

Michael I. Tribelsky*¹, and Sergei I. Anisimov†²

¹*Department of Applied Physics, Faculty of Engineering, Fukui University,
Bunkyo 3-9-1, Fukui 910-8507, Japan*

*and Max-Planck-Institut für Physik komplexer Systeme,
Nöthnitzer Straße 38, D-01187 Dresden, Germany*

²*Landau Institute for Theoretical Physics RAN, Kosygin St., Moscow 117334, Russia*

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We employ the fact that for most media the region where the adiabatic second derivative of volume over pressure is anomalously small and may become negative is narrow to develop the phenomenological theory of shock waves of rarefaction analogous to that for weak shock waves of compression. The corresponding generalized Burgers equation is derived and analyzed. An exact solution of the equation is obtained and discusses.

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It is well known that the sign of variation of pressure and density in a weak shock wave is associated with the sign of $(\partial^2 V / \partial p^2)_s$. Usually the sign of the derivative is positive, and accordingly the shock wave is a wave of compression. However, in some specific regions, e.g., close to a critical point, $(\partial^2 V / \partial p^2)_s$ may become negative. It creates prerequisites for a shock wave of rarefaction (SWR) to come into being. Despite the fact that the above passage may be found in any shock wave text-book, see, for example [1], much to our surprise, nobody has payed attention that narrowness of the region where SWR may exist implies that *such a shock wave is always weak*, which makes the problem of description of the structure of this wave tractable within the framework of the macroscopic hydrodynamics.

In the present Letter the theory of weak shock waves is applied to the problem. As a result the generalized Burgers equation, which describes a hydrodynamic flow in the region, where the derivative $(\partial^2 V / \partial p^2)_s$ may change sign is derived. The equation is valid for both signs of the derivative and therefore in addition to SWR describes non-steady rarefaction waves linked to the shock in the case when the pressure variation imposed by the initial conditions is bigger than the region of existence of SWR. Some particular steady solutions of the equation describing different types of SWR are obtained and analysed. To be precise we must stipulate that we do not consider any singularities related to the very vicinity to the critical point. In other words it is implied the system is far enough from it to avoid any effect of the singularities.

To begin with let us consider the conventional Burgers equation, which governs a weak shock wave of compression. In the laboratory coordinate frame for a wave advancing in the negative direction of x -axis the equation reads [1]

$$\frac{\partial p'}{\partial t} - c_1 \frac{\partial p'}{\partial x} - \alpha p' \frac{\partial p'}{\partial x} = \alpha c_1^3 \frac{\partial^2 p'}{\partial x^2}, \quad (1)$$

where p' is the pressure variation, so that the pressure in the wave profile p equals to $p_1 + p'$ (here and in what fol-

lows subscript 1 indicates the value of the corresponding quantity in the unperturbed medium, i.e., at $x \rightarrow -\infty$, while subscript 2 will stand for the value of the same quantity behind the shock wave, at $x \rightarrow \infty$), $c(p')$ is the velocity of sound,

$$\alpha = \frac{c_1^3}{2V^2} \left(\frac{\partial^2 V}{\partial p^2} \right)_s,$$

V denotes the specific volume ($V \equiv 1/\rho$), a is a dissipative constant [1]

$$a \equiv \frac{1}{2\rho c_1^3} \left[\left(\frac{4}{3}\eta + \zeta \right) + \kappa \left(\frac{1}{c_v} - \frac{1}{c_p} \right) \right],$$

η and ζ stand for the coefficients of viscosity, κ for the thermal conductivity and $c_{v,p}$ for the corresponding specific heats.

Eq. (1) may be regarded as leading approximation to expansion of a certain more general nonlinear dissipative equation in powers of both weak nonlinearity and weak dissipation. It is important that according to the meaning of the leading approximation the nonlinear term in Eq. (1) is considered in non-dissipative limit — dissipative corrections to it have additional smallness and may be neglected.

