aligned}$$

the qutrit analogue of the CNOT gate.

$$\mathcal{C}|0000\ldots 0\rangle = \underbrace{\{\text{eigenstates of } T_{a_1b_1} T_{a_2b_2} \cdots T_{a_Nb_N}\}}_{\text{stabiliser states.}}$$

Adding randomness (including feed-forward measurement) allows mixtures too; then our accessible part of the space of density matrices is:

$$\left\{\sum_j p_j |s_j\rangle\langle s_j| \,:\, |s_j\rangle \in \mathcal{C}|00\ldots 0\rangle,\ p_j \geq 0,\ \sum_j p_j = 1\right\} \equiv \mathrm{STAB}\left[(\mathbb{C}^3)^{\otimes N}\right],$$

the convex hull of stabiliser states.

This is not the full space, so $\{K, H, S\}$ are not universal.^{K}

To get a universal set, we could add, e.g.,

$$T = \begin{pmatrix} \xi^{-1} & & \\ & 1 & \\ & & \xi \end{pmatrix} \qquad \xi = e^{2i\pi/9} \qquad \text{"T-gate".}$$

$\{C, T\}|00\ldots0\rangle \simeq C|\psi\rangle$, where $|\psi\rangle$ is "magic", i.e. in some sense has already had the hard stuff done.

How do we measure this property? Via a "magic monotone", which doesn't increase under Clifford gates or projective measurement of Pauli operators. One such is called "mana":

a) Define 'phase space point operators':

$$A_{\underline{b}} \equiv 3^{-N} T_{\underline{b}} \left[\sum_{\underline{a}} T_{\underline{a}} \right] T_{\underline{b}}^{\dagger}.$$

These obey:

$$\mathrm{Tr}\left(A_{\underline{b}} A_{\underline{c}}\right) = 3^N \delta_{\underline{b}\underline{c}} \quad \text{(orthogonality)}$$

$$3^{-N} \sum_{\underline{b}} A_{\underline{b}} = \mathbb{I} \quad \text{(completeness).}$$

b) Expand the density matrix of interest in this basis:

$$\rho = \sum_{\underline{a}} \underbrace{W_\rho(\underline{a})}_{\text{"discrete Wigner function"}} A_{\underline{a}}.$$

The coefficients $W_\rho(\underline{a})$ are all real; also,

$$\sum_{\underline{a}} W_\rho(\underline{a}) = \mathrm{Tr}\,\rho = 1,$$

and

$$\mathrm{Tr}(\rho_1\rho_2) = 3^N \sum_{\underline{a}} W_{\rho_1}(\underline{a}) W_{\rho_2}(\underline{a}).$$

c) Define the mana of ρ, $M(\rho)$, as

$$\boxed{M(\rho) \equiv \log \sum_{\underline{a}} |W_\rho(\underline{a})|}.$$

If $\rho = |\psi\rangle\langle\psi|$, with $|\psi\rangle$ a stabiliser state, it is a theorem that $W_\rho(\underline{a}) \geq 0$; then $\sum_{\underline{a}} |W_\rho(\underline{a})| = \sum_{a} W_\rho(\underline{a}) = 1$, and the mana is zero. So $M(\rho)$ measures the "non-stabiliserness" of the state, and thus the "non-Cliffordness" of the circuit needed to produce it.

Cautionary note: $M(\rho)$ is defined only when the on-site Hilbert space dimension is odd — so it can be used for qutrits, but not qubits.

Properties of mana:

$$M(\rho_A \otimes \rho_B) = M(\rho_A) + M(\rho_B) \qquad \text{(additivity)}$$

$$M(\rho) \leq \tfrac{1}{2}(N\log 3 - S_2) \qquad S_2 : 2^{\text{nd}} \text{ Rényi entropy}$$

Interpretation: mana is a lower bound on the number of non-Clifford rotations needed to prepare the state.

Proof: Apply non-Clifford rotations via magic state injection, i.e. take an input state plus n 'magic state' ancillas $|a\rangle$ and apply a protocol of Clifford gates and Pauli measurements — this yields the desired non-Clifford rotation with probability p. So for Q non-Clifford rotations, we need to start with

$$|00\ldots0\rangle \otimes |a\rangle^{\otimes(nQp^{-Q})} \simeq |00\ldots0\rangle \otimes |a\rangle^{\otimes(nQ)}.$$

$\because$ need to retry if it fails; can eliminate by taking n large enough.