In our case $|(\partial^2 V / \partial p^2)_s|$ is anomalously small (it even can vanish). For this reason the lowest nonlinearity $\alpha p'(\partial p' / \partial x)$ may not correspond to the leading nonlinear term any more. It means that higher order (in p') terms must be taken into account. On the other hand, the region, where $(\partial^2 V / \partial p^2)_s < 0$ is narrow, therefore the third derivative $(\partial^3 V / \partial p^3)_s$ also has a certain smallness in this region. Only the fourth derivative $(\partial^4 V / \partial p^4)_s$ does not have any smallness there. Thus, all terms of order of $O((p')^2)$, $O((p')^3)$ and $O((p')^4)$ may produce contributions of the same order and must be taken into account simultaneously. It is important that dissipative corrections to the nonlinear terms remain negligible and all such terms may be considered in the non-dissipative limit, in the same manner as it is for the conventional

Burgers equation. All the above mentioned is related exclusively to nonlinear terms. Regarding the dissipative term on the right hand side of Eq. (1), there are no specific reasons for it to be small and no correction to this term is required.

To obtain the desired nonlinear corrections to the left hand side of Eq. (1) let us consider an arbitrary adiabatic flow in the form of a traveling wave. Due to the adiabatic condition the profile of entropy of such a flow should not change and since the state of the medium ahead of the shock front is spatially homogeneous with $s = const$, we obtain that $s = const$ anywhere. Then, we may employ the general solution of the continuity and Euler equations valid for simple waves [1] and reduce these equations to the following single equation

$$\frac{\partial p'}{\partial t} + [v(p') \pm c(p')] \frac{\partial p'}{\partial x} = 0, \quad (2)$$

where the flow velocity v is given by the expression

$$v = \pm \int_0^{p'} \frac{dp}{c(p)\rho(p)}. \quad (3)$$

Sign plus in Eqs. (2)–(3) corresponds to the wave advancing in the positive x direction, minus – to negative. According to our choice of the direction of the wave propagation in what follows we take sign minus.

It is important to stress that the reduction of the set of hydrodynamic equations to Eqs. (2)–(3) is an *exact* result valid for *any* nonlinear dependence $c(p)$ and $\rho(p)$. However, if p' is small one can expand these functions about the point $p' = 0$. It is seen straightforwardly that if one truncates the expansion at zero term, it reduces Eqs. (2)–(3) to the linear acoustic equation. The truncation at the term of order $O(p')$ yields the left hand side of Eq. (1). Increasing the order of the truncation one can obtain corrections to the conventional Burgers equation with *any* given accuracy.

Thus, the desired generalization of Burgers equation has the form

$$\frac{\partial p'}{\partial t} + u(p') \frac{\partial p'}{\partial x} = c_1^3 a \frac{\partial^2 p'}{\partial x^2}, \quad (4)$$

where u stands for the velocity of a point in the wave profile. For our choice of the propagation direction $u(p') = v(p') - c(p')$ [2].

To get the equation in the explicit form one must expand $u(p')$ in powers of small p' retaining the terms up to $O((p')^3)$. However, before doing that it is worth analyzing some general features of Eq. (4).

First of all let us obtain the general expression for the velocity v_s of a steady traveling shock, when $p'(x, t) \rightarrow p'(x + v_s t) \equiv p'(\xi)$. After trivial change of variables $(x, t) \rightarrow \xi$, Eq. (4) may be easily integrated. Then, taking into account the boundary conditions ($p' \rightarrow 0$, $\partial p'/\partial \xi \rightarrow 0$ at $\xi \rightarrow -\infty$ and $p' \rightarrow p_2 - p_1$, $\partial p'/\partial \xi \rightarrow 0$ at $\xi \rightarrow \infty$), we arrive at the following expression for v_s

$$v_s = -\frac{1}{p_2 - p_1} \int_0^{p_2 - p_1} u(p) dp \equiv -\langle u \rangle, \quad (5)$$

where $\langle \dots \rangle$ denotes average over p' [we remind that $u = v - c < 0$ see also Eq. (3)].

Let us show now that v_s given by Eq. (5) satisfies the evolutionary conditions. The conditions say that in the co-moving coordinate frame the velocity of the flow in front of the shock should be bigger than c_1 , while the velocity behind the shock should be smaller than c_2 . In the laboratory coordinate frame the medium in front of the shock is in the rest state with zero velocity. Accordingly, in the co-moving coordinate frame the flow velocity in front of the shock is v_s . Thus, the first evolutionary condition says $v_s > c_1$. Behind the shock the flow velocity in the laboratory frame is v_2 . Consequently, the second condition yields $v_s + v_2 < c_2$. Bearing in mind Eq. (5) both the conditions may be written as follows

$$c_1 \equiv -u_1 < -\langle u \rangle < c_2 - v_2 \equiv -u_2 \quad (6)$$

Finally, taking into account that for a simple wave and the chosen direction of propagation the sign of derivative du/dp' is opposite to that for $(\partial^2 V/\partial p^2)_s$ [1], i.e., for negative $(\partial^2 V/\partial p^2)_s$ function $-u(p')$ monotonously increases with decrease of p' , and that for SWR p_2 is smaller than p_1 , we reduce inequality (6) to evident.

Let us proceed with the derivation of the generalized Burgers equation. From all the above-mentioned it is clear that in the discussed region the derivative $(\partial^2 V/\partial p^2)_s$ may be approximated as follows

$$(\partial^2 V/\partial p^2)_s \approx \frac{1}{\rho_1} \frac{1}{(\rho_1 c_1^2)^2} \left[-\epsilon^2 + \frac{\mu^6}{4} \left(\frac{p - p_m}{\rho_1 c_1^2} \right)^2 \right]. \quad (7)$$

Here p_m is the value of p , corresponding to the local minimum of the derivative, ϵ is a small dimensionless quantity $\mu = O(1)$ and power μ^6 as well as the numerical coefficient are introduced for convenience of further notations.

According to Eq (7), $(\partial^2 V/\partial p^2)_s < 0$ at $p_m - \Delta < p < p_m + \Delta$, where $\Delta = 2\rho_1 c_1^2 \epsilon / \mu^3 = O(\epsilon)$. Then, expanding $u(p')$ in powers of p' , taking into account that for SWR p' is of order ϵ , or smaller and dropping terms higher than $O(\epsilon^3)$, after some calculations we reduce the general equation (4) to the form of Eq. (1), where term $\alpha p'$ should be replaced by the following expression

$$\alpha p' \rightarrow \frac{\rho_1^2 c_1^3}{2} \left[\left(\frac{\partial^2 V}{\partial p^2} \right)_s p' + \frac{1}{2} \left(\frac{\partial^3 V}{\partial p^3} \right)_s p'^2 + \frac{1}{6} \left(\frac{\partial^4 V}{\partial p^4} \right)_s p'^3 \right],$$

and all derivatives $(\partial^n V/\partial p^n)_s$ are taken at $p = p_1$.

It is more convenient, however, to rewrite the equation in more universal dimensionless form. Let us introduce new variables

$$y \equiv \frac{p'}{\Delta}; \quad \tau \equiv \frac{\epsilon^6}{\mu^6 c_1 a} t; \quad \zeta \equiv \frac{\epsilon^3}{\mu^3 c_1^2 a} (x + c_1 t).$$

In these variables involving expression (7) one can reduce the generalized Burgers equation to the following final form

$$\frac{\partial y}{\partial \tau} - f(y) \frac{\partial y}{\partial \zeta} = \frac{\partial^2 y}{\partial \zeta^2}, \quad (8)$$

where

$$f(y) \equiv (z^2 - 1)y + zy^2 + \frac{y^3}{3}; \quad z \equiv \frac{p_1 - p_m}{\Delta}. \quad (9)$$

To examine solutions of this equation, which correspond to SWR we should supplement it with the boundary conditions $\partial y / \partial \zeta \rightarrow 0$ at $\zeta \rightarrow \pm\infty$; $y \rightarrow y_1 = 0$ at $\zeta \rightarrow -\infty$; $y \rightarrow y_2 = \text{const} < 0$ at $\zeta \rightarrow \infty$ and bear in mind that $y \leq 0$; $-1 \leq z \leq 1$. To study steady SWR it is convenient to introduce a traveling coordinate $\eta \equiv \zeta + \nu\tau$, where in accordance with Eq (5) the dimensionless velocity ν equals the following expression