But, by additivity,

$$M\left(|00\ldots0\rangle \otimes |a\rangle^{\otimes(nQ)}\right) = nQ\,M(|a\rangle).$$

So to prepare a state ρ with mana $M(\rho)$, we need

$$nQ\,M(|a\rangle) \geq M(\rho) \Rightarrow Q \geq \frac{M(\rho)}{nM(|a\rangle)}.$$

number of non-Clifford gates included in the circuit.

(N.B. — I think $\geq$ should be $\leq$ in (21) of ref. [2].)

4. Case study: $\mathbb{Z}_3$ Potts model

$$H = -\sin\theta \sum_j \left(Z_j^\dagger Z_{j+1} + \text{H.c.} \right) - \cos\theta \sum_j \left(X_j^\dagger + X_j \right).$$

(Analogue of transverse-field Ising model.) Duality $\theta \to \pi/2 - \theta$; continuous transition at $\theta_c = \pi/4$, with order parameter

$$m = \langle Z + Z^\dagger \rangle.$$

Critical point described by CFT with central charge $4/5$.

How does the ground-state mana of this model depend on θ?

i) $\theta = \pi/2$: classical Potts model, $H = -\sum_j (Z_j^\dagger Z_{j+1} + \text{H.c.})$

$$= -2\sum_j \left(|00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22| \right)_{j,j+1}.$$

Ground states: $|00\ldots0\rangle$, $|11\ldots1\rangle$, $|22\ldots2\rangle$: mana zero.

ii) $\theta = 0$: independent qutrits in a field, $H_j = -X_j^\dagger - X_j$

$$= -\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \text{ ground state } \underbrace{\frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle + |2\rangle\right)}_{\text{mana zero.}}$$

Product state, so mana is again zero.

What about in between?

Ref. [1] addresses this using DMRG. (Not so clear that this will be reliable near the critical point, but see ref. [1] for arguments that it's okay.) For results, see their Fig. 1. [SLIDE].

Is this mana removable by local unitary operations? (Less interesting if so.) Try a 'two-point' version:

$$m_{cc}(A,B) \equiv m(\rho_{A\cup B}) - \frac{1}{2}\left[m(\rho_A) + m(\rho_B)\right].$$

(N.B. m is mana density, not total mana — hence the factor of ½.) For results, see their Fig. 2. [SLIDE].

Interpretation: toy model of $\rho_{A\cup B}$. For $\theta = \frac{\pi}{2}$ (pure FM),

$$|\Omega_0\rangle = \sum_n X_i^n X_j^n |00\rangle. \qquad \text{(cat state)}$$

Reducing θ causes hybridisation; write this as

$$\rho_{toy} = \frac{1}{3}(1-\alpha_{ij})\,\rho_i \otimes \rho_j + \alpha_{ij}|N_2\rangle\langle N_2|,$$

where

$$|N_2\rangle \propto \sum_n X^n\left(2|00\rangle - |11\rangle - |22\rangle\right).$$

We can infer the form of α_{ij} from the connected correlation function,

$$\langle Z_i^\dagger Z_j + \text{H.c.}\rangle - 2\,\mathrm{Re}\left[\langle Z_i^\dagger\rangle\langle Z_j\rangle\right] \propto \alpha_{ij},$$

so using known Potts-model results,

$$\alpha_{ij} \propto \begin{cases} b + c\,e^{-|i-j|/\xi} & \theta > \pi/4 \quad \text{(FM)} \\ |i-j|^{-2\Delta} & \theta = \pi/4 \quad \text{(critical)} \\ e^{-|i-j|/\xi} & \theta < \pi/4 \quad \text{(paramagnet)}, \end{cases}$$

with $2\Delta = 4/15$. Mana measures how far $\rho_{A\cup B}$ is from $\rho_A \otimes \rho_B$, so is proportional to α_{ij} at criticality,

$$m_{cc}(A) = C_1 |i-j|^{-2\Delta} - \varepsilon.$$

[offset because $m(\rho_A) = m(\rho_B)$ assumed, which isn't quite true]

This fits the lower panel in Fig. 2 well.

Extra complications for upper panel, where m_{cc} hits zero at a finite separation: this is because $\rho_{A\cup B}$ enters the space of operators with a positive Wigner representation before it reaches STAB — see paper for a fuller discussion.