$$\nu = \frac{z^2 - 1}{2} y_2 + \frac{zy_2^2}{3} + \frac{y_2^3}{12}$$

Then, integration of Eq. (8) yields

$$\eta = -12 \left[-\frac{\log |y|}{y_2 y_3 y_4} + \frac{\log |y - y_2|}{y_2 (y_3 - y_2) (y_4 - y_2)} + \frac{\log |y - y_3|}{y_3 (y_3 - y_4) (y_3 - y_2)} + \frac{\log |y - y_4|}{y_4 (y_4 - y_3) (y_4 - y_2)} \right], \quad (10)$$

where $y_{3,4}$ are the roots of the equation

$$y^2 + (4z + y_2)y + y_2^2 + 4zy_2 + 6(z^2 - 1) = 0 \quad (11)$$

Let us show that $y_{3,4}$ are always real and one of these root is smaller than y_2 , while the other is bigger than y_1 (we remind that by definition $y_1 = 0$). In other words the left hand side of Eq. (11) is negative at $y_2 < y < 0$ and any possible values of z and y_2 . To prove it note that for SWR pressure p_2 must satisfy the obvious condition $p_m - \Delta \leq p_2 \leq p_1$, which in the dimensionless variables is transformed into the inequality

$$-(z + 1) \leq y_2 \leq 0.$$

Due to the fact that y^2 enters into Eq. (11) with a positive coefficient to prove negativeness of this polynomial at $y_2 < y < 0$ it suffices to examine its values at the marginal points $y = y_{1,2}$. At $y = y_1 = 0$ we obtain

$$y_2^2 + 4zy_2 + 6(z^2 - 1). \quad (12)$$

For the same reason it is sufficient to inspect the values of polynomial (12) at the marginal values of y_2 , namely at $y_2 = -(z + 1)$, $y_2 = 0$. It is straightforward to see that the marginal values of Eq. (12) are negative at $|z| < 1$. Negativeness of polynomial (11) at $y = y_2$ is proved in

the same manner. The proved relative position of points $y_{1,2,3,4}$ guarantees that the derivative $dy/d\eta$ is negative at $y_2 < y < y_1$ for any possible values of z and y_2 , i.e., the profile of the steady SWR is a monotonously decreasing function of η .

To end this Letter we present several particular versions of the general solution (10), when the dependence $y(\eta)$ may be obtained explicitly.

(i) $z = 1$ $y_2 = -2$, which corresponds the maximal possible amplitude of SWR ($p_1 = p_m + \Delta$, $p_2 = p_m - \Delta$).

$$\eta = -\frac{3}{2} \log \left| \frac{2 + y}{y} \right| - \frac{3}{2\sqrt{5}} \log \frac{y + 1 + \sqrt{5}}{\sqrt{5} - 1 - y}.$$

(ii) $z = 1$ $|y_2| \ll 1$ $p_1 = p_m + \Delta$, $p_1 - p_2 \ll \Delta$. In this case the leading approximation in small $|y_2|$ yields

$$y = y_2 \sqrt{\frac{1 + \tanh(y_2^2 \eta / 3)}{2}}$$

(iii) $z = -(1 + y_2)$, $|y_2| \ll 1$ [$p_1 = p_m - \Delta - (p_1 - p_2)$, $p_1 - p_2 \ll \Delta$]. The leading approximation in $|y_2|$ in this case results in the following profile

$$y = y_2 \left(1 - \sqrt{\frac{1 - \tanh(y_2^2 \eta / 3)}{2}} \right)$$

Note, that while in the present Letter only steady solutions of Eq. (8)–(9) are discussed, the equation itself describes much more broad spectrum of problems of non-steady flows, including such very interesting issues, as evolution of arbitrary initial profiles, collision of shocks, etc. The corresponding study is in progress and results will be reported elsewhere.

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* Electronic address: tribel@scroll.apphy.fukui-u.ac.jp

† Electronic address: anisimov@itp.ac.ru

[1] L. D. Landau, and E. M. Lifshitz, *Fluid Mechanics - 2nd ed.* (Butterworth-Heinemann, Oxford, 1995).

[2] For a wave traveling in the opposite direction we would have $u(p') = v(p') + c(p')$